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THE FITTING LENGTH OF A FINITE SOLUBLE GROUP AND THE NUMBER OF CONJUGACY CLASSES OF ITS MAXIMAL METANILPOTENT SUBGROUPS

BY
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1. **Introduction.** It is known that the Fitting length $h(G)$ of a finite soluble group G is bounded in terms of the number $\nu(G)$ of the conjugacy classes of its maximal nilpotent subgroups. For $|G|$ odd, a bound on $h(G)$ in terms of $\nu(G)$ was discussed in Lausch and Makan [6]. In the case when the prime 2 divides $|G|$, a logarithmic bound on $h(G)$ in terms of $\nu(G)$ is obtained in [7]. The main purpose of this paper is to show that the Fitting length of a finite soluble group is also bounded in terms of the number of conjugacy classes of its maximal metanilpotent subgroups. In fact, our result is rather more general.

Let F be a saturated formation of finite soluble groups, which is also a Fischer class. Then there is a uniquely determined set π of primes such that $N_\pi \leq F \leq S_\pi$, where N_π is the class of all finite nilpotent π-groups and S_π the class of all finite soluble π-groups (see Hartley [5, §3.3, Remark 1]). Let $F_\pi^0=\{1\}$, the class consisting of the trivial groups, let $F_\pi^1=F_\pi=S_{\pi'}F$ and, for an integer $k>1$, let $F_\pi^k=F_\pi^{k-1}F_\pi$. (If X and Y are two classes of groups, we define XY to be the class of groups G which is an extension of an X-group by a Y-group.) One can easily check that F_π^k is both a saturated formation and a Fischer class, for each $k\geq 0$.

More precisely, we show:

THEOREM. *For each integer $n>1$, the* F*-length $l(G)$ of a finite soluble group G is at most $\mu_n(G)+n-1$, where $\mu_n(G)$ is the number of conjugacy classes of maximal F_π^n-subgroups of G.*

All groups considered in this paper are finite and soluble. For the definitions and basic facts about saturated formations, Fitting classes and various subgroups related to both these classes which will enter our discussion, we refer the readers to Carter and Hawkes [1], Fischer, Gaschütz and Hartley [3], Gaschütz [4], Hartley [5] and Wright [8].

Given a saturated formation X which is also a Fitting class, a series

$$1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_m = G$$

of normal subgroups of a group G is called an **X-series** of G if for each $i=1, 2, \ldots, m$, either $G_i/G_{i-1} \in X$ or $G_i/G_{i-1} \in S_{\pi'}$, where π is the uniquely determined set of

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primes such that $N_\pi \leq X \leq S_\pi$. The **X-length** of G is defined to be the smallest number of X-factors in any X-series of G. Observe that, if $X = N$, the class of all finite nilpotent groups, then the X-length of a group is the familiar Fitting length of the group.

2. **A necessary and sufficient condition for an injector to be a projector.** In this section, we establish a result which we need in proving the theorem and which might also be of independent interest.

PROPOSITION 2.1. *Let* E *be a Fischer class which is also a saturated formation, and let V be an* E*-injector of a group H. Then, V is an* E*-projector (i.e., an* E*-covering subgroup) of H if and only if V is a maximal* E^2*-subgroup of H.*

For the proof of Proposition 2.1 we need the following result which has been proved independently by Graham Chambers [2] and the author (see [7]). Chambers obtains this result as a special case of his more general result, namely Theorem 3 in [2].

THEOREM 2.2. *Let* X *be a Fischer class and* Y *a saturated formation. Let D be the* Y*-normalizer of a group G corresponding to a Sylow system Σ of G and let V be the* X*-injector of G into which Σ reduces. Then D and V are permutable subgroups of G and, moreover, DV avoids the* Y*-eccentric,* X*-avoided chief factors of G and covers the rest.*

In Theorem 2.2, we need not assume that Y contains the class N. In that case, D is defined as in Wright [8] with respect to an integrated screen. Since, in view of the corollary to Lemma 3 in Hartley [5], V is strongly pronormal in G (see [2], for definition), Theorem 2.2 is clearly a special case of Theorem 3 in Chambers [2].

We can now prove Proposition 2.1, but before we do so, we wish to make the following remark.

REMARK. Though the various results from Carter and Hawkes [1], which we will use in the course of the proof of Proposition 2.1, are proved there for saturated formations containing the class N, they also hold for an arbitrary saturated formation.

