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REFLEXIVE OPEN MAPPINGS ON GENERALIZED GRAPHS

STANISLAW MIKLOS

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Abstract

In this paper we show that a locally connected and locally compact metric image of a generalized graph under a reflexive open mapping is a generalized graph; further, we characterize all acyclic generalized graphs X with the property that any locally one-to-one reflexive open mapping of X into a Hausdorff space is globally one-to-one. Several problems are posed and some examples are given.

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Whyburn [11, page 182] has shown that an open image of a graph (that is, a one-dimensional connected polytope) is a graph. Eberhart, Fugate and Gordh generalize this result, showing that the weakly confluent image of a graph is a graph [5, Theorem II.6]. In [1], we proved that the metric open image of a generalized graph (that is, a connected space embeddable into a graph) is a generalized graph, too. A generalization of open mappings is the class of reflexive open mappings. Recall that a (continuous) mapping f from a topological space onto another topological space is said to be reflexive open if $f^{-1}(f(U))$ is open whenever U is open (see [4, page 597]). Therefore the following problem is natural.

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Problem 1. Characterize the class of all images of generalized graphs under reflexive open mappings.

An important subclass of reflexive open mappings are one-to-one mappings (see [3]). So we have

Problem 2. Characterize the class of all images of generalized graphs under one-to-one mappings.

In this paper we give a partial answer to Problems 1 and 2. Namely we reduce the class of images of generalized graphs under one-to-one mappings and also reflexive open mappings to locally connected and locally compact metric spaces. Moreover, we characterize all acyclic generalized graphs X with the property that any locally one-to-one reflexive open mapping of X into a Hausdorff space is globally one-to-one.

Lelek and McAuley [9, page 320] have proved the following

Theorem A (Lelek and McAuley). *If a locally connected and locally compact metric space Y is a one-to-one continuous image of the line, then Y is homeomorphic to one of the five objects* (I)–(V) *listed in Figure* 1.

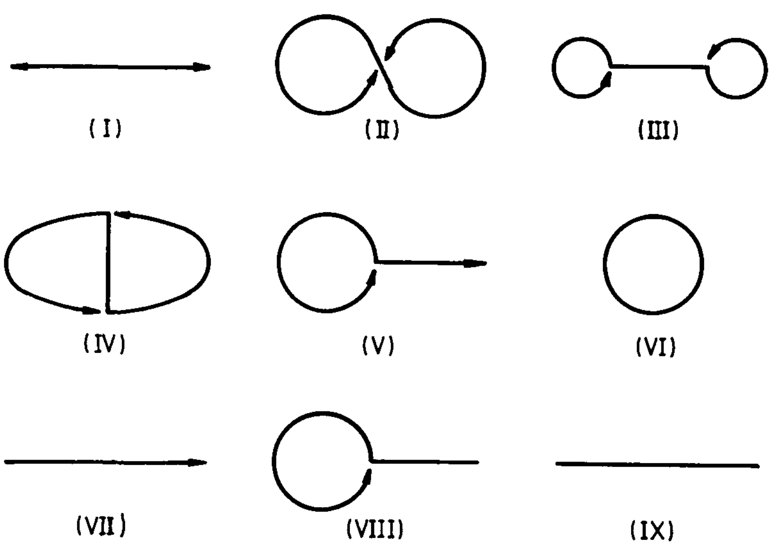

Figure 1

By an open arc we mean the homeomorphic image of the line and by a half-open arc we mean the homeomorphic image of the half-line.

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Applying these notions and using the same argument as in the proof of [9, Theorem 1] we get

THEOREM 1. *If a locally connected and locally compact metric space Y is a one-to-one continuous image of the half-open arc, then Y is homeomorphic to one of the three objects* (VI)–(VIII) *of Figure* 1.

Now we prove a proposition which will be used later to obtain a stronger result (Theorem 2 below).

