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REAL-TIME MONITORING WITH RCA MODELS

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We propose a family of *weighted* statistics based on the CUSUM process of the WLS residuals for the online detection of changepoints in a Random Coefficient Autoregressive model, using both the standard CUSUM and the Page-CUSUM process. We derive the asymptotics under the null of no changepoint for all possible weighing schemes, including the case of the standardized CUSUM, for which we derive a Darling–Erdös-type limit theorem; our results guarantee the procedure-wise size control under both an open-ended and a closed-ended monitoring. In addition to considering the standard RCA model with no covariates, we also extend our results to the case of exogenous regressors. Our results can be applied irrespective of (and with no prior knowledge required as to) whether the observations are stationary or not, and irrespective of whether they change into a stationary or nonstationary regime. Hence, our methodology is particularly suited to detect the onset, or the collapse, of a bubble or an epidemic. Our simulations show that our procedures, especially when standardising the CUSUM process, can ensure very good size control and short detection delays. We complement our theory by studying the online detection of breaks in epidemiological and housing prices series.

1. INTRODUCTION

Economic history is full of events where stationary observations suddenly become explosively increasing, or—vice versa—where explosive trends crash. A classical example are financial bubbles, where periods of "tame" fluctuations are followed by exuberant growth, in turn then followed by a collapse which is typically modeled as a stationary regime (see e.g., Harvey, Leybourne, and Sollis, 2017 and Phillips and Shi, 2018). Historical examples include the "Tulipmania" in the Netherlands in the 1630s and the "South Sea" bubble of 1720. More recent instances include the Japan's real estate and stock market bubble of the 1980s, which after the collapse turned into a negative period, the so called "Lost Decade"; and, in the US, the "Dotcom" bubble of the 1990s and the housing bubble between 1996 and 2006. Similar phenomena, with observations undergoing a change in

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persistence or even more radical changes in their nature (from stationary to explosive and vice versa), are also encountered in applied sciences. For example, in epidemiology, the onset of a pandemic is characterized by a sudden, explosive growth in the number of cases, followed by a return to a stationary regime when the pandemic subsides. Hence the importance of developing tools allowing to test for changes in persistence, or a switch between regimes.

In particular, a research area where this issue has been paid particular attention is financial econometrics, where several methodologies have been developed to test (ex-ante or ex-post) for the onset or collapse of a financial bubble. In the interest of a concise discussion, we refer the reader to the seminal articles on ex-post detection by Phillips, Wu, and Yu (2011), and Phillips, Shi, and Yu (2015a), and also to Skrobotov (2023) for a review. In addition, several contributions deal with the real-time detection of bubble episodes see, *inter alia*, Homm and Breitung (2012), Phillips, Shi, and Yu (2015b), and a recent contribution by Whitehouse, Harvey, and Leybourne (2023); also, we refer to a recent and comprehensive review on the issue of online changepoint detection by Aue and Kirch (2024). A common trait to this literature is that it relies on an AutoRegressive (AR) framework as the workhorse model. Using an AR specification has several advantages: it is a parsimonious and well-studied set-up, and it naturally lends itself to modeling both stationary and nonstationary regimes. Phillips and Yu (2011) show that an AR model with explosiveness is an adequate representation for bubble behavior under very general assumptions. Moreover, an AR specification lends itself to constructing tests for bubble behavior using the Dickey–Fuller test and its variants-e.g., the Augmented DF test (ADF; see Diba and Grossman, 1988), the sup-ADF test (Phillips et al., 2011; Phillips and Yu, 2011), and the generalized sup-ADF test (Phillips et al., 2015a; Phillips et al., 2015b). From a technical viewpoint, however, using an AR model is fraught with difficulties when monitoring for changes from an explosive to a stationary regime, although several promising solutions have been proposed such as the reverse regression approach by Phillips and Shi (2018). In addition to the ability of detecting changes, ensuring a timely detection is important. To the best of our knowledge, no optimality results exist; indeed, typical procedures are based on *unweighted* CUSUM (or related) statistics, which are not designed to ensure optimal detection timings (see e.g., Aue and Horváth, 2004). A recent contribution by Otto and Breitung (2023) considers a (backward) CUSUM-type test, which is similar, in spirit, to the so-called "Page-CUSUM" approach (see also below for a further description of this approach), showing that this methodology has excellent power as far as ex-post detection is concerned; however, even this contribution is not designed for possibly nonstationary data, and its performance in terms of timely online detection is yet to be explored.

Main contributions of this article

We address the general issue of online, real-time detection of changes in persistence and also between regimes (i.e., from stationarity to nonstationarity and vice versa) by using a Random Coefficient Autoregressive (RCA) model, viz.

$$y_i = \left(\beta_i + \epsilon_{i,1}\right) y_{i-1} + \epsilon_{i,2}, \qquad (1.1)$$

where y_0 denotes an initial value. The RCA model was firstly studied by Anděl (1976) and Nicholls and Quinn (2012). It is nonlinear in nature, due to the fact that the "marginal effect" of y_{i-1} onto y_i is allowed to change over time, and also due to the well-documented fact that "linear techniques", such as e.g., first-differencing to obtain stationarity, in general cannot work (Leybourne, McCabe, and Tremayne, 1996); hence, it is one of the possible specifications belonging in the wider class of nonlinear models for time series (see Fan and Yao, 2008), which have been proposed "as a reaction against the supremacy of linear ones,a situation inherited from strong, though often implicit, Gaussian assumptions" (Akharif and Hallin, 2003). The RCA model is particularly well-suited to modeling time series with potentially explosive behavior; for example, in the seminal paper by Diba and Grossman (1988), a rational bubble is modeled as having RCA dynamics (see their equation (17)). Model (1.1) also nests, as a special case corresponding to having $\beta_i = 1$, the so-called Stochastic Unit Root model (STUR; see Granger and Swanson, 1997), and it also allows for the possibility of (conditional) heteroskedasticity in y_i; Tsay (1987) shows that the widely popular ARCH model by Engle (1982) can be cast into the RCA setup, which therefore can be viewed as a second-order equivalent; and Ling (2004) studies the conditions under which the RCA model is equivalent in distribution to the Double AutoRegressive (DAR) model[1]. Two major advantages of the RCA model are that: (a) it is possible to test whether a nonlinear specification like the one in (1.1) is adequate as opposed to a linear, AutoRegressive specification (e.g., using the test by Akharif and Hallin, 2003, or Horváth and Trapani, 2019); and, (b) it is possible to construct estimators of β_0 that are always asymptotically normal, irrespective of whether y_i is stationary or nonstationary (Aue and Horváth, 2011). Although the literature has developed several contributions on ex-post changepoint detection using an RCA model (e.g. Horváth and Trapani, 2023a), to the best of our knowledge there are still significant limitations in the context of online detection,e.g., Na, Lee, and Lee (2010), Li, Tian, and Qi (2015) and Li et al. (2015) all consider sequential changepoint detection using unweighted CUSUM-based statistics, but only under the maintained assumption of stationarity, which precludes the ability to detect changes from stationary to exponential behavior, and vice versa.

In this article, we fill the gaps discussed above, by proposing a family of statistics based on the Weighted Least Squares (WLS) estimator, and in particular on the *weighted* CUSUM process of the WLS residuals, designed to ensure a faster detection than unweighted statistics. We study the standard CUSUM process, and the so-called "Page-CUSUM" (Fremdt, 2015); and both "open-ended" and "closed-ended" procedures, where sequential monitoring goes on indefinitely, or stops at a pre-specified point in time, respectively. We make at least five contributions

[1] See also our discussion in Section 7 below for a more detailed comparison between the two models.

to the current literature on online changepoint detection. First, we derive the limiting distribution of our statistics under the closed-ended case; typically, in the literature, the critical values obtained under the open-ended case are employed even for the closed-ended case, as a (conservative) choice, ultimately leading to loss of power. Second, we study the case of a closed-ended procedure with a "very short" monitoring horizon, adapting the boundary function for this case, and deriving the corresponding limiting distribution; this case has not been considered in the literature before, but it is of practical relevance because the researcher may prefer to carry out monitoring for a short time, and then—in the absence of changepoints—restart the procedure afresh. Third, in the case of the *standardized* CUSUM,[2] we propose an approximation to compute critical values which offers a superior alternative to asymptotic critical values, whose accuracy is marred by the notoriously weak convergence to the Extreme Value Distribution; simulations show that our approximation works extremely well even in small samples, and for all cases considered, offering size control and short detection delays. Fourth, we derive the limiting distribution of the *weighted* Page-CUSUM statistics, showing also that such a limit is a well-defined random variable. Fifth, as well as studying the basic RCA model, we also develop the full-blown theory (irrespective of whether y_i is stationary or not) for the case where there are covariates in (1.1); to the best of our knowledge, this is the first contribution to deal with this case. As a final remark, we would like to add that a major advantage of the RCA set-up is that our statistics can be employed under both stationarity and explosiveness with no modifications required; indeed, no prior knowledge on the stationarity of the observations is required. Hence, our methodology can be applied to detect changes in the persistence of a stationary series, or changes from stationarity to a non-stationary regime, or vice versa from a non-stationary regime to a stationary one. This has important practical consequences, allowing for e.g., faster responses from public health authorities in the presence of a pandemic, or from policy makers in the presence of inflationary shocks, or a more accurate date stamping of bubble onsets and collapses. In conclusion, we point out that our article is part of a wider research program by the authors on inference in the context of the RCA model, and we would like to comment on its relation to previous contributions. In particular, this contribution is naturally related to the papers by Horváth and Trapani (2023a), Horváth and Trapani (2023b) and Horváth, Trapani, and VanderDoes (2024), where ex-post tests (based on the maximally selected CUSUM process, the $\mathcal{L}_p$-norms of the CUSUM process, and the maximally selected Likelihood Ratio test, respectively) are proposed; in turn, these are bound to be helpful when verifying the so-called "non-contamination assumption", which stipulates that no changepoint occurs during the training period. Horváth and Trapani (2019) propose a test for the null hypothesis that, in (1.1), $Var\left(\epsilon_{i,1}\right) = 0$, i.e. for the null that an RCA specification is not appropriate, and an AR model is preferable; in turn, the theory

[2]We define "standardized CUSUM" the CUSUM process weighted using weights proportional to its standard deviation, as opposed to the "weighted CUSUM" which uses lighter weights, see also below for a formal definition.

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developed in this article requires $Var\left(\epsilon_{i,1}\right) > 0$, at least in the nonstationary case, or the asymptotic theory developed hereafter is no longer valid. Hence, the test by Horváth and Trapani (2019) should be implemented at the outset, in order to verify that an RCA model is indeed adequate for the data. Finally, we would like to bring the reader's attention to Horváth and Trapani (2016), where several (anti)concentration inequalities, which are the building block of our asymptotics, are derived.

The remainder of the article is organized as follows. We introduce our main model, and the test statistics, in Section 2. The asymptotics under the null and the alternative is reported in Section 3. We consider the extension of the RCA model to the case of exogenous regressors in Section 4. Monte Carlo evidence is in Section 5; Section 6 contains two empirical applications, to Covid-19 hospitalization data and house prices. Section 7 concludes. In the Supplementary Material, we report further Monte Carlo evidence (to complement Section 5), further empirical evidence (to complement Section 6), the extension of our theory to the case of deterministics, and all technical lemmas and proofs.

NOTATION. Henceforth, we use: '$\rightarrow$' for the ordinary limit; '$\xrightarrow{\mathcal{P}}$' for convergence in probability; 'a.s.' for 'almost surely'; '$\xrightarrow{a.s.}$' for almost sure convergence; and '$\|\cdot\|$' for the Euclidean norm of vectors and matrices. Finally, $\{W(t), 0 \leq t \leq 1\}$ denotes a standard Wiener process. Other notation is introduced later on in the article.

2. SEQUENTIAL MONITORING OF RCA MODELS

Recall the RCA model

$$y_i = \left(\beta_i + \epsilon_{i,1}\right) y_{i-1} + \epsilon_{i,2}, \tag{2.1}$$

and recall that y_0 denotes an initial value. We begin by laying out our first assumption on the behavior of the innovations $\epsilon_{i,1}$ and $\epsilon_{i,2}$, which is required throughout the article in all cases considered.