Proof of Proposition 2.1. Suppose first that V is an E-projector of H and let L be an E^2-subgroup of H which contains V. By Satz 2 in Fischer, Gaschütz and Hartley [3], V is an E-injector of L and so $L_E \leq V$, where L_E is the unique largest normal E-subgroup, or the E-radical of L. On the other hand, $L/L_E \in E$. Thus, by our assumption, $L = L_E V = V$ and we have shown that V is a maximal E^2-subgroup of H.

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Suppose next that V is a maximal E^2-subgroup of H and proceed by induction on $|H|$ to show that V is an E-projector of H. If $H \in \mathrm{E}$, $V=H$ and we are done. If $H \notin \mathrm{E}$, H has an E-crucial chief factor, say R/S. It will be sufficient to show that V covers H/R; for, in that case, VS is an E-crucial maximal subgroup of H and by induction V is an E-projector of VS and therefore that of H by Theorem 5.4 of Carter and Hawkes [1].

Let $N=N_H(V \cap R)$. If $N<H$, V is, by induction, an E-projector of N and therefore covers N/N^{E}, where N^{E} is the smallest normal subgroup of N such that $N/N^{\mathrm{E}} \in \mathrm{E}$. But, by the Frattini argument N covers $H/R \in \mathrm{E}$; hence $N^{\mathrm{E}} \leq R$ and V covers H/R, as required. Hence assume $V \cap R \lhd H$. Let Σ be a Sylow system of H which reduces into V and let D be the E-normalizer of H corresponding to Σ. By Theorem 2.2, $DV=VD=D(V \cap R)$. Thus, $DV/V \cap R \cong D/D \cap V \cap R \in \mathrm{E}$, whence $DV \in \mathrm{E}^2$ and $D \leq V$. But then D, and therefore V, covers H/R, as required. The proof is complete.

3. **Proof of the Theorem.** Throughout this section Y will denote the class F_π, X will denote the class F_π^{n-1} and H will denote the class F_π^n.

We begin with the following lemma.

LEMMA 3.1. *If V is an* H-*injector of G, then $G_{\mathrm{X}}=V_{\mathrm{X}}$. Moreover, V/G_{X} is a* Y-*injector of G/G_{X}.*

Proof. The first part of the lemma is a consequence of Lemma 10 in Hartley [5].

Next, let $(N/G_{\mathrm{X}}) \lhd \lhd (G/G_{\mathrm{X}})$ and consider $N \cap V/G_{\mathrm{X}}$. Since V is an H-injector of G, $N \cap V$ is an H-injector of N, and so, by the first part, $(N \cap V)_{\mathrm{X}}=N_{\mathrm{X}}$. Thus, since $N \cap V$ is a maximal H-subgroup of N, it follows that $(N \cap V)/N_{\mathrm{X}}$ is a maximal Y-subgroup of N/N_{X}. But $N_{\mathrm{X}}=G_{\mathrm{X}}$. Hence, $N \cap V/G_{\mathrm{X}}$ is a maximal Y-subgroup of N/G_{X}. Since N/G_{X} was an arbitrary subnormal subgroup of G/G_{X}, this shows that V/G_{X} has the defining properties of Y-injectors, and so V/G_{X} is a Y-injector of G/G_{X}, as required.

With the help of Proposition 2.1 and Lemma 3.1, we can next prove the following lemma.

LEMMA 3.2. *Let V be as in Lemma 3.1. If V/G_{Y} is a maximal* H-*subgroup of G/G_{Y}, then V/G_{X} is a* Y-*projector of G/G_{X}.*

Proof. By Lemma 3.1, V/G_{X} is a Y-injector of G/G_{X}. Moreover, since the F-length of $G_{\mathrm{X}}/G_{\mathrm{Y}}$ is at most $n-2$, it follows, by our assumption, that V/G_{X} is a maximal Y^2-subgroup of G/G_{X}. Thus, by Proposition 2.1, V/G_{X} is a Y-projector of G/G_{X}, as required.

The following lemma is a straightforward generalization of Lemma 1 in Lausch and Makan [6] and provides a basis for induction argument in the proof of the theorem.

