VARIOUS VARIETIES[1]

Dedicated to the memory of Hanna Neumann

SHEILA OATES MACDONALD

(Received 7 June 1972; revised 29 August 1972)

Communicated by M. F. Newman

The study of varieties of universal algebras[2] which was initiated by Birkhoff in 1935, [2], has received considerable attention during the past decade; the question of particular interest being: "Which varieties have a finite basis for their laws?" In that paper Birkhoff showed that the laws of a finite algebra which involve a bounded number of variables are finitely based, so it is not altogether surprising that finite algebras have received their share of this attention.

In 1951 Lyndon, [9], showed that any two-element algebra has a finite basis for its laws, but Murskii's example, [12], of a three-element groupoid which does not have a finite basis for its laws showed that Lyndon's result is best possible as far as general algebras are concerned. The Oates-Powell theorem, [14], showed that the laws of a finite group are finitely-based, and the Kovács and Newman proof of this, [7], has been adapted by Kruse, [8], to prove the analogous result for finite rings. Perkins, [15], has proved the result for commutative semigroups, and McKenzie, [11], for finite lattices. A recent announcement by Baker, [1], shows that McKenzie's result can be extended to any finite algebra belonging to a variety of algebras all of whose congruence lattices are distributive. Finally it is rumoured that someone in Moscow has proved the result for finite Lie algebras.

Perhaps the most disturbing counterexample is that of a six-element semigroup given by Perkins, [15]. However the lattice of congruences of a semigroup is not, in general, even modular, whereas the congruences of groups, rings and Lie algebras are permutable, and the lattice of congruences of a lattice is distributive. Since both of these properties imply modularity, a plausible conjecture might be :-

[1] This is a version of an invited address given on 19th May, 1972 at the Annual Meeting of the Australian Mathematical Society, at Armidale.

[2] Definitions concerning universal algebras may be found in Cohn's book, [3], and concerning varieties in Hanna Newmann's book, [13].

CONJECTURE 1. *If $\mathfrak{V}$ is any variety of algebras whose congruence lattices are modular then any finite algebra in $\mathfrak{V}$ has a finite basis for its laws.* (This conjecture seems to have been arrived at independently by several people, including Kirby A. Baker and myself.)

All known finite algebras having an infinite basis for their laws contain as subalgebra the two-element semigroup, $A = \{0, a\}$, in which all products are zero. The congruence lattice of the direct product of countably many copies of A is the lattice of all equivalence relations on an infinite set and this satisfies no non-trivial lattice identity.[3] On the basis of this observation S. Burris has made the rather more optimistic conjecture:

CONJECTURE 2. *If $\mathfrak{V}$ is any variety of algebras whose congruence lattices satisfy some non-trivial lattice identity then any finite algebra in $\mathfrak{V}$ has a finite basis for its laws.*

What possible methods of proof are available? Having regard to the aforementioned theorem of Birkhoff the straightforward method would seem to be to show that any given law is derivable from the laws involving a bounded number of variables, and, indeed, this is precisely what McKenzie does in [11], where he proves:-

THEOREM. *If L is a finite lattice of order l, $m = [\log_2 l]$ and $n = 1 + l^m$ then the laws of L can be derived from those involving at most n variables.*

The earlier results were proved by an indirect method. To explain this I need a few definitions:

A *section* of an algebra A is H/ρ where H is a subalgebra of A, and ρ is a congruence on H. It is a proper section if it is not equal to the whole of A.

A *critical algebra* is a finite algebra which is not contained in the variety generated by its proper sections.

A *Cross variety* is a variety, $\mathfrak{V}$, with the following properties:

(i) finitely-generated algebras in $\mathfrak{V}$ are finite,
(ii) there are (up to isomorphism) only finitely many critical algebras in $\mathfrak{V}$,
(iii) $\mathfrak{V}$ has a finite basis for its laws.