Assumption 2.1. *(i)*$\{\left(\epsilon_{i,1}, \epsilon_{i,2}\right), -\infty < i < \infty\}$ are independent and identically distributed random variables with (a) $E\epsilon_{i,1} = E\epsilon_{i,2} = 0$; (b) $0 < E\epsilon_{i,1}^2 = \sigma_1^2 < \infty$ and $0 < E\epsilon_{i,2}^2 = \sigma_2^2 < \infty$; (c) $E\epsilon_{i,1}\epsilon_{i,2} = 0$; (d) $E|\epsilon_{i,1}|^\kappa < \infty$ and $E|\epsilon_{i,2}|^\kappa < \infty$ for some $\kappa > 4$; *(ii)* y_0 is independent of $\{\left(\epsilon_{i,1}, \epsilon_{i,2}\right), -\infty < i < \infty\}$ and such that $E|y_0|^\kappa < \infty$ for some $\kappa > 0$.

Part *(i)* of the assumption is standard, and we refer to Horváth and Trapani (2023a), inter alia, as an example. Indeed, the condition $E\epsilon_{i,1}\epsilon_{i,2} = 0$ could also be lifted (see e.g. Conlisk, 1974, Hill and Peng, 2014, and Regis, Serra, and van den Heuvel (2022, Sect. 4.2)), although this is not the classical RCA model of Nicholls and Quinn (2012) which we study here; in our case, a technical difficulty would be how to derive the exact rate of divergence of $|y_i|$ (following and merely adapting the proof of Lemma A.4 in Horváth and Trapani (2016) would not be straightforward).

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Further, as is typical in this literature, Assumption 2.1 states that the innovations $\epsilon_{i,1}$ and $\epsilon_{i,2}$ are independent across i. We use this assumption throughout the article, in order not to overshadow the main arguments; however, we note that it is possible to relax the independence assumption of part *(ii)*. Indeed, Horváth and Trapani (2023a) extend the RCA model to the case of dependent innovations, assuming that both $\epsilon_{i,1}$ and $\epsilon_{i,2}$ are weakly dependent.[3] In such a case, it can be shown that all our results still hold, and the only impact of weak dependence is on the asymptotic variance $\mathfrak{s}^2$ defined in (2.6); Horváth and Trapani (2023a) derive an estimator of the asymptotic variance $\mathfrak{s}^2$ valid under weak dependence.[4]

It is well-known (Aue, Horváth, and Steinebach, 2006) that, in (2.1), the stationarity or lack thereof of y_i is determined by the value of $E\log\left|\beta_0+\epsilon_{0,1}\right|$:

- if $-\infty \leq E\log\left|\beta_0+\epsilon_{0,1}\right| < 0$, then y_i converges exponentially fast to a strictly stationary solution $\left\{\bar{y}_i, -\infty < i < \infty\right\}$ for all initial values y_0;
- if $E\log\left|\beta_0+\epsilon_{0,1}\right| > 0$, then y_i is nonstationary with $|y_i| \stackrel{a.s.}{\to} \infty$ exponentially fast (Berkes, Horváth, and Ling, 2009);
- if $E\log\left|\beta_0+\epsilon_{0,1}\right| = 0$, then $|y_i| \stackrel{\mathcal{P}}{\to} \infty$, but at a rate slower than exponential (see Horváth and Trapani (2016, Lem. A.4)).

We further assume that the autoregressive parameter β_i is constant over the training segment $\{y_i, 1 \leq i \leq m\}$.

Assumption 2.2. $\beta_i = \beta_0$ for $1 \leq i \leq m$.

The requirement in Assumption 2.2 is known as the *non-contamination assumption* (see Chu, Stinchcombe, and White, 1996). The assumption is testable by using an ex-post changepoint test; for example, one could use the test by Horváth and Trapani (2023a) which: is designed specifically for the RCA set-up; can be applied irrespective of whether y_i is stationary or not; and—similarly to our proposed sequential tests—is based on the CUSUM process of the WLS residuals.

We subsequently test for the null hypothesis that, as new data come in after m, β_i remains constant, viz.

$$H_0 : \beta_0 = \beta_{m+1} = \beta_{m+2} = \dots \tag{2.2}$$

Several techniques are available to estimate (2.1). Berkes et al. (2009) and Aue and Horváth (2011) study Quasi Maximum Likelihood (QML) estimation under both stationarity and nonstationarity, see also Hill, Li, and Peng (2016), who study an Empirical Likelihood estimator, and Horváth et al. (2024), for an application of QML to ex-post changepoint detection. Koul and Schick (1996) (see also Schick, 1996) show that WLS, which is computationally simpler, is first-order equivalent to QML. Hereafter, we use the following WLS estimator, computed using the

[3] With also the "weak exogeneity" requirements that $E\left(\epsilon_{i,1}y_{i-1}^2/\left(1+y_{i-1}^2\right)\right)=0$ and $E\left(\epsilon_{i,2}y_{i-1}/\left(1+y_{i-1}^2\right)\right)=0$.

[4] Indeed, in an unreported robustness check for our empirical applications in Section 6, we also carry out our tests using the estimator of $\mathfrak{s}^2$ derived under dependence; results are unchanged, which can be taken as a heuristic indication that the *i.i.d.* assumption provides a good approximation at least as far as our datasets are concerned.

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observations on the training segment $\{y_i, 1 \leq i \leq m\}$ is the solution to

$$\widehat{\beta}_m = \arg\min_{\beta} \sum_{i=2}^{m} \frac{(y_i - \beta y_{i-1})^2}{1 + y_{i-1}^2},$$

corresponding to

$$\widehat{\beta}_m = \left(\sum_{i=2}^{m} \frac{y_{i-1}^2}{1 + y_{i-1}^2} \right)^{-1} \left(\sum_{i=2}^{m} \frac{y_i y_{i-1}}{1 + y_{i-1}^2} \right). \tag{2.3}$$

We note that the choice of the weights $\left(1 + y_{i-1}^2\right)^{-1}$ is not unique; the rationale for our choice in (2.3) is based on considering the "error term" $\epsilon_{i,1} y_{i-1} + \epsilon_{i,2}$, whose (conditional) variance is $Var\left(\epsilon_{0,1}\right) y_{i-1}^2 + Var\left(\epsilon_{0,2}\right)$, whence the weights containing the y_{i-1}^2 term. The use of $\left(1 + y_{i-1}^2\right)^{-1}$ is proposed in Schick (1996), where it is also shown that efficiency is attained when using $Var\left(\epsilon_{0,1}\right)$ and $Var\left(\epsilon_{0,2}\right)$, or consistent estimators thereof. However, in our context we also allow for nonstationarity, and in this case $Var\left(\epsilon_{0,2}\right)$ cannot be estimated consistently (see e.g., Horváth and Trapani (2019, Lem. A10)).

2.1. Standard Weighted CUSUM-based Detectors

The building block of our statistics are the weighted WLS residuals defined as

$$\frac{\left(y_i - \widehat{\beta}_m y_{i-1}\right) y_{i-1}}{1 + y_{i-1}^2}, \quad \text{for } i = m+1, \ldots$$

The weighted WLS residuals have been employed, for the purpose of ex-post changepoint detection, in Horváth and Trapani (2023a), and have one main advantage: their (suitably normed) partial sums converge to a Gaussian process irrespective of whether the observations y_i are stationary or not. Intuitively, this is due to the fact that, under nonstationarity, the y_is are "big", but the presence of the similarly "big" weight $1 + y_{i-1}^2$ at the denominator balances the whole ratio, ensuring that the residuals are bounded, this fact is also known as "variance-induced stationarity", and it has been exploited also in other, similar contexts such as DAR models (Cavaliere and Rahbek, 2021).

Consider now the absolute value of the CUSUM process of the weighted WLS residuals, customarily known as the *detector*:

$$Z_m(k) = \left| \sum_{i=m+1}^{m+k} \frac{\left(y_i - \widehat{\beta}_m y_{i-1}\right) y_{i-1}}{1 + y_{i-1}^2} \right|, \quad k \geq 1. \tag{2.4}$$

Heuristically, under the null of no change, the residuals have zero mean; hence, the partial sum process inside the absolute value in the definition of $Z_m(k)$ should also fluctuate around zero with increasing variance. Conversely, in the presence of a break (at, say, k^*), $\widehat{\beta}_m$ is a biased estimator for the "new" autoregressive parameter β_{m+k^*+1}; thus, $Z_m(k)$ should have a drift term. Hence, a break is flagged if $Z_m(k)$

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exceeds a threshold. We call such a threshold the *boundary function*, and propose the following family

$$g_{m,\psi}(k) = c_{\alpha,\psi} \jmath m^{1/2} \left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}. \quad \textbf{(2.5)}$$

In (2.5), $0 \le \psi \le 1/2$, and $\jmath$ is defined as

$$\jmath^2 = \begin{cases} a_1\sigma_1^2 + a_2\sigma_2^2, & \text{if } -\infty \le E\log\left|\beta_0+\epsilon_{0,1}\right| < 0, \\ \sigma_1^2, & \text{if } E\log\left|\beta_0+\epsilon_{0,1}\right| \ge 0, \end{cases} \quad \textbf{(2.6)}$$

with σ_1^2 and σ_2^2 defined in Assumption 2.1, and

$$a_1 = E\left(\frac{\overline{y}_0^2}{1+\overline{y}_0^2}\right)^2, \quad \text{and} \quad a_2 = E\left(\frac{\overline{y}_0}{1+\overline{y}_0^2}\right)^2,$$

where $\left\{\overline{y}_i, -\infty < i < \infty\right\}$ is the stationary solution of (2.1). At this point, some clarification about the vocabulary employed hereafter—and, in particular, about the notions of "weights" and "weighted CUSUM"—is in order. In (2.5), the term $\left(\frac{k}{m+k}\right)^{\psi}$ is a weight which is designed to reduce the detection delay and, as Aue and Horváth (2004) and Aue et al. (2008), *inter alia*, show, reduces such delay more and more as ψ increases from 0 to $1/2$. Hereafter, we will always refer to the ratio $Z_m(k)/g_{m,\psi}(k)$ as the weighted CUSUM, and to $\left(\frac{k}{m+k}\right)^{\psi}$ as the weight function or weighting scheme, which is understood to be a function of ψ.

On account of (2.4) and (2.5), a changepoint is found at a stopping time $\tau_{m,\psi}$ defined as

$$\tau_{m,\psi} = \begin{cases} \inf\{k \ge 1 : Z_m(k) \ge g_{m,\psi}(k)\}, \\ \infty, \text{ if } Z_m(k) < g_{m,\psi}(k) \text{ for all } 1 \le k < \infty. \end{cases} \quad \textbf{(2.7)}$$

The constant $c_{\alpha,\psi}$ in (2.5) is chosen so as to ensure that: (a) under the null, the procedure-wise probability of Type I Error does not exceed a user-chosen value α, viz. $\lim_{m\to\infty} P\left\{\tau_{m,\psi} = \infty | H_0\right\} = \alpha$; and (b) under the alternative, $\lim_{m\to\infty} P\left\{\tau_{m,\psi} < \infty | H_A\right\} = 1$.