PROPOSITION. *Let X be a generalized graph, Y a locally connected and locally compact metric space, and let f be a one-to-one mapping from X onto Y. Then Y is a generalized graph.*

PROOF. If X is either an open arc or a half-open arc, then Theorems A and 1 imply that $f(X)$ is a generalized graph. Thus let X be an arbitrary generalized graph different from an open arc and a half-open arc. Further, if X is a graph, then f is a homeomorphism, whence $f(X)$ is a graph. If X is not a graph (that is, if X is not compact), then there exists a graph G and a homeomorphic embedding $h: X \to G$ such that $G \backslash h(X)$ is a finite set of end points of G (see [1, Theorem 1(iii)]). We put $G' = h(X)$ and $G \backslash G' = \{e_1, e_2, \dots, e_n\}$. For $i = 1, 2, \dots, n$, let e_i' denote a ramification point of G' adjacent to e_i, and let each arc $e_i'e_i$ be ordered from e_i' to e_i. Further, let $a_1^i, a_2^i, a_3^i, \dots$ be an infinite increasing sequence tending to e_i, where $a_1^i = e_i'$ and $a_j^i \in e_i'e_i \backslash \{e_i\}$, whenever $j \in \{1, 2, 3, \dots\}$. For $i = 1, 2, \dots, n$ we consider the set $H^i = \bigcap_{j=1}^{\infty} \operatorname{cl}\{g(x): x \geq a_j^i\}$, where $g: G' \to Y$ is defined by $g = fh^{-1}$. Since Y is locally connected and locally compact, we conclude that H^i is either empty or a singleton (compare with the proof of [9, Theorem 1]). Now, if $H^i = \varnothing$ for $i = 1, 2, \dots, n$, then g is a homeomorphism, whence Y is a generalized graph. If we have $H^i = \{p\}$ for some $i \in \{1, 2, \dots, n\}$, then the mapping g is not a homeomorphism because the function $g^{-1}: Y \to G'$ is not continuous at the point p. Thus, if $q \in g^{-1}(p)$, then $\operatorname{ord}_q G' < \operatorname{ord}_p Y$, where $\operatorname{ord}_z Z$ denotes the Menger-Urysohn order at a point z in a space Z (see [11, page 48] or [8, page 274]). The point p is contained in the intersection of at most n sets H^i. Thus, since g is one-to-one we have $\operatorname{ord}_p Y \leq \operatorname{ord}_q G' + n$. Further, since the number of the sets H^i which are non-empty is finite and G' is a generalized graph, we have that (1) all points of Y are of some finite order, and (2) almost all points of Y are of order two. Since Y is connected as a continuous image of a connected space, it must be a generalized continuum, that is, a connected and locally compact space. Hence by (1), (2) and [1, Theorem 1(v)] Y is a generalized graph. The proof is complete.

The proposition can be generalized as follows.

THEOREM 2. *Let X be a generalized graph, Y a locally connected and locally compact metric space, and let f be a reflexive open mapping from X onto Y. Then Y is a generalized graph.*

PROOF. From [12, Theorem 5] (compare with [4, Theorem 3.6]), f factors uniquely as $f = hm$ where $m: X \to m(X)$ is an open mapping and $h: m(X) \to Y$ is a one-to-one mapping. Since X is a generalized graph and Y is metric, $m(X)$ is also a generalized graph (see [1, Theorem 2]). Thus, by the proposition, Y is a generalized graph and the proof is complete.

REMARK 1. It is not difficult to check that local connectedness is an essential condition in all theorems above (compare a remark in [9, page 321]). For instance, the "Warsaw circle" (that is, the curve $\sin(1/x)$) is a one-to-one and continuous image of the half-line. However, by [6, Theorem 2] we infer that local connectedness, in all theorems above, can be replaced by the condition that Y is connected im kleinen or, more generally, by aposyndesis of Y. Further, by [6, Theorem 4] it can be seen that planability of Y can be substituted for local compactness in all theorems above.

Jungck [7, page 43] has proved that if a locally connected and a locally compact metric space Y is the image of the line under a reflexive open map which is either locally or globally one-to-one, then Y is homeomorphic to one of the six objects (I)–(VI) above (Figure 1). From [12, Theorem 5] it is easy to show the following

THEOREM 3. *If a locally connected and locally compact metric space Y is the image of the line under a reflexive open mapping, then Y is homeomorphic to one of the nine objects* (I)–(IX) *above* (*see Figure* 1).

Let a graph G, containing at least one end point e, be given. A generalized graph $G\backslash\{e\}$ will be called a 1-generalized graph.

Using this notion we formulate the next result.