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LEMMA 3.3. *Let $N \trianglelefteq G$. Then every maximal* H*-subgroup of G/N is the image in G/N of a suitable maximal* H*-subgroup of G. In particular, $\mu_n(G/N) \leq \mu_n(G)$. If moreover, $\mu_n(G/N) = \mu_n(G)$, then the image in G/N of every maximal* H*-subgroup of G is a maximal* H*-subgroup of G/N.*

Proof. Let W/N be a maximal H-subgroup of G/N. Since H is a saturated formation, W has an H-projector V, say (see Gaschütz [4]). Also, since $W/N \in \mathrm{H}$, $W = VN$. Let V^* be a maximal H-subgroup of G which contains V. Clearly $W/N = NV/N \leq NV^*/N \in \mathrm{H}$. Thus, since W/N is a maximal H-subgroup of G/N, it follows that $NV^*/N = W/N$. In particular, since V is a maximal H-subgroup of W, $V^* = V$. The rest of the lemma now follows.

We can now complete the proof of the theorem as follows:

We proceed by induction on $|G|$. Thus, we can assume that $\mu_n(G/G_{\mathrm{Y}}) = \mu_n(G)$. Also, we can assume that $l(G) > n$ since otherwise the result is trivially true. Let V be an H-injector of G. Then, since $\mu_n(G/G_{\mathrm{Y}}) = \mu_n(G)$, V/G_{Y} is, by Lemma 3.3, a maximal H-subgroup of G/G_{Y}. Hence, by Lemma 3.2, V/G_{X} is a Y-projector of G/G_{X}. Since, by Hilfssatz 2.2 in Gaschütz [4], Y-projectors of G are invariant under homomorphisms of G, it follows then that V/G_{H} is a Y-projector of G/G_{H}.

Next, let W/G_{Y} be an H-injector of G/G_{Y}. Then, since $V \not\geq G_{\mathrm{A}}$, where $\mathrm{A} = \mathrm{Y}^{n+1}$, V/G_{Y} and W/G_{Y} belong to two distinct conjugacy classes of maximal H-subgroups of G/G_{Y}. Hence, W/G_{Y^2} is not a maximal H-subgroup of G/G_{Y^2}; for, otherwise, W/G_{H} is, in view of Lemma 3.2, a Y-projector of G/G_{H}, and, therefore, conjugate to V/G_{H}, contrary to the fact that V/G_{Y} and W/G_{Y} are not conjugate in G/G_{Y}. Thus, it follows that $\mu_n(G/G_{\mathrm{Y}^2}) < \mu_n(G)$. In fact, $\mu_n(G/G_{\mathrm{Y}^2}) \leq \mu_n(G) - 2$, since $V/G_{\mathrm{Y}^2} < VG_{\mathrm{A}}/G_{\mathrm{Y}^2} \in \mathrm{H}$. Hence, by the induction hypothesis, $l(G/G_{\mathrm{Y}^2}) \leq n + (\mu_n(G) - 2) - 1$. Since $l(G_{\mathrm{Y}^2}) \leq 2$, it follows finally that $l(G) \leq n + \mu_n(G) - 1$, and so we are done.

REMARK. The theorem is not true for $n = 1$ as the case when $\mathrm{F} = \mathrm{N}$ and $\mu_1(G) = 2$ shows (see the corollary following the proof of the main theorem in [6]).

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REFERENCES

1. R. W. Carter and T. O. Hawkes, *The* F*-normalizers of a finite soluble group*, J. Algebra **5** (1967), 175–202.

2. G. A. Chambers, *On f-Prefrattini Subgroups*, Canad. Math. Bull. **15** (1972), 345–348.

3. B. Fischer, W. Gaschütz and B. Hartley, *Injektoren endlicher auflösbarer Gruppen*, Math. Z. **102** (1967), 337–339.

4. W. Gaschütz, *Zur Theorie der endlichen auflösbaren Gruppen*, Math. Z. **80** (1963), 300–305.

5. B. Hartley, *On Fischer's dualization of formation theory*, Proc. London Math. Soc. (3) **19** (1969), 193–207.

6. H. Lausch and A. Makan, *On a relation between the Fitting length of a soluble group and the number of conjugacy classes of its maximal nilpotent subgroups*, Bull. Austral. Math. Soc. **1** (1969), 3–10.

7. A. R. Makan, *On some aspects of finite soluble groups*, Ph.D. thesis, The Australian National University, Canberra, April 1971.

8. C. R. B. Wright, *On screens and* L-*Izers of finite solvable groups*, Math. Z. **115** (1970), 273–282.

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