In (2.7), the monitoring goes on indefinitely ("open-ended"). On the other hand, in some applications it may be desirable to stop the sequential monitoring procedure after m^* period ("closed-ended"), e.g., in order to extend or update the training period. In this case, we use the same detector $Z_m(k)$ defined in (2.4), but we modify (2.7) as

$$\tau^*_{m,\psi} = \begin{cases} \inf\{1 \le k \le m^* : Z_m(k) \ge g^*_{m,\psi}(k)\}, \\ m^*, \text{ if } Z_m(k) < g^*_{m,\psi}(k) \text{ for all } 1 \le k \le m^*, \end{cases} \quad \textbf{(2.8)}$$

with boundary function

$$g^*_{m,\psi}(k) = c^*_{\alpha,\psi} \jmath m^{1/2}(1+k/m)(k/(m+k))^{\psi}. \quad \textbf{(2.9)}$$

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The boundary function $g^*_{m,\psi}(k)$ is suitable for the case where the monitoring horizon m^* is "long", i.e., when it goes on for a period which is at least proportional (or more than proportional) to the size of the training sample m. If a shorter monitoring horizon is considered, where $m^* = o(m)$, then we use the same detector as above, $Z_m(k)$, but the boundary function needs to be modified as

$$\overline{g}_{m,\psi}(k) = \overline{c}_{\alpha,\psi}\, \jmath\left(m^*\right)^{1/2-\psi} k^{\psi}. \tag{2.10}$$

In this case, the stopping rule is defined as

$$\overline{\tau}_{m,\psi} = \begin{cases} \inf\{1 \le k \le m^* : Z_m(k) \ge \overline{g}_{m,\psi}(k)\}, \\ m^*, \text{ if } Z_m(k) < \overline{g}_{m,\psi}(k) \text{ for all } 1 \le k \le m^*. \end{cases} \tag{2.11}$$

2.2. Page-CUSUM Detectors

In a series of recent contributions, Fremdt (2015), Kirch and Stoehr (2022a), and Kirch and Stoehr (2022a) study a different family of detectors, known as the "Page-CUSUM" processes, designed to offer a shorter detection delay:

$$Z^{\dagger}_m(k) = \max_{1\le \ell \le k} \left| \sum_{i=m+\ell}^{m+k} \frac{\left(y_i - \widehat{\beta}_m y_{i-1}\right) y_{i-1}}{1 + y^2_{i-1}} \right|, \quad k \ge 1. \tag{2.12}$$

Intuitively, this family of detectors searches for the "worst-case scenario" at each point in time k, and therefore should guarantee faster detection in the presence of a changepoint. Consistently with the approach studied in this article, we consider *weighted* versions of $Z^{\dagger}_m(k)$; the corresponding stopping rules are defined as:

$$\tau^{\dagger}_{m,\psi} = \begin{cases} \inf\{k \ge 1 : Z^{\dagger}_m(k) \ge g_{m,\psi}(k)\}, \\ \infty, \text{ if } Z^{\dagger}_m(k) < g_{m,\psi}(k) \text{ for all } 1 \le k < \infty, \end{cases} \tag{2.13}$$

with $g_{m,\psi}(k)$ defined in (2.5), for an open-ended procedure (replacing the critical value $c_{\alpha,\psi}$ with $c^{\dagger}_{\alpha,\psi}$);

$$\tau^{*\dagger}_{m,\psi} = \begin{cases} \inf\{1 \le k \le m^* : Z^{\dagger}_m(k) \ge g^*_{m,\psi}(k)\}, \\ m^*, \text{ if } Z^{\dagger}_m(k) < g^*_{m,\psi}(k) \text{ for all } 1 \le k \le m^*, \end{cases} \tag{2.14}$$

with $g^*_{m,\psi}(k)$ defined in (2.9), for the case of a closed-ended procedure with a long monitoring horizon (replacing the critical value $c^*_{\alpha,\psi}$ with $c^{*\dagger}_{\alpha,\psi}$); and

$$\overline{\tau}^{\dagger}_{m,\psi} = \begin{cases} \inf\{1 \le k \le m^* : Z^{\dagger}_m(k) \ge \overline{g}_{m,\psi}(k)\}, \\ m^*, \text{ if } Z^{\dagger}_m(k) < \overline{g}_{m,\psi}(k) \text{ for all } 1 \le k \le m^*, \end{cases} \tag{2.15}$$

with $\overline{g}_{m,\psi}(k)$ defined in (2.10), for the case of a closed-ended procedure with a short monitoring horizon (replacing the critical value $\overline{c}_{\alpha,\psi}$ with $\overline{c}^{\dagger}_{\alpha,\psi}$).

3. ASYMPTOTICS

We begin by listing a set of technical assumptions, which complement Assumption 2.1.

Assumption 3.1. If $E\log\left|\beta_0+\epsilon_{0,1}\right|<0$, it holds that $P\left(\bar{y}_0=0\right)<1$.

Assumption 3.2. If $E\log\left|\beta_0+\epsilon_{0,1}\right|\geq 0$, it holds that: *(i)* $\epsilon_{0,2}$ has a bounded density; *(ii)* $\left\{\epsilon_{i,1},-\infty<i<\infty\right\}$ and $\left\{\epsilon_{i,2},-\infty<i<\infty\right\}$ are independent.

Assumption 3.3. If $E\log\left|\beta_0+\epsilon_{0,1}\right|>0$, it holds that $P\left(\epsilon_{0,2}=x\right)<1$ for all $-\infty<x<\infty$.

Assumption 3.1 is required, essentially, in order to avoid degeneracy in the denominator of $\widehat{\beta}_m$ defined in (2.3); examples of this restriction can be found also in Horváth and Trapani (2023a). Assumption 3.2, and in particular the independence between $\epsilon_{i,1}$ and $\epsilon_{i,2}$ is needed specifically in order to derive the anti-concentration bound in Horváth and Trapani (2016, Lem. A.4). Assumption 3.3, together with part *(ii)* of Assumption 3.2, ensures that $|y_i|\overset{a.s.}{\to}\infty$, ruling out that it could be identically equal to zero, and it is taken from Berkes et al. (2009) (see their equations (3.4) and (3.5)). Note that, according to Assumption 3.3, it suffices that $\epsilon_{0,2}$ is a nondegenerate random variable.

3.1. Asymptotics Under the Null

We derive the weak limits of our test statistics. Results differ depending on the choice of the weight ψ, on whether an open-ended or a closed-ended monitoring scheme is used (and, in the latter case, on the length of the monitoring horizon), and on whether the detector is constructed using the standard CUSUM or the Page-CUSUM. In all cases, we derive nuisance free limiting distributions, from which critical values can be obtained by simulation for a given desired nominal significance level, α.

3.1.1. *Weighted standard CUSUM-based detectors.* We begin by studying the standard CUSUM detectors defined in Section 2.1, starting with the open-ended case, also studied in Horváth et al. (2004) in the context of a linear regression.

THEOREM 3.1. *We assume that Assumptions 2.1 and 2.2 hold, and either: (i)* $E\log\left|\beta_0+\epsilon_{0,1}\right|<0$ *holds and Assumption 3.1 hold; or (ii)* $E\log\left|\beta_0+\epsilon_{0,1}\right|=0$ *and Assumption 3.2 hold; or (iii)* $E\log\left|\beta_0+\epsilon_{0,1}\right|>0$ *and Assumptions 3.2–3.3 hold. Then, under* H_0 *it holds that, for all* $\psi<1/2$

$$\lim_{m\to\infty}P\left\{\tau_{m,\psi}=\infty\right\}=P\left\{\sup_{0<u\leq 1}\frac{|W(u)|}{u^{\psi}}<c_{\alpha,\psi}\right\}.$$

Theorem 3.1 offers the limiting distribution (and, therefore, an approximation for critical values) of our procedure in the open-ended case; in essence, the result is the same as in Horváth et al. (2004, Thm. 2.1), of which it is an extension to the case of an RCA model. Importantly, the theorem holds irrespective of whether the observations $\{y_i, -\infty < i < \infty\}$ form a stationary sequence or not.

We now turn to the closed-ended case. We begin by considering the case of (2.8), where the monitoring goes on for a sufficient amount of time:

$$m^* = O\left(m^{\lambda}\right) \text{ for some } \lambda \geq 1, \quad \text{and} \quad \lim_{m\to\infty} \frac{m^*}{m} = m_0 \in (0,\infty] > 0. \tag{3.1}$$

In this case, we use the boundary function $g^*_{m,\psi}(k)$ defined in (2.9), and the corresponding stopping rule (2.8). Define $m_* = m_0/(1+m_0)$ if $m_0 < \infty$, and $m_* = 1$ if $m_0 = \infty$.

THEOREM 3.2. *We assume that the conditions of Theorem 3.1, and (3.1), are satisfied. Then, under H_0 it holds that, for all $\psi < 1/2$*

$$\lim_{m\to\infty} P\left\{\tau^*_{m,\psi} = \infty\right\} = P\left\{\sup_{0<u\leq m_*} \frac{|W(u)|}{u^{\psi}} < c^*_{\alpha,\psi}\right\}. \tag{3.2}$$

Theorem 3.2 extends the results in Theorem 3.1 to the closed-ended case; to the best of our knowledge, this result is new, and it has important practical implications. Indeed, in real applications, the monitoring horizon can be viewed to be always closed, since sooner or later the researcher will stop. Hence, in real applications, even when m^* is "much bigger" than m, m_* will always be smaller than 1. In turn, this entails that critical values derived using Theorem 3.1 are bound to overstate (sometimes slightly, sometimes more decidedly) the true critical values, thereby yielding a loss of power. On the contrary, the result in Theorem 3.2 does not suffer from this issue: the applied user can mechanically compute m_* after deciding m and m^*, and simulate critical values based on (3.2).

We now study the case of (2.11), in a closed-ended set-up where monitoring stops after very few steps, viz.

$$m^* \to \infty, \quad \text{and} \quad \lim_{m\to\infty} \frac{m^*}{m} = 0. \tag{3.3}$$

In this case, we use the boundary function $\overline{g}_{m,\psi}(k)$ defined in (2.10), and the corresponding stopping rule defined in (2.11).

THEOREM 3.3. *We assume that the conditions of Theorem 3.1 and (3.3) are satisfied. Then, under H_0 it holds that, for all $\psi < 1/2$*

$$\lim_{m\to\infty} P\left\{\overline{\tau}_{m,\psi} = \infty\right\} = P\left\{\sup_{0<u\leq 1} \frac{|W(u)|}{u^{\psi}} < \overline{c}_{\alpha,\psi}\right\}. \tag{3.4}$$

Theorem 3.3 completes the theory spelt out in Theorems 3.1 and 3.2 by considering a very short, closed-ended monitoring procedure. This result is also new in the literature; note that the critical value defined in (3.4) is the same, for a given nominal level α, as the one in Theorem 3.1; however, as also noted above, comparing equations (2.5) and (2.10), in the case of a short closed-ended monitoring procedure the boundary function differs, and therefore the decision rule, as to whether to mark a break or not, also differs.

We now turn to the case $\psi = 1/2$, studying the closed-ended monitoring procedure under both (3.1) and (3.3). Define

$$\gamma(x) = \sqrt{2\log x} \quad \text{and} \quad \delta(x) = 2\log x + \frac{1}{2}\log\log x - \frac{1}{2}\log\pi, \tag{3.5}$$

and

$$c^{*}_{\alpha,0.5} = \overline{c}_{\alpha,0.5} = \frac{x+\delta(\log m^{*})}{\gamma(\log m^{*})}. \tag{3.6}$$

THEOREM 3.4. *We assume that the conditions of Theorem 3.1 are satisfied. Then, under H_0 for all $-\infty < x < \infty$:*

- *if (3.1) holds, then it holds that* $\lim_{m\to\infty} P\left\{\tau^{*}_{m,0.5} = m^{*}\right\} = \exp(-\exp(-x))$*;*
- *if (3.3) holds, then it holds that* $\lim_{m\to\infty} P\left\{\overline{\tau}_{m,0.5} = m^{*}\right\} = \exp(-\exp(-x))$.

Theorem 3.4 completes our theory, considering (under any length for the monitoring horizon) the case where the weight $\psi = 1/2$. The result is a typical Darling–Erdös theorem (Darling and Erdős, 1956), and it is essentially the same as Horváth, Kokoszka, and Steinebach (2007, Thm 1.1); however, similarly to Theorem 3.1, the result in Theorem 3.4 does not require that the observations $\{y_i, -\infty < i < \infty\}$ be stationary. Upon inspecting the proofs of Theorems 3.1–3.3 and 3.4, the limiting distributions of the weighted CUSUM with $\psi < 1/2$, and of the standardized CUSUM with $\psi = 1/2$ are asymptotically independent,[5] and therefore the two procedures, in principle, can be combined.

Theorem 3.4 offers an explicit formula to compute asymptotic critical values; however, these are bound to be inaccurate due to the slow convergence to the Extreme Value distribution. In particular, simulations show that, in finite samples, asymptotic critical values overstate the true values thus leading to low power. A possible correction can be proposed, similarly to Gombay and Horváth (1996), as follows. Define $h_{m^*} = h_{m^*}(m^*)$ such that, as $m^* \to \infty$

$$h_{m^*} \to \infty \quad \text{and} \quad h_{m^*}/m^* \to 0, \tag{3.7}$$

and let $\phi_m = (m^* + h_{m^*})/(2h_{m^*})$. Let also $\widehat{c}_{\alpha,0.5} \to \infty$ denote the solution of

$$\frac{\widehat{c}_{\alpha,0.5}\exp\left(-\frac{1}{2}\widehat{c}^{2}_{\alpha,0.5}\right)}{(2\pi)^{1/2}}\left(\log\phi_m + \frac{4-\log\phi_m}{\widehat{c}^{2}_{\alpha,0.5}}\right) = \alpha. \tag{3.8}$$

[5] Intuitively, this is because $W(\cdot)$ has independent increments, and the limiting distribution, in the case of $0 \le \psi < 1/2$, is determined by the 'central' values of $W(\cdot)$; conversely, when $\psi = 1/2$, the limiting law is determined by values at the very beginning of $W(\cdot)$.