THEOREM 4. *Any locally one-to-one reflexive open mapping of an acyclic generalized graph X into a Hausdorff space Y is globally one-to-one if and only if X is either a graph or a generalized graph.*

PROOF. First we prove that, if an acyclic generalized graph X is a graph or a 1-generalized graph, then any locally one-to-one reflexive open mapping f from X into a Hausdorff space Y is globally one-to-one. Indeed, if X is

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a graph, then f is open by [4, Corollary 3.7.2], whence it is a local homeomorphism (for the definition see [11, page 199]) on X. Acyclicity of X implies that f is a homeomorphism (see [2, Corollary 2]). If X is an acyclic 1-generalized graph we suppose on the contrary that there is some f which is locally one-to-one but not globally one-to-one. Thus there are two different points a, b in X with $f(a) = f(b)$. Since an arc ab is compact and f is locally one-to-one, there exists an arc cd in ab such that $f(c) = f(d)$ and f is globally one-to-one on the half-open arc $cd \setminus \{d\}$. But then $f(cd)$ is homeomorphic to a simple closed curve S. We consider $f^{-1}(S)$. Since S is compact in the Hausdorff space Y, it is closed. Thus $f^{-1}(S)$ is closed in X. Let C be a component of $f^{-1}(S)$ containing the arc cd. Note that C is atroidic. Otherwise, C contains a ramification point r. Since f is locally one-to-one, there is a closed neighbourhood U of r such that the partial mapping $f|U\colon U \to S$ is globally one-to-one. Since U contains a ramification point, also S contains a ramificiation point, a contradiction. Further, since X is a 1-generalized graph, C is either an arc or a half-open arc, and hence C contains at least one end point e (of C). Let V be an open neighborhood of e such that the partial mapping $f|\overline{V}$ is one-to-one. Further, consider a component K of $f^{-1}(f(V))$ such that $K \cap V = \varnothing$ and $K \cap cd \neq \varnothing$. Note that if $f(e) = f(c) = f(d)$, then $K \cap cd = \{p\}$, where $p \in f^{-1}(f(e))$, and if $f(e) \neq f(c) = f(d)$, then $K \cap cd$ is either an arc or a half-open arc with an end point $q \in f^{-1}(f(e))$. Thus $K \cap cd$ is not open in cd, and moreover we conclude K is not open in X. Hence $f^{-1}(f(V))$ is not open in X, contrary to reflexive openness of f. This contradiction shows that f is globally one-to-one.

Second, let X be an acyclic generalized graph which is neither a graph nor a 1-generalized graph. We show that there exists a locally one-to-one reflexive open mapping from X into a Hausdorff space Y which is not globally one-to-one. Indeed, since X is neither a graph nor a 1-generalized graph, it contains an open arc A which is closed in X (see the definition of an open arc). Let ab be an arc in A whose interior contains all ramification points of X contained in A, and let B and C be components of $A \setminus ab$. Define a mapping f on X in such a way that the partial mappings $f|X \setminus B$ and $f|X \setminus C$ are homeomorphisms and $f(A)$ is homeomorphic to a simple closed curve.

It is easy to verify that $Y = f(X)$ is a generalized graph and f is open (thus reflexive open) and locally one-to-one because $f|X \setminus B$ and $f|X \setminus C$ are homeomorphisms. However, f is not globally one-to-one because $f(A)$ is homeomorphic to the simple closed curve. The proof is complete.

As a consequence of Theorem 4 we get the following corollary.

Corollary [7, Theorem (9.4)]. *Any locally one-to-one and reflexive open map f of the half-line into a Hausdorff space is globally one-to-one.*

Remark 2. Acyclicity of the generalized graph X is a necessary hypothesis in Theorem 4 even if we additionally assume that X is a graph. Indeed, for X and Y being graphs as in Figure 2, it is easy to define a locally one-to-one reflexive open mapping from X onto Y which is not globally one-to-one. However any locally one-to-one reflexive open mapping on a graph is a local homeomorphism. Graphs admitting a local homeomorphism which is not a homeomorphism are characterized in [10].

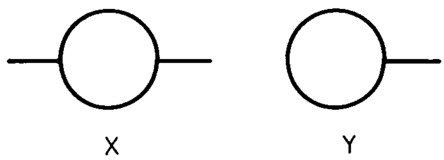

Figure 2

Problem 3. Characterize all non-compact generalized graphs X with the property that any locally one-to-one and reflexive open mapping from X into a Hausdorff space is globally one-to-one.

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Institute of Mathematics
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław
Poland