The proofs of 51.41 and 51.52 in Neumann's book, [13], may readily be adapted to the more general case to show that a Cross variety is generated by a finite algebra (namely the direct product of its critical algebras) and that a subvariety of a Cross variety is Cross, so that embedding our finite algebra in a Cross variety would give the required result. Unfortunately, although the variety generated by a finite algebra always has property (i), ([2]), it is by no means obvious

[3] I am indebated to Ralph Freese and J. B. Nation for providing me with this proof of Burris's observation.

that, even if it has a finite basis for its laws, it necessarily contains only finitely many critical algebras, although all known examples satisfy this condition. The only case in which this is not an integral part of the proof is when congruence lattices are distributive, but Jónsson, [6], has shown that in this case a subdirectly irreducible algebra in the variety generated by a finite algebra is a section of that algebra. Hence there are only finitely many subdirectly irreducible algebras, and, a fortiori, only finitely many critical algebras, in such a variety. (Baker, Freese and Nation drew this result to my attention.) However it seems that the above method might be viable at least in the case where congruences are permutable since then there is a considerable number of results such as the Jordan-Hölder theorem which can be called upon, Cohn [3], 11.6.

If we are to work directly with the laws then it is likely that the following results which link properties of algebras in a variety with properties of their congruence lattices will be of use. Indeed, the second one has been used by Baker.

THEOREM (Mal'cev, [10]). *The congruence of algebras in a variety are permutable if and only if there exists a derived ternary operation, p such that:*

$$P1 \qquad p(a,a,b) = b,$$

$$P2 \qquad p(a,b,b) = a.$$

THEOREM (Jónsson, [6]). *The lattices of congruences of algebras in a variety are distributive if and only if there exist derived ternary operations* $d_0, d_1, \cdots, d_n$ *such that:*

$$D1 \qquad d_0(a,b,c) = a \ \text{ and } \ d_n(a,b,c) = c,$$

$$D2 \qquad d_i(a,b,a) = a \, (i = 0, 1, \cdots, n),$$

$$D3 \qquad d_i(a,a,b) = d_{i+1}(a,a,b) \ (i \text{ even}),$$

$$D4 \qquad d_i(a,b,b) = d_{i+1}(a,b,b) \ (i \text{ odd}).$$

THEOREM (Day, [4]). *The lattices of congruences of algebras in a variety are modular if and only if there exist derived quaternary operations,* $m_0, m_1, \cdots, m_n$ *such that:*

$$M1 \qquad m_0(a,b,c,d) = a \ \text{ and } \ m_n(a,b,c,d) = d,$$

$$M2 \qquad m_i(a,b,b,a) = a \ (i = 0, 1, \cdots, n),$$

$$M3 \qquad m_i(a,b,b,d) = m_{i+1}(a,b,b,d) \ (i \text{ odd}),$$

$$M4 \qquad m_i(a,a,d,d) = m_{i+1}(a,a,d,d) \ (i \text{ even}).$$

(Here a derived operation is one built up from the original operations in the algebra, for example in groups $p(a,b,c) = ab^{-1}c$ satisfies the conditions of the first theorem, and $m_0(a,b,c,d) = a, m_1(a,b,c,d) = ac^{-1}ba^{-1}d, m_2(a,b,c,d) = d$ satisfy the conditions of the third theorem, with $n = 2$.)

Day reports that R. Wille has raised the general problem: *Can any non-trivial lattice identity that holds for all the congruence lattices of the algebras in a given variety be characterised by a sequence of derived operations?* An affirmative answer to this would tie in with Burris's conjecture.

On the other hand, if these conjectures are false where should we start looking for a counterexample? Results of Evans, [5], suggest that varieties of loops and quasigroups can be quite badly behaved, so perhaps this would be a fruitful area to search, especially as these have permutable congruence lattices. Indeed, defining a quasigroup as an algebra with three binary operations, multiplication, $x.y$, left division $x\backslash y$, and right division, x/y with the laws:

$$x.(x\backslash y) = y = x\backslash(x.y) \text{ and } (x/y).y = x = (x.y)/y$$

we have that the appropriate derived ternary operation is

$$p(a,b,c) = (a(b\backslash k))/(c\backslash k) \text{ where } k \text{ is any fixed element.}$$

Thus a quasigroup counterexample would demolish both conjectures.

I hope that these remarks about varieties generated by finite algebras will inspire somebody to produce a reasonably complete classification of those which have a finite basis for their laws and also to determine whether they are Cross.

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Department of Mathematics
University of Queensland
Brisbane
Australia