THEOREM 3.5. *We assume that the conditions of Theorem 3.4 are satisfied. Then, replacing* $c^*_{\alpha,0.5}$ *and* $\overline{c}_{\alpha,0.5}$ *with* $\widehat{c}_{\alpha,0.5}$ *defined in (3.8), it holds that* $\lim_{m\to\infty} P\left\{\tau^*_{m,0.5} = m^*\right\} = \alpha$ *and* $\lim_{m\to\infty} P\left\{\overline{\tau}_{m,0.5} = m^*\right\} = \alpha$ *respectively.*

Theorem 3.5 offers an alternative way of approximating critical values for the case where $\psi = 1/2$; whilst it is a relatively standard Gaussian approximation (see also the book by Csörgő and Horváth, 1993), to the best of our knowledge this is the first time that such an extension is considered in the case of sequential monitoring. The choice of h_{m^*} is a matter of tuning; qualitatively, as h_{m^*} increases, critical values become smaller (thus making the procedure more conservative) and vice versa. Our simulations indicate that $h_{m^*} = (\log m^*)^{1/2}$ yields the best results in terms of size and power.

3.1.2. *Weighted Page-CUSUM detectors.* We now consider the use of the Page-CUSUM detector $Z^{\dagger}_m(k)$ defined in (2.12), using the stopping rules defined in (2.13)–(2.15). Let $\{W_1(x), x \geq 0\}$ and $\{W_2(x), x \geq 0\}$ denote two independent standard Wiener processes.

THEOREM 3.6. *We assume that the conditions of Theorem 3.1 are satisfied. Then, under* H_0*, for all* $\psi < 1/2$

- *it holds that*

$$\lim_{m\to\infty} P\left\{\tau^{\dagger}_{m,\psi} = \infty\right\} = P\left\{\sup_{0<x<\infty} \frac{\sup_{0\leq t\leq x}\left|\left(W_2(x) - W_2(t)\right) - (x-t)W_1(1)\right|}{(1+x)\left(x/(1+x)\right)^{\psi}} < c^{\dagger}_{\alpha,\psi}\right\}; \quad \mathbf{(3.9)}$$

- *if, in addition, (3.1) holds, then it holds that*

$$\lim_{m\to\infty} P\left\{\tau^{\dagger *}_{m,\psi} = \infty\right\} = P\left\{\sup_{0<x\leq m_0} \frac{\sup_{0\leq t\leq x}\left|\left(W_2(x) - W_2(t)\right) - (x-t)W_1(1)\right|}{(1+x)\left(x/(1+x)\right)^{\psi}} < c^{\dagger *}_{\alpha,\psi}\right\}; \quad \mathbf{(3.10)}$$

- *if, in addition, (3.3) holds, then it holds that*

$$\lim_{m\to\infty} P\left\{\overline{\tau}^{\dagger}_{m,\psi} = \infty\right\} = P\left\{\sup_{0<x\leq 1} \frac{\sup_{0\leq t\leq x}\left|\left(W(x) - W(t)\right)\right|}{x^{\psi}} < \overline{c}^{\dagger}_{\alpha,\psi}\right\}. \quad \mathbf{(3.11)}$$

Theorem 3.6 offers the full-blown asymptotic theory for the weighted Page-CUSUM detectors; the same considerations as above hold, e.g., the case under (3.1) studied in (3.10), where a closed-ended scheme is considered, may be more realistic and therefore give a better approximation than the case of an open-ended scheme as considered in (3.9).

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3.1.3. *LRV estimation.* In order to apply all the results above, we require an estimate of $\mathfrak{s}^2$ defined in (2.6). This can be constructed using the data in the training sample as

$$\widehat{\mathfrak{s}}_m^2 = \frac{1}{m}\sum_{i=2}^{m}\left(\frac{\left(y_i - \widehat{\beta}_m y_{i-1}\right) y_{i-1}}{1+y_{i-1}^2}\right)^2. \tag{3.12}$$

COROLLARY 3.1. *Under the conditions of Theorem 3.1, there exists a* $\zeta > 0$ *such that* $\widehat{\mathfrak{s}}_m^2 = \mathfrak{s}^2 + O_P\left(m^{-\zeta}\right)$.

Corollary 3.1 holds irrespective of whether y_i is stationary or not, and therefore, practically, it entails that $\widehat{\mathfrak{s}}_m^2$ can be used with no prior knowledge of the stationarity or not of y_i. Furthermore, we note that $\widehat{\mathfrak{s}}_m^2$ does not depend on ψ, so the result in Corollary 3.1 can be applied for all $0 \le \psi \le 1/2$, and for both open-ended and closed-ended procedures.

3.2. Asymptotics Under the Alternative

We consider the following alternative, where the deterministic part of the autoregressive coefficient of (2.1) undergoes a change

$$y_i = \begin{cases} \left(\beta_0 + \epsilon_{i,1}\right) y_{i-1} + \epsilon_{i,2} & 1 \le i \le m + k^*, \\ \left(\beta_A + \epsilon_{i,1}\right) y_{i-1} + \epsilon_{i,2} & i > m + k^*, \end{cases} \tag{3.13}$$

where $\beta_0 \neq \beta_A$ and k^* is the time of change. In (3.13), we do not put any constraints on the values of β_0 and β_A. Hence, under (3.13), the observations could transition from a stationary regime to another stationary regime (if both $E\log\left|\beta_0 + \epsilon_{0,1}\right| < 0$ and $E\log\left|\beta_A + \epsilon_{0,1}\right| < 0$); from a nonstationary regime to another nonstationary regime (if both $E\log\left|\beta_0 + \epsilon_{0,1}\right| \ge 0$ and $E\log\left|\beta_A + \epsilon_{0,1}\right| \ge 0$); or either regime can be stationary and the other one nonstationary (which arises if $E\log\left|\beta_0 + \epsilon_{0,1}\right| < 0$ and $E\log\left|\beta_A + \epsilon_{0,1}\right| \ge 0$, implying a switch from stationarity to nonstationarity, or if $E\log\left|\beta_0 + \epsilon_{0,1}\right| \ge 0$ and $E\log\left|\beta_A + \epsilon_{0,1}\right| < 0$, implying a switch from nonstationarity to stationarity). We entertain the possibility that the amplitude of change may depend on the (training) sample size m, thus defining

$$\Delta_m = \beta_A - \beta_0. \tag{3.14}$$

THEOREM 3.7. *We assume that (3.13) holds with* $k^* = O(m)$. *Under the conditions of Theorem 3.1, if it holds that*

$$\lim_{m\to\infty} m^{1/2}\left|\Delta_m\right| = \infty, \tag{3.15}$$

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then it holds that $\lim_{m\to\infty} P\left\{\tau_{m,\psi} < \infty | H_A\right\} = 1$*, for all* $\psi < 1/2$*. Under the conditions of Theorem 3.4, if it holds that*

$$\lim_{m\to\infty} \frac{m^{1/2}\left|\Delta_m\right|}{\sqrt{\log\log m}} = \infty, \tag{3.16}$$

then it holds that $\lim_{m\to\infty} P\left\{\tau^*_{m,0.5} < \infty | H_A\right\} = 1$*. The same results hold under the conditions of Theorem 3.2; and under the conditions of Theorem 3.3 upon replacing m with* m^* *in (3.15) and (3.16). The same results also hold, for* $\psi < 1/2$*, using the procedures based on the Page-CUSUM detector* $Z^{\dagger}_m(k)$ *defined in (2.13)–(2.15).*

Condition (3.15) entails that $|\Delta_m|$ can drift to zero, but not too fast (i.e., power is ensured as long as breaks are "not too small"), for all $\psi < 1/2$. When using $\psi = 1/2$, condition (3.16) suggests that there is a (minor) loss of power.

3.2.1. *Detection delays.* In this section, we study the limiting law of the detection delay $\tau_{m,\psi} - k^*$; we only consider detectors based on the weighted CUSUM, to make a direct comparison with related results derived by Aue and Horváth (2004) and Aue et al. (2008).

Define

$$\xi^{(1)}_{m,\psi} = \left(\frac{c_{\alpha,\psi} m^{1/2-\psi}}{|\Delta_m|}\right)^{1/(1-\psi)}, \quad \text{and} \quad \xi^{(2)}_{m,\psi} = \frac{\varsigma_{(2)}}{1-\psi}\frac{(\xi^{(}_{m,\psi}1))^{1/2}}{|\Delta_m|}, \tag{3.17}$$

for the case where $0 \le \psi < 1/2$, and

$$\xi^{(1)}_{m,0.5} = \frac{2\varsigma^2_{(2)}\log\log m}{\Delta^2_m}, \quad \text{and} \quad \xi^{(2)}_{m,0.5} = \left(\frac{2}{\log\log m}\right)^{1/2}\xi^{(1)}_{m,0.5}, \tag{3.18}$$

for the case where $\psi = 1/2$, where $\varsigma^2_{(2)}$ is defined as the counterpart to (2.6) after the changepoint, viz.

$$\varsigma^2_{(2)} = \begin{cases} a_{1,(2)}\sigma^2_1 + a_{2,(2)}\sigma^2_2, \text{ if } -\infty \le E\log\left|\beta_A + \epsilon_{0,1}\right| < 0, \\ \sigma^2_1, \text{ if } E\log\left|\beta_A + \epsilon_{0,1}\right| \ge 0, \end{cases}$$

with

$$a_{1,(2)} = E\left(\frac{\overline{y}^2_{0,(2)}}{1+\overline{y}^2_{0,(2)}}\right)^2 \quad \text{and} \quad a_{2,(2)} = E\left(\frac{\overline{y}_{0,(2)}}{1+\overline{y}^2_{0,(2)}}\right)^2,$$

and $\overline{y}^2_{0,(2)}$ is the stationary solution of $y_i = \left(\beta_A + \epsilon_{i,1}\right)y_{i-1} + \epsilon_{i,2}$ when $E\log\left|\beta_A + \epsilon_{0,1}\right| < 0$.

Let $\Phi(x)$ denote the standard normal distribution function.

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THEOREM 3.8. *We assume that the conditions of Theorem 3.1 are satisfied, and that*

$$\lim_{m\to\infty} |\Delta_m| = 0, \quad and \quad \lim_{m\to\infty} m^{1/2} |\Delta_m| = \infty, \tag{3.19}$$

and $k^* = O\left(m^{\theta}\right)$, *for some* $0 \leq \theta < ((1-2\psi)/(2(1-\psi)))^2$. *Then, under* H_A *it holds that*

$$\lim_{m\to\infty} P\left\{\frac{\tau_{m,\psi} - \xi^{(1)}_{m,\psi}}{\xi^{(2)}_{m,\psi}} \leq x\right\} = \Phi(x), \tag{3.20}$$

for all $0 \leq \psi < 1/2$. *The same result holds under the conditions of Corollary 3.2 when replacing* $\tau_{m,\psi}$ *with* $\tau^*_{m,\psi}$.

THEOREM 3.9. *We assume that the conditions of Theorem 3.4 are satisfied, and that*

$$c_0 (\log m)^{-\epsilon} \leq |\Delta_m| \leq c_1, \quad and \quad k^* = o\left(\frac{(\log\log m)^{1/2}}{\Delta_m^2}\right), \tag{3.21}$$

for some $c_0, c_1 > 0$ *and* $\epsilon > 0$. *Then, under* H_A *it holds that*

$$\lim_{m\to\infty} P\left\{\frac{\tau^*_{m,0.5} - \xi^{(1)}_{m,0.5}}{\xi^{(2)}_{m,0.5}} \leq x\right\} = \Phi(x). \tag{3.22}$$

Theorems 3.8 and 3.9 state that the stopping times, $\tau_{m,\psi}$ (and $\tau^*_{m,\psi}$, in the case of a closed-ended monitoring procedure) and $\tau^*_{m,0.5}$ follow a Gaussian distribution. After some algebra, it can be verified that, as $m \to \infty$, it holds that

$$\tau_{m,\psi} \approx \left(\frac{c_{\alpha,\psi}}{|\Delta_m|}\right)^{1/(1-\psi)} m^{(1-2\psi)/(2(1-\psi))}, \quad \text{and} \quad \tau^*_{m,0.5} \approx \frac{2\mathfrak{s}^2_{(2)} \log\log m}{\Delta_m^2}, \tag{3.23}$$

and the same holds for $\tau^*_{m,\psi}$, *mutatis mutandis*. Given that, in the assumptions of Theorems 3.8 and 3.9, k^* is of smaller order of magnitude than $\tau_{m,\psi}$ and $\tau^*_{m,0.5}$, (3.23) can be interpreted also as the delay in detecting a changepoint. When $0 \leq \psi < 1/2$, the delay is proportional to a polynomial function of m, which declines as ψ approaches $1/2$; the shortest detection time is found for $\psi = 1/2$, where fixed size changepoints are found with a delay proportional to as little as $O(\log\log m)$. These results hold for all values of $E\log\left|\beta_i + \epsilon_{0,1}\right|$, and therefore they can be employed irrespective of whether y_i is stationary or explosive. Hence, as an example also mentioned above, in the context of monitoring for the onset (or crash) of financial bubbles, $\tau_{m,\psi}$ and $\tau^*_{m,0.5}$ can be used for the purpose of "date-stamping": if e.g. an explosive sequence is found to change into a stationary regime (or if, vice versa, a stationary sequence is found to change into an explosive one), the stopping times $\tau_{m,\psi}$ or $\tau^*_{m,0.5}$ will be the estimate for the end of the bubble (or of the onset thereof, respectively).

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4. MONITORING THE RCA MODEL WITH COVARIATES

In a recent contribution, Astill et al. (2023) argue in favor of adding covariates to the basic AR specification, showing theoretically and empirically that this results in better (and quicker) detection of bubble episodes. Hence, we modify (2.1) as

$$y_i = \left(\beta_i + \epsilon_{i,1}\right) y_{i-1} + \lambda_0^{\intercal} \mathbf{x}_i + \epsilon_{i,2}, \tag{4.1}$$

where y_0 is an initial value and $\mathbf{x}_i \in \mathbb{R}^p$. Equation (4.1) is, essentially, a dynamic model with exogenous covariates, with λ_0 constant over time.

In order to monitor for the stability of the autoregressive coefficient, we propose again a detector based on the WLS loss function

$$\mathcal{G}_m(\beta, \lambda) = \sum_{i=2}^{m} \frac{\left(y_i - \beta y_{i-1} - \lambda^{\intercal} \mathbf{x}_i\right)^2}{1 + y_{i-1}^2}. \tag{4.2}$$

The estimators of β_0 and λ_0 are defined as $\left(\widehat{\beta}_m, \widehat{\lambda}_m\right) = \arg\min_{\beta,\lambda} \mathcal{G}_m(\beta, \lambda)$, and satisfy

$$\frac{\partial}{\partial \beta} \mathcal{G}_m\left(\widehat{\beta}_m, \widehat{\lambda}_m\right) = -2 \sum_{i=2}^{m} \frac{\left(y_i - \widehat{\beta}_m y_{i-1} - \widehat{\lambda}_m^{\intercal} \mathbf{x}_i\right) y_{i-1}}{1 + y_{i-1}^2} = 0, \tag{4.3}$$

which suggests the following detector

$$Z_m^X(k) = \left| \sum_{i=m+1}^{m+k} \frac{\left(y_i - \widehat{\beta}_m y_{i-1} - \widehat{\lambda}_m^{\intercal} \mathbf{x}_i\right) y_{i-1}}{1 + y_{i-1}^2} \right|. \tag{4.4}$$

The following assumptions complement Assumptions 2.1–3.3.

Assumption 4.1. *(i)* $E(\mathbf{x}_i) = 0$, and $E\|\mathbf{x}_i\|^{\kappa_1} < \infty$ for some $\kappa_1 > 4$; *(ii)* $\mathbf{x}_i = \mathbf{g}(\eta_i, \eta_{i-1}, \ldots)$, where $\mathbf{g}: \mathcal{S}^{\infty} \to \mathbb{R}^p$ is a non-random, measurable function and $\{\eta_i, -\infty < i < \infty\}$ are *i.i.d.* random variables with values in the measurable space $\mathcal{S}$ and $\left(E\left\|\mathbf{x}_i - \mathbf{x}_{i,j}^*\right\|^{\kappa_1}\right)^{1/\kappa_1} \le c_0 j^{-\kappa_2}$, with some $c_0 > 0$ and $\kappa_2 > 2$, where $\mathbf{x}_{i,j}^* = \mathbf{g}\left(\eta_i, \ldots, \eta_{i-j+1}, \eta_{i-j,i,j}^*, \eta_{i-j-1,i,j}^* \cdots\right)$ where $\left\{\eta_{\ell,i,j}^*, -\infty < \ell, i, j < \infty\right\}$ are *i.i.d.* random copies of η_0, independent of $\{\eta_i, -\infty < i < \infty\}$.

Assumption 4.2. $\left\{\left(\epsilon_{i,1}, \epsilon_{i,2}\right), -\infty < i < \infty\right\}$ and $\{\eta_i, -\infty < i < \infty\}$ are independent.

Assumption 4.3. If $E \log\left|\beta_0 + \epsilon_{0,1}\right| > 0$, it holds that $P\left\{\left(\beta_0 + \epsilon_{0,1}\right) y_0 + \lambda^{\intercal} \mathbf{x}_0 + \epsilon_{0,2} = x\right\} = 0$ for all $-\infty < x < \infty$.

Assumption 4.1 states that the regressors $\mathbf{x}_i$ form a decomposable Bernoulli shift, i.e., a weakly dependent, stationary process which can be well-approximated by an m-dependent sequence. The concepts of Bernoulli shift and decomposability appeared first in Ibragimov (1962) (see also Wu, 2005 and Berkes, Hörmann, and Schauer, 2011). Bernoulli shifts have proven a convenient way to model dependent

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time series, mainly due to their generality and to the fact that they are much easier to verify than e.g., mixing conditions: Aue et al. (2009) and Liu and Lin (2009), *inter alia*, provide numerous examples of such DGPs, which include ARMA models, ARCH/GARCH sequences, and other nonlinear time series models (such as e.g., random coefficient autoregressive models and threshold models). Indeed, under stationarity, y_i itself can be approximated by a decomposable Bernoulli shift (Horváth and Trapani, 2023a). Assumption 4.2 states that the exogenous variables $\mathbf{x}_i$ are independent of the innovations $\epsilon_{i,1}$ and $\epsilon_{i,2}$, and Assumption 4.3 is similar to Assumption 3.3, ensuring $|y_i| \stackrel{a.s.}{\to} \infty$.

Finally, we note that, in the presence of covariates, we need to exclude the boundary case $E\log\left|\beta_0+\epsilon_{0,1}\right|=0$; this is because we would need an exact (and large enough) rate of divergence for $|y_i|$ as $i\to\infty$, but this result is not available in the case $E\log|\beta_0+\epsilon_{0,1}|=0$ (see also Horváth and Trapani (2019, Thm. 4), and the discussion thereafter). The boundary function is defined as

$$g_{m,\psi}^{(x)}(k)=c_{\alpha,\psi}^{(x)}\mathcal{s}_x^2 m^{1/2}\left(1+\frac{k}{\mathcal{s}_{x,d}^2 m}\right)\left(\frac{k}{\mathcal{s}_{x,d}^2 m+k}\right)^{\psi}, \tag{4.5}$$

where $c_{\alpha,\psi}^{(x)}$ is a critical value, and

$$\mathcal{s}_x^2=\begin{cases}\mathcal{s}_{x,2}^2/\mathcal{s}_{x,1}, \text{ if } -\infty\le E\log\left|\beta_0+\epsilon_{0,1}\right|<0,\\ \sigma_1, \text{ if } E\log\left|\beta_0+\epsilon_{0,1}\right|>0,\end{cases} \tag{4.6}$$

$$\mathcal{s}_{x,d}^2=\begin{cases}\mathcal{s}_{x,2}^2/\mathcal{s}_{x,1}^2, \text{ if } -\infty\le E\log\left|\beta_0+\epsilon_{0,1}\right|<0,\\ 1, \text{ if } E\log\left|\beta_0+\epsilon_{0,1}\right|>0,\end{cases}$$

with

$$\mathcal{s}_{x,1}^2=\mathbf{a}^\intercal\mathbf{QCQa}, \quad \text{and} \quad \mathcal{s}_{x,2}^2=\sigma_1^2 E\left(\frac{\bar{y}_0^2}{1+\bar{y}_0^2}\right)^2+\sigma_2^2 E\left(\frac{\bar{y}_0}{1+\bar{y}_0^2}\right)^2, \tag{4.7}$$

where $\mathbf{a}$, $\mathbf{Q}$ and $\mathbf{C}$ are defined in (C.26)-(C.28) in the Supplementary Material. The stopping rule is

$$\tau_{m,\psi}^{(x)}=\begin{cases}\inf\{k\ge 1: Z_m^X(k)\ge g_{m,\psi}^{(x)}(k)\},\\ \infty, \text{ if } Z_m^X(k)<g_{m,\psi}^{(x)}(k) \text{ for all } 1\le k<\infty,\end{cases} \tag{4.8}$$

and

$$\tau_{m,\psi}^{*(x)}=\begin{cases}\inf\{k\ge 1: Z_m^X(k)\ge g_{m,\psi}^{(x)}(k)\},\\ m^*, \text{ if } Z_m^X(k)<g_{m,\psi}^{(x)}(k) \text{ for all } 1\le k\le m^*,\end{cases} \tag{4.9}$$

for an open-ended and a closed-ended monitoring procedure respectively.

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THEOREM 4.1. *We assume that Assumptions 2.1, 4.1, and 4.2 are satisfied, and either (i)* $E\log|\beta_0+\epsilon_{0,1}|<0$, *or (ii)* $E\log|\beta_0+\epsilon_{0,1}|>0$ *and Assumption 4.3 hold. Then, under* H_0, *the results of Theorems 3.1, 3.2, and 3.3 hold for* $\tau^{(x)}_{m,\psi}$, $\tau^{*(x)}_{m,\psi}$, *and* $\overline{\tau}^{(x)}_{m,\psi}$ *respectively.*

THEOREM 4.2. *We assume that the conditions of Theorem 4.1 are satisfied. Then, for* $\psi=1/2$, *under* H_0, *the same results as in Theorem 3.4 hold.*

Along the same lines as in Section 3.1.2, it is possible to construct weighted monitoring schemes based on the detector

$$Z^{\dagger(X)}_m(k)=\max_{1\le\ell\le k}\left|\sum_{i=m+\ell}^{m+k}\frac{\left(y_i-\widehat{\beta}_m y_{i-1}-\widehat{\lambda}_m^{\intercal}\mathbf{x}_i\right)y_{i-1}}{1+y_{i-1}^2}\right|,\quad k\ge 1. \tag{4.10}$$

THEOREM 4.3. *We assume that Assumptions 2.1, 4.1, and 4.2, and (3.1), are satisfied, and either (i)* $E\log|\beta_0+\epsilon_{0,1}|<0$, *or (ii)* $E\log|\beta_0+\epsilon_{0,1}|>0$ *and Assumption 4.3 hold. Then, under* H_0, *the results of Theorem 3.6 hold.*

In all the results above, the limiting behavior of the stopping time is the same as in the absence of covariates; however, this does not mean that these do not play a role, since they build into the recursion that defines y_i. As far as Theorem 4.2 is concerned, the same approximation for critical values as in (3.8) can be used.

Under the alternative that the deterministic part of the autoregressive root changes,

$$y_i=\begin{cases}\left(\beta_0+\epsilon_{i,1}\right)y_{i-1}+\lambda_0^{\intercal}\mathbf{x}_i+\epsilon_{i,2} & 1\le i\le m+k^*,\\ \left(\beta_A+\epsilon_{i,1}\right)y_{i-1}+\lambda_0^{\intercal}\mathbf{x}_i+\epsilon_{i,2} & i>m+k^*,\end{cases} \tag{4.11}$$

the same results as in Section 3.2 hold, as summarized in the following theorem.

THEOREM 4.4. *We assume that the conditions of Theorem 4.1 are satisfied. Then, under (4.11), the same results as in Theorem 3.7 hold.*

5. SIMULATIONS

We provide some Monte Carlo evidence and guidelines on implementation; further details and results are reported in Section A of the Supplementary Material, where, in particular, we consider the case of covariates (Section A.1 of the Supplementary Material), the case of a smooth break (Section A.2 of the Supplementary Material), and a further investigation of power versus breaks of variable magnitude in the cases of STUR and explosive processes (Section A.3 of the Supplementary Material). We consider the following Data Generating Process, based on (4.1)

$$y_i=\left(\beta_0+\epsilon_{i,1}\right)y_{i-1}+\lambda_0 x_i+\epsilon_{i,2}, \tag{5.1}$$

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for $1 \leq i \leq m+1{,}000$, where we simulate $\epsilon_{i,1}$ and $\epsilon_{i,2}$ as independent of one another and *i.i.d.* with distributions $N\left(0,\sigma_1^2\right)$ and $N\left(0,\sigma_2^2\right)$ respectively, discarding the first $1{,}000$ observations to avoid dependence on initial conditions. We used $\sigma_1^2 = 0.01$ in all experiments, and we have considered three cases: in **Case I**, we set $\beta_0 = 0.5$, with $E\log\left|\beta_0+\epsilon_{i,1}\right| = -0.717$, corresponding to a stationary regime; in **Case II**, we set $\beta_0 = 1.05$, with $E\log\left|\beta_0+\epsilon_{i,1}\right| = 0.044$, corresponding to a mildly explosive regime; and in **Case III**, we set $\beta_0 = 1$, with $E\log\left|\beta_0+\epsilon_{i,1}\right| = -0.007$, indicating a stationary, but 'on the cusp' process (this case corresponds to the STUR model). The variance of the idiosyncratic shock is $\sigma_2^2 = 0.5$ in Case I, and $\sigma_2^2 = 0.1$ in Cases II and III; in unreported simulations, using different values does not result in any significant changes, save for the (expected) fact that tests have better properties (in terms of size and power) for smaller values of σ_2^2. When covariates are used, we set $\lambda_0 = 1$ and generate x_i as $i.i.d.N(0,1)$. Critical values (for a nominal level equal to 5%) are computed using the results in Theorem 3.2 when using the weighted CUSUM with $\psi < 1/2$, and Theorem 3.6*(ii)* when using Page-CUSUM statistics. When using the weighted CUSUM with $\psi = 1/2$, we use both the asymptotic critical values $c_{\alpha,0.5}$ defined in (3.6), and the approximation $\widehat{c}_{\alpha,0.5}$ defined as the solution of (3.8), with $h_{m^*} = (\log m^*)^{1/2}$. Results are based on $1{,}000$ replications.

When considering Case I—i.e., when considering stationarity—we compare our tests with the test by Otto and Breitung (2023) to assess the relative merits of our approach. Whilst we refer to Section 3.3 in the original paper by Otto and Breitung (2023) for the full-fledged details, here we offer a brief description of how the test works, focusing on the case of no covariates for simplicity. During the training period, we estimate an AR(1) model based on $y_i = \beta y_{i-1} + u_i$, so that $u_i = \epsilon_{i,1}y_{i-1} + \epsilon_{i,2}$, with $\widehat{\beta}_m = \left(\sum_{i=2}^m y_{i-1}^2\right)^{-1}\sum_{i=2}^m y_{i-1}y_i$. Subsequently, for each point in time $m+1 \leq i \leq m+m^*$, we define the recursive residuals $\widehat{u}_i = y_i - \widehat{\beta}_i y_{i-1}$, and the weighted residuals $y_{i-1}\widehat{u}_i$,[6] using the detector

$$OB(k) = \max_{1\leq \ell \leq k}\left|\frac{1}{m^{1/2}\widehat{\sigma}_m\widehat{C}_m^{1/2}}\sum_{i=m+\ell}^{m+k}\frac{y_{i-1}\widehat{u}_i}{\left[1+\left(\sum_{s=2}^{i-1}y_{s-1}^2\right)^{-1}y_{i-1}^2\right]^{1/2}}\right|; \tag{5.2}$$

note that, in essence, this is a Page-CUSUM detector along the lines of equation (2.12). In (5.2), we have defined

$$\widehat{\sigma}_m^2 = \frac{1}{m}\sum_{i=2}^m \frac{(y_{i-1}\widehat{u}_i)^2}{1+\left(\sum_{s=2}^{i-1}y_{s-1}^2\right)^{-1}y_{i-1}^2} \quad \text{and} \quad \widehat{C}_m = \frac{1}{m}\sum_{i=2}^m y_{i-1}^2,$$

[6] In principle, it would be possible to also consider WLS estimation of β, and therefore use the corresponding weighted WLS residuals. However, preliminary Monte Carlo evidence suggests that this approach, whilst feasible, does not work as well as using a plain OLS estimator *cum* weighted OLS residuals.

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with stopping rule

$$\tau_{OB} = \begin{cases} \inf\{k \geq 1 : OB(k) \geq g_{OB}(k)\}, \\ m^*, \text{ if } OB(k) < g_{OB}(k) \text{ for all } 1 \leq k \leq m^*, \end{cases}$$

and boundary function

$$g_{OB}(k) = c_{\alpha,OB}\left[1 + 2\left(\frac{k-\ell}{m}\right)\right], \tag{5.3}$$

where $c_{\alpha,OB}$ is a critical value.[7] Note that the procedure by Otto and Breitung (2023) requires $m^* > m$, so it is not suitable for short horizon closed-ended procedures.

Empirical rejection frequencies are in Tables 1–3, where we consider the case where, in (5.1), $\lambda_0 = 0$, i.e., a "pure" RCA without covariates. The empirical rejection frequencies—with some exceptions which can be ascribed to sampling error—clearly tend to home towards their confidence band $[0.036, 0.064]$ as m and m^* both increase (note that numbers are to be compared across the same value of the ratio m^*/m, as the nature of the results in e.g., Theorem 3.2 indicate). In the case of stationarity (Table 1), the procedure-wise probability of Type I Errors is always controlled when using $\psi = 1/2$; the asymptotic critical values $c_{\alpha,0.5}$ defined in (3.6) lead to under-rejection, as can be expected in light of the slow convergence to the asymptotic distribution; conversely, their approximation using $\widehat{c}_{\alpha,0.5}$ defined as the solution of (3.8) achieves size control in all cases considered. When using $\psi < 1/2$, both in the case of the CUSUM and the Page-CUSUM, the procedure-wise probability of Type I Errors is also controlled, but larger sample sizes m are required. In the explosive case (Table 2), our procedures tend to under-reject whenever using $\psi = 1/2$ (both with $c_{\alpha,0.5}$ and $\widehat{c}_{\alpha,0.5}$), although this seems to slowly improve as m increases, for all lengths of the monitoring horizon m^*; results are less conservative when using $\psi < 1/2$, again both in the case of the CUSUM and the Page-CUSUM. On the other hand, in the STUR case (Table 3), empirical rejection frequencies tend to be higher than in the other cases, which may be explained as a consequence of the fact that the behavior of $|y_i|$ becomes unstable, as it sits on the boundary between the stationary and the explosive regimes. The procedure-wise probability of Type I Errors is controlled in all cases when using $\psi = 1/2$ and $c_{\alpha,0.5}$, whereas using $\widehat{c}_{\alpha,0.5}$ requires either large m (≥ 200), or not overly long monitoring horizons when $m < 200$ (in those cases, using $m^* \leq m$ always results in size control). When $\psi < 1/2$, size control is more problematic and the procedure tends to be oversized, unless m^* is "small" compared to m. Whilst results are broadly similar for the CUSUM and the Page-CUSUM, we note that the latter yields marginally improved size control over the former.

The main message of Tables 1–3 is that, broadly speaking, using the CUSUM with $\psi = 1/2$ seems the preferred solution across all cases considered. The

[7] See e.g., Table 2 in Otto and Breitung (2023).

TABLE 1. Empirical rejection frequencies under the null of no changepoint and no covariates - Case I, $\beta_0 = 0.5$.

		Weighted CUSUM			Standardized CUSUM		Weighted Page-CUSUM			OB
	ψ	0	0.25	0.45	0.5		0	0.25	0.45	
					$c_{\alpha,0.5}$	$\widehat{c}_{\alpha,0.5}$				
m	m^*									
	25	0.047	0.057	0.045	0.023	0.045	0.045	0.057	0.054	
50	50	0.068	0.081	0.066	0.034	0.055	0.060	0.080	0.069	0.049
	100	0.064	0.079	0.070	0.036	0.053	0.058	0.052	0.061	0.058
	200	0.092	0.107	0.086	0.038	0.057	0.087	0.089	0.083	0.051
	50	0.063	0.068	0.055	0.025	0.048	0.061	0.066	0.054	
100	100	0.057	0.067	0.075	0.027	0.056	0.059	0.066	0.063	0.043
	200	0.062	0.062	0.066	0.029	0.048	0.059	0.055	0.060	0.037
	400	0.062	0.061	0.054	0.028	0.050	0.062	0.064	0.061	0.045
	100	0.060	0.064	0.053	0.020	0.034	0.050	0.056	0.058	
200	200	0.049	0.063	0.058	0.023	0.044	0.045	0.058	0.055	0.049
	400	0.057	0.057	0.060	0.023	0.042	0.055	0.057	0.052	0.051
	800	0.049	0.050	0.056	0.024	0.048	0.047	0.062	0.067	0.047

choice between $c_{\alpha,0.5}$ and $\widehat{c}_{\alpha,0.5}$ essentially depends on m (and m^*), but in the vast majority of the cases considered, using $\widehat{c}_{\alpha,0.5}$ offers excellent size control without having an overly conservative procedure. Our results suggest avoiding overly long monitoring horizons when $m \leq 100$. In these cases, monitoring can be carried out firstly over a short horizon and, if no changepoints are detected, the procedure can be started afresh including the previous monitoring horizon within the training sample. In the Supplementary Material, we report empirical rejection frequencies in the presence of covariates in (5.1). Whilst results are essentially the same in the stationary and in the explosive case, in the STUR case (Table A.1 in the Supplementary Material) the procedures tend to (sometimes massively) over-reject. The best results are obtained with $\psi = 1/2$ and $c_{\alpha,0.5}$, which achieves size control at least when $m > 50$ (or when $m \leq 50$, but the monitoring horizon is not too long, i.e., $m^* \leq m$). This suggests that, when covariates are included in the basic RCA specification, $\psi = 1/2$ and $c_{\alpha,0.5}$ should be employed—especially when trying to detect the inception of a bubble from a near stationary regime.

We would like to emphasize that all the cases considered above are fully under the researcher's control: whether to include covariates or not, and the length of the monitoring horizon m^*, can be decided *a priori*; as mentioned above, preliminary analysis on y_i during the training period offers further help.

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TABLE 2. Empirical rejection frequencies under the null of no changepoint and no covariates - Case II, $\beta_0 = 1.05$.

			Weighted CUSUM			Standardized CUSUM		Weighted Page-CUSUM		
		ψ	0	0.25	0.45	0.5		0	0.25	0.45
						$c_{\alpha,0.5}$	$\widehat{c}_{\alpha,0.5}$			
m	m^*									
	25		0.035	0.042	0.026	0.011	0.022	0.027	0.037	0.030
50	50		0.046	0.051	0.040	0.012	0.029	0.044	0.054	0.039
	100		0.033	0.049	0.027	0.009	0.023	0.028	0.026	0.025
	200		0.055	0.063	0.044	0.018	0.034	0.051	0.056	0.047
	50		0.044	0.043	0.030	0.008	0.020	0.033	0.038	0.027
100	100		0.033	0.036	0.039	0.015	0.025	0.034	0.034	0.028
	200		0.054	0.052	0.041	0.009	0.019	0.051	0.040	0.024
	400		0.059	0.050	0.046	0.014	0.035	0.052	0.055	0.044
	100		0.047	0.049	0.034	0.015	0.027	0.039	0.040	0.038
200	200		0.041	0.055	0.050	0.022	0.036	0.041	0.055	0.047
	400		0.051	0.054	0.053	0.015	0.037	0.049	0.046	0.046
	800		0.042	0.042	0.047	0.013	0.023	0.037	0.048	0.047

Under the alternative, monitoring is carried out for $m+1 \leq i \leq m+m^*$, with the same specifications as in Cases I–III and

$$y_i = \left(\beta_0 + \Delta I\,(i \geq m+1) + \epsilon_{i,1}\right) y_{i-1} + \lambda_0 x_i + \epsilon_{i,2}. \tag{5.4}$$

We set: $\Delta = 0.5$ under Case I, with $\beta_A = 0.75$ and $E \log \left|\beta_A + \epsilon_{i,1}\right| = -0.298$, thus having a change in persistence with y_i stationary under the null and the alternative; $\Delta = -0.1$ under Case II, with $\beta_A = 0.969$ and $E \log \left|\beta_A + \epsilon_{i,1}\right| = -0.063$, thus having a change from nonstationarity to stationarity; and $\Delta = 0.1$ under Case III, with $\beta_A = 1.1$ and $E \log \left|\beta_A + \epsilon_{i,1}\right| = 0.089$, thus having a change from stationarity to nonstationarity. Median delays and empirical rejection frequencies for $m = 200$, under the case of no covariates ($\lambda_0 = 0$), are in Table 4.

Under stationarity, using $\psi = 0.45$ yields the best results in terms of delay and power, with little difference between the CUSUM and the Page-CUSUM (with the latter occasionally delivering higher power, if not shorter detection delays); results are anyway satisfactory also when using $\psi = 1/2$ with $\widehat{c}_{\alpha,0.5}$, which is always either the best or the second best in terms of detection delay. The same results are observed in the explosive and in the STUR cases. The latter is particularly remarkable: using $\psi = 1/2$ with $\widehat{c}_{\alpha,0.5}$ delivers essentially the same performance as using $\psi = 0.45$; however, according to Table 3, the latter choice yields an oversized procedure, thus making its performance under the alternative less reliable. Hence,

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TABLE 3. Empirical rejection frequencies under the null of no changepoint and no covariates - Case III, $\beta_0 = 1$.

		Weighted CUSUM			Standardized CUSUM		Weighted Page-CUSUM		
	ψ	0	0.25	0.45	0.5		0	0.25	0.45
					$c_{\alpha,0.5}$	$\widehat{c}_{\alpha,0.5}$			
m	m^*								
	25	0.070	0.069	0.053	0.028	0.047	0.062	0.072	0.058
50	50	0.068	0.073	0.055	0.031	0.044	0.066	0.076	0.058
	100	0.092	0.100	0.078	0.053	0.070	0.089	0.076	0.076
	200	0.111	0.114	0.092	0.055	0.073	0.109	0.098	0.091
	50	0.070	0.068	0.049	0.020	0.037	0.064	0.070	0.049
100	100	0.075	0.082	0.072	0.038	0.057	0.077	0.078	0.069
	200	0.095	0.099	0.085	0.047	0.067	0.092	0.083	0.077
	400	0.111	0.100	0.089	0.057	0.076	0.106	0.106	0.092
	100	0.060	0.066	0.051	0.024	0.040	0.057	0.058	0.052
200	200	0.093	0.102	0.096	0.045	0.059	0.090	0.097	0.083
	400	0.076	0.079	0.071	0.032	0.060	0.078	0.076	0.072
	800	0.093	0.087	0.082	0.040	0.057	0.085	0.096	0.088

Table 4 essentially indicates that using $\psi = 1/2$ with $\widehat{c}_{\alpha,0.5}$ yields the best results in terms of timeliness of detection and power. As a final remark, detection delays worsen as m^* increases; this is a natural phenomenon, since critical values (see e.g., Corollary 3.2) increase with m^*, thus making detection, *ceteris paribus*, more infrequent. In Table A.2 in the Supplementary Material, we report median delays and empirical rejection frequencies in the presence of covariates in the basic RCA specification. Our guidelines based on size control suggest using $\psi = 1/2$ with $c_{\alpha,0.5}$ in this case; median delays appear very satisfactory, although the cases $\psi = 1/2$ with $\widehat{c}_{\alpha,0.5}$ and especially $\psi = 0.45$ perform better in several cases. We also refer to Section A.2 of the Supplementary Material for a Monte Carlo investigation of power in the presence of a smooth break, and to Section A.3 of the Supplementary Material for a more detailed set of simulations on the effect of Δ on power in the STUR and nonstationary case.

In conclusion, the results in this section show that online changepoint detection in an RCA model based on weighted CUSUM statistics seems to work very well both under stationarity and nonstationarity. Broadly speaking, in the latter case using the *standardized* CUSUM (i.e., $\psi = 1/2$) is the preferred choice; in the former case, choosing ψ close to (but smaller than) $1/2$, appears to yield the best compromise between size control and timely detection.

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Table 4. Median delays and empirical rejection frequencies under alternatives—no covariates.

DGP	m^*	Weighted CUSUM $\psi=0$	Weighted CUSUM $\psi=0.25$	Weighted CUSUM $\psi=0.45$	Standardized CUSUM 0.5 $c_{\alpha,0.5}$	Standardized CUSUM 0.5 $\widehat{c}_{\alpha,0.5}$	Weighted Page-CUSUM $\psi=0$	Weighted Page-CUSUM $\psi=0.25$	Weighted Page-CUSUM $\psi=0.45$	OB
Case I (β_0=0.5)	100	54 (0.705)	44 (0.695)	35.5 (0.618)	37 (0.465)	34 (0.552)	55 (0.700)	44 (0.690)	35.5 (0.644)	
	200	82 (0.821)	63 (0.822)	49 (0.769)	57 (0.633)	51 (0.704)	79 (0.818)	62 (0.819)	47 (0.779)	55 (0.834)
	400	105 (0.947)	80 (0.941)	61 (0.894)	74 (0.784)	62 (0.847)	105 (0.940)	80.5 (0.926)	59 (0.885)	77 (0.881)
	800	135 (0.964)	103 (0.952)	69 (0.923)	90 (0.843)	75.5 (0.889)	135 (0.959)	89 (0.958)	60 (0.933)	115 (0.843)
Case II (β_0=1.05)	100	20 (1.000)	14 (1.000)	11 (1.000)	12 (1.000)	11 (1.000)	20 (1.000)	14 (1.000)	11 (1.000)	
	200	26 (1.000)	16 (1.000)	11 (1.000)	13 (1.000)	11 (1.000)	25 (1.000)	16 (1.000)	10 (1.000)	
	400	29 (1.000)	18 (1.000)	11 (1.000)	13 (1.000)	11 (1.000)	30 (1.000)	18 (1.000)	11 (1.000)	
	800	32 (1.000)	19 (1.000)	12 (1.000)	13 (1.000)	12 (1.000)	32 (1.000)	18 (1.000)	10 (1.000)	
Case III (β_0=1)	100	29 (0.994)	23 (0.993)	20 (0.991)	24 (0.989)	21 (0.990)	30 (0.996)	23 (0.995)	19 (0.993)	
	200	37 (1.000)	26 (1.000)	19 (1.000)	24 (1.000)	21 (1.000)	37 (1.000)	26 (1.000)	19 (1.000)	
	400	42 (1.000)	29 (1.000)	20 (1.000)	25 (1.000)	21 (1.000)	42 (1.000)	29 (1.000)	20 (1.000)	
	800	47 (1.000)	31 (1.000)	21 (1.000)	25 (1.000)	22 (1.000)	47 (1.000)	29 (1.000)	21 (1.000)	

Note: For each DGP, we report the *median* detection delay for only the cases where a changepoint is detected (thus leaving out the cases where no changepoint is detected). Numbers in round brackets represent the empirical rejection frequencies.

6. EMPIRICAL APPLICATIONS

We validate our approach through two empirical applications, both involving (potentially) nonstationary data. In Section 6.1, we consider a 'pure' RCA model with no covariates, and use it to detect daily Covid-19 hospitalizations; in Section 6.2, we consider and RCA model with covariates, and use it to detect changes in the dynamics of house prices.[8]

[8]Further results can be found in Section B of the Supplementary Material. Here, we would also like to point out that, by way of robustness analysis, we have also carried out our empirical exercises using a weighted-sum-of-covariance estimator for the long-run variance $\mathfrak{s}^2$, using the formulas suggested in Horváth and Trapani (2023a, eqns. (3.5)–(3.7)). Results are essentially the same, which indirectly confirms the assumption that $\left\{\left(\epsilon_{i,1}, \epsilon_{i,2}\right), -\infty < i < \infty\right\}$ is an *i.i.d.* sequence.

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Table 5. Online changepoint detection for Covid-19 daily hospitalization - England data.

Changepoint 1	Changepoint 2	Changepoint 3
28 August 2020	4 November 2020	27 January 2021
$\widehat{\beta} = 0.995$	$\widehat{\beta} = 1.010$	$\widehat{\beta} = 1.002$
[11 April 2020–15 August 2020]	[29 August 2020–29 October 2020]	[5 November 2020–31 December 2020]

Note: The series ends at 30 January 2021. We use the logs of the original data (plus one, given that, in some days, hospitalizations are equal to zero): no further transformations are used.

For each changepoint, we report the left WLS estimates of β_0 - i.e., the value of β_0 *prior* to the breakdate; we report the sample on which estimation was performed in square brackets, based on the changepoints identified using the test by Horváth and Trapani (2023a).

6.1. Sequential Monitoring of Covid-19 Hospitalizations

We consider daily data on Covid-19 hospitalizations. There is a huge literature on the application of time series methods to the reduced form of epidemiological models; Shtatland and Shtatland (2008), *inter alia*, advocate using a low-order autoregression as an approximation of the popular SIR model, especially as a methodology for the early detection of outbreaks; and Horváth et al. (2024) apply the RCA model to a similar dataset, detecting several changepoints. In this context, it is important to check whether the observations change from an explosive to a stationary regime (meaning that the epidemic is slowing down), or vice versa whether they change from a stationary to an explosive regime (corresponding to a surge in the epidemic). Our dataset consists of daily data for England recorded between 19 March 2020 and 30 January 2021, i.e., spanning the period from approximately the first lockdown (which was announced on 23 March 2020), until after the third and last lockdown of 6 January 2021. We transform the series into logs (plus one since on some days the number of hospitalization is zero); no further pre-processing is applied.[9]

We use a "pure" RCA specification with no covariates. The purpose of our exercise is to ascertain whether public health authorities, back in 2020, could have benefited from the use of a sequential monitoring procedure to flag changes in the dynamics of daily hospitalizations, thus informing their decisions. We have used $\psi = 1/2$ in $g_{m,\psi}(k)$ in (2.5); based on the empirical rejection frequencies reported in Tables 1–3, we use the critical values $\widehat{c}_{\alpha,0.5}$ defined in (3.8);[10] in all scenarios we set the length of the monitoring horizon always equal to the training sample size, i.e., $m = m^*$, which, according to the results in Tables 1–3, always ensures size control.

[9]The data are available from https://ourworldindata.org/grapher/uk-daily-covid-admissions?tab=chart\&stackMode=absolute\&time=2020-03-29.latest\&region=World

See also https://www.instituteforgovernment.org.uk/sites/default/files/2022-12/timeline-coronavirus-lockdown-december-2021.pdf, for a timeline of the UK Government decisions on lockdown and closures.

[10]Results with different values of ψ - and with $\psi = 1/2$ and asymptotic critical values—are available upon request.

By way of preliminary analysis (and also to assess how our online detection methodology fares), in Section B.2 of the Supplementary Material we apply the test by Horváth and Trapani (2023a) to the whole sample (Table B.2, in the Supplementary Material). In particular, after the closure of the education and hospitality sectors in all UK nations announced on 23 March 2020 (and implemented 3 days later), a changepoint is found at 10 April 2020, indicating that the lockdown had started to "bite" after that date; subsequently, another changepoint is found towards the end of August, which can be naturally interpreted as the beginning of the second wave in the UK, after an increase in travelling during the holiday season. Hence, we use, as the training sample, the period between 11 April 2020 and 15 August 2020, with $m = 127$. We find evidence of a changepoint on 28 August 2020. As can be seen in Table B.2 in the Supplementary Material, the ex-post test by Horváth and Trapani (2023a) finds a break at August 26th. Clearly, a direct comparison between the two tests is not meaningful, since ex-post detection uses the information contained in the full sample, whereas online detection cannot make use of it—thus putting the latter at a disadvantage compared to the former. Still, there is (only) a two days delay between the two procedures. Moreover, the August changepoint was officially acknowledged by the Prime Minister on 18 September 2020; hence, our sequential monitoring procedure could have brought forward public health decisions. The value of β after the break is above 1, indicating explosive dynamics in Covid-19 hospitalizations. Table B.2 in the Supplementary Material indicates that there was a further break on 29 October 2020, corresponding to the lockdown at the end of that month. Hence, we restart our monitoring procedure, and we use the training sample 29 August 2020 till 29 October 2020, with $m = 61$; under an explosive regime, this should suffice to ensure size control and short detection delay, according to the results in Tables 3 and 4. Results are reported in Table 5. A changepoint is flagged on 4 November 2020, with a decline in β indicating the effect of the closures and also of the growing concerns about a second wave. The break is detected very close to (and indeed *before*) the lockdown, suggesting that lockdowns tended to occur when a turning point in hospitalizations had already occurred, or was "in the making". Such evidence can be read in conjunction with the results in Table B.2 in the Supplementary Material, and also with Wood (2022), who, albeit with a different methodology and focus, finds similar results for hospital deaths. Finally, we carry out monitoring using a training sample between 5 November 2020, till 31 December 2020, thus having $m = 58$. We find a changepoint on 27 January 2021, i.e., two weeks later the break found with the ex-post test by Horváth and Trapani (2023a), and three weeks later after the national lockdown announced on 6 January 2021. In Section B.2 of the Supplementary Material, we also report the findings when using the test by Otto and Breitung (2023), which confirm our findings here (although we point out that, strictly speaking, the test by Otto and Breitung, 2023 is not designed for an RCA model, nor is it designed for the case of nonstationary data).

In conclusion, the empirical evidence presented above shows that our RCA-based approach to online changepoint detection is suitable to detect the onset (and the receding) of a pandemic with short delays, thus being a recommended item in the toolbox of public health decision-makers.

6.2. Sequential Monitoring of Housing Prices

We apply our sequential detection procedures to the online detection of a change in housing prices in Los Angeles at the end of the first decade of the century. Horváth and Trapani (2023a) apply an RCA-based, ex-post changepoint test, and find evidence of such a bubble starting around 4 February 2009, when, after a period of "hard landing", prices stabilized;[11] we also refer to a related paper by Horváth, Liu, and Lu (2022), who use monthly data. We use Los Angeles as a case study to check how timely online detection is, and also to assess the robustness of our results to the choice of the training and monitoring sample sizes, and the benefit of adding covariates. In our application, following Horváth and Trapani (2023a), we use (logs of) daily housing prices.[12]

All our monitoring exercises start at 15 January 2009; Horváth and Trapani (2023a) find no changepoints between 4 May 2006, and 3 February 2009, which entails that the non-contamination assumption during the training period is satisfied in all cases considered. We monitor for changepoints using several alternative models. In addition to the basic RCA specification, with no covariates, we also use different combinations of regressors, including: two variables that are closely related to the risk-free interest rate, taking into account the opportunity cost of capital (the Moody's Seasoned Aaa Corporate Bond Yield, and the 10 Year US Treasury Constant Maturity Rate); a measure of volatility (namely, the VXO volatility index); and a measure of real economic activity (we use the Lewis–Mertens–Stock Weekly Economic Indicator, WEI[13]). All regressors are taken from the FRED St Louis website. The risk-free interest rate proxies and the volatility measure are transformed into logs. Applying standard unit root tests, and also the test by Trapani (2021) designed for the RCA model, to these explanatory variables, we find overwhelming evidence of a unit root in all of them during the period 28 March 2008, corresponding to the earliest starting point for the training sample, and 30 October 2009, corresponding to the latest ending point for the monitoring horizon (see Section B.3 of the Supplementary Material for details); hence, we employ their (demeaned) first differences. As far as the WEI is concerned, it comes at a weekly frequency, and we use its weekly value for each day by way of disaggregation; other disaggregation methods could be employed following the Mixed-Data Sampling literature (MIDAS; see e.g., Ghysels, 2018), e.g., using a weighted average interpolation, but we found that these did not make virtually any difference in our results (in fact, marginally worsening the detection delay).

[11] See Section B.3 of the Supplementary Material for a plot of the data.

[12] We use the daily data constructed by Bollerslev, Patton, and Wang (2016), and we refer to that paper for a description of the datasets.

[13] See Lewis et al. (2022) for a thorough description.

TABLE 6. Online changepoint detection for Los Angeles daily housing prices.

Model: $y_i=(\beta_i+\epsilon_{i,1})y_{i-1}+\epsilon_{i,2}$			Model: $y_i=(\beta_i+\epsilon_{i,1})y_{i-1}+\lambda_1x_{1,i}+\lambda_2x_{2,i}+\epsilon_{i,2}$		
m^*	100	200	m^*	100	200
m			m		
100	9 June 2009 [no changepoint found]	15 June 2009	100	2 June 2009	2 June 2009
200	9 June 2009 [no changepoint found]	15 June 2009	200	3 June 2009	10 June 2009
Model: $y_i=(\beta_i+\epsilon_{i,1})y_{i-1}+\lambda_1x_{1,i}$ Model: $+\lambda_2x_{2,i}+\lambda_3x_{3,i}+\epsilon_{i,2}$			$y_i=(\beta_i+\epsilon_{i,1})y_{i-1}+\lambda_1x_{1,i}+\lambda_2x_{2,i}+\lambda_3x_{3,i}+\lambda_4x_{4,i}+\epsilon_{i,2}$		
m^*	100	200	m^*	100	200
m			m		
100	2 June 2009	2 June 2009	100	18 May 2009	18 May 2009
200	3 June 2009	10 June 2009	200	2 June 2009	2 June 2009

Note: For each combination of m and m^*, we report the estimated breakdate. For all combinations of m and m^*, monitoring starts on 15 January 2009. When $m=100$, the training sample covers the period 20 August 2008, till 14 January 2009; when $m=200$, the training sample covers the period 28 March 2008, till 14 January 2009. Similarly, when $m^*=100$, the monitoring horizon stops at 9 June 2009; when $m^*=200$, the monitoring horizon stops at October 30th, 2009.

We have used the following notation for the regressors: $x_{1,i}$ denotes the 10 Year US Treasury Constant Maturity Rate, $x_{2,i}$ denotes the Moody's Seasoned Aaa Corporate Bond Yield, $x_{3,i}$ is the VXO volatility index, and $x_{4,i}$ is the WEI. Horváth and Trapani (2023a) find evidence of a changepoint on 3 February 2009, applying ex-post changepoint detection to the period 5 January 1995–23 October 2012. The deterministic part of the autoregressive coefficient, β, is found to be equal to 0.99931 in the period before the changepoint, and 1.00007 afterwards.

No further pre-processing is carried out. As far as implementation is concerned, we have used $\psi=1/2$ in $g^{(x)}_{m,\psi}(k)$ in (4.5); in the case considered, the data have an autoregressive coefficient whose deterministic part is close to unity during the training period; in Table 6, we report results using the asymptotic critical values $c_{\alpha,0.5}$.[14]

Results in Table 6 show that adding covariates can potentially lead to meaningful improvements in terms of detection delay: a measure of economic activity such as the WEI index, in particular, seems to contain relevant information to model the dynamics of house prices. We also note that detection delays seem to be around 4 months, which may be a consequence of the relatively small change in β before and after the change. Interestingly, results are unaffected by the size of the training sample m, but they do differ if m^* changes. This is a purely mechanical effect, due to the increase in the asymptotic critical values as m^* increases, as also noted in Section 5.

[14] In Table B.5 in the Supplementary Material, we report results obtained with $\widehat{c}_{\alpha,0.5}$, which show that differences, if any, are minimal.

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7. DISCUSSION AND CONCLUSIONS

We propose a family of weighted CUSUM statistics for the online detection of changepoints in a Random Coefficient Autoregressive model. Our statistics are, in particular, based on the CUSUM process of the WLS residuals, and we study both the standard CUSUM, and the so-called Page-CUSUM monitoring schemes, which is designed to offer higher power/shorter detection delay. In the case of the standard CUSUM, we also study the standardized version using $\psi = 1/2$, obtaining, under the null, a Darling–Erdös limit theorem. In this case, seeing as the asymptotic critical values are a poor approximation leading to an overly conservative procedure, we also propose an approximation which works well in finite samples, avoiding overrejections. As our simulations show, the use of weighted statistics is particularly beneficial under the alternative, with detection delays decreasing as ψ increases. Whilst, for the ease of exposition, we focus on the RCA case with no covariates, we also extend our theory to include exogenous regressors, which are allowed to be weakly dependent according to a very general definition of dependence. Simulations show that our procedures broadly guarantee size control; indeed, our experiments indicate that, for any given training sample size m and monitoring horizon m^*, it is always possible to choose the appropriate weighing scheme to ensure the best balance between size control and timely detection. This is reinforced by our empirical illustrations, showing that when our methodology is applied to the online detection of epidemiological and housing data, it manages to find breaks very quickly, even when compared against ex-post detection methodologies.

Importantly, building on the well-known fact that, in an RCA model, WLS inference on the deterministic part of the autoregressive parameter is always Gaussian, irrespective of whether the observations form a stationary or nonstationary sequence, our monitoring procedures can be applied to virtually any type of data: stationary, or with explosive dynamics, or on the boundary between the two regimes, with no modifications required. Hence, our methodology is particularly suited, as a leading example, to the detection of both the onset, or the collapse, of a bubble (when using financial data), or of a pandemic (when using epidemiological data). Other cases also can be considered, e.g., monitoring for changes in the persistence of a stationary series such as inflation, and the extension of the basic RCA specification to include covariates should enhance the applicability of our methodology.

This article is part of a wider research program. Our methodologies and proposed tests can also be extended to other nonlinear models where the phenomenon known as “volatility induced stationarity” holds.[15] A prime example is the Double AutoRegressive model (DAR) studied by Ling (2004) and Cavaliere and Rahbek (2021), inter alia, defined as

$$y_t = \beta y_{t-1} + \eta_t \left(\sigma_1^2 + \sigma_2^2 y_{t-1}^2\right)^{1/2}, \tag{7.1}$$

[15] We are grateful to the Co-Editor for suggesting this possible extension.

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where η_t is i.i.d., with mean zero and unit variance. Letting the information set available up to $t-1$ be defined as F_{t-1}, it is easy to see that $E(y_t|\mathcal{F}_{t-1}) = \beta y_{t-1}$, and $Var(y_t|\mathcal{F}_{t-1}) = \sigma_1^2 + \sigma_2^2 y_{t-1}^2$, which entails that the DAR model is second-order equivalent to the RCA of (2), and that testing for the constancy of β in (7.1) is tantamount to testing for the constancy of the conditional mean function, similarly to the case of an RCA model. Furthermore, Ling (2004) notes that, under the assumption of Gaussian η_t, the DAR and the RCA models are equivalent in distribution. Heuristically, this suggests that Maximum and Quasi Maximum Likelihood based inference for (2) and (7.1), studied in Aue and Horváth (2011) and Cavaliere and Rahbek (2021) respectively, should yield equivalent results; in turn, seeing as the WLS estimator is first-order equivalent to QML in the context of (2), the same can be expected when applying the WLS estimator to (7.1). This suggests that our tests, which are designed for (2), could be applied to test for the constancy of β even if the true DGP is the one in (7.1). By the same token, our methodologies can be applied to monitoring Garch-type processes (Horvath, Trapani, and Wang, 2024).

Further, in Section 4 we have considered (for the first time in the literature, to the best of our knowledge) an RCA model with covariates. In Assumption 4.2, we need to rule out any form of endogeneity in order for our asymptotics to hold. However, the literature has investigated the issue of monitoring using models with endogenous covariates; in particular, Kurozumi (2017) studies a family of procedures based on the weighted CUSUM process based on residuals using both the Instrumental Variable (IV) and the OLS estimators; Kurozumi (2017) shows that (in the presence of endogeneity) procedures based on the OLS residuals can have higher power than IV-based ones, at least when the break occurs early in the monitoring period. This immediately suggests that, even in our context, estimation and changepoint testing are two separate tasks: the former requires IV (or a similar technique that controls for endogeneity), whereas the latter seems to be better served using a Least Squares estimator, as also discussed in Perron and Yamamoto (2014) and Casini and Perron (2019).[16] Extensions of the results in Section 4 to the case of endogenous regressors are currently under investigation by the authors.

SUPPLEMENTARY MATERIAL

The supplementary material for this article can be found at https://doi.org/10.1017/S0266466625000052.

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[16] We are grateful to an anonymous Referee for bringing this very important extension to our attention.

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