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SOME REMARKS ON PRINCIPAL PRIME IDEALS

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Abstract

In this paper we investigate principal prime ideals in commutative rings. Among other things we characterize the principal prime ideals that are both minimal and maximal and characterize the maximal ideals of a polynomial ring that are principal. Our main result is that if (p) is a principal prime ideal of an atomic ring R, then $\mathrm{ht}(p) \leq 1$.

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In this paper we study principal prime ideals in commutative rings. Throughout, all rings will be commutative rings with identity. Let R be a commutative ring. A nonunit $p \in R$ is *prime* (often called a principal prime) if $p \mid ab$ $(a, b \in R)$ implies $p \mid a$ or $p \mid b$, or equivalently, (p) is a prime ideal of R. Now a nonunit $a \in R$ is *irreducible* (or an *atom*) if $a = bc$ $(b, c \in R)$ implies $a \sim b$ or $a \sim c$, where $x \sim y$ denotes being associate, that is, $(x) = (y)$. In addition, R is said to be *atomic* if each nonunit of R is a finite product of atoms. If R satisfies the ascending chain condition on principal ideals (ACCP), then R is atomic, but the converse is false. Of course, a principal prime p is irreducible, but the converse is false. One of our main results is that for a principal prime ideal (p) in an atomic ring R, we have $\mathrm{ht}(p) \leq 1$.

Examples of principal prime ideals that come to mind (besides (0) in an integral domain) are the height-one primes of a unique factorization domain (UFD) (or equivalently, (a) where a is irreducible), the maximal ideal of an n-dimensional discrete valuation domain, or the maximal ideal of a special principal ideal ring (SPIR) such as $\mathbb{Z}/p^n\mathbb{Z}$. (Recall that a UFD is an atomic integral domain in which any two factorizations of a nonzero nonunit element into atoms are unique up to order and associates and a SPIR is a principal ideal ring with one nonzero prime ideal and that prime ideal is nilpotent.) In these examples, the principal prime ideals are either maximal or have height less than or equal to one. We first give an example to show that in a (quasilocal) domain a principal prime ideal can have arbitrary height and co-height.

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EXAMPLE 1. (A principal prime ideal (p) with $\mathrm{ht}(p) = n$ and $\mathrm{coht}(p) = m$ where $0 \leq n, m \leq \infty$.) Let $(V, (p))$ be an n-dimensional discrete valuation domain ($0 \leq n \leq \infty$). For $0 \leq m \leq \infty$, let $\bar{X} = \{X_1, \ldots, X_m\}$ be a set of indeterminates over V ($\bar{X} = \emptyset$ if $m = 0$ and $\bar{X} = \{X_1, X_2, \ldots\}$ if $m = \infty$). Then $pV[\bar{X}]$ is a principal prime ideal of $V[\bar{X}]$ with $\mathrm{ht}\, pV[\bar{X}] = n$ (as $V[\bar{X}]_{pV[\bar{X}]} = V(\bar{X})$ has $\dim V(\bar{X}) = \dim V = n$) and $\mathrm{coht}\, pV[\bar{X}] = \dim V[\bar{X}]/pV[\bar{X}] = \dim(V/(p))[\bar{X}] = m$. Thus in the quasilocal domain $D = V[\bar{X}]_{(p,\bar{X})}$ we have $\mathrm{ht}\, pD = n$ and $\mathrm{coht}\, pD = m$. (For the existence of an n-dimensional discrete valuation domain and the fact that $\dim V = \dim V(\bar{X})$, the reader may consult [4, Corollary 18.5] and [4, Proposition 18.7], respectively.)

It is well known that if (p) is a principal prime ideal with p regular (that is, not a zero divisor), then $J = \bigcap_{n=1}^{\infty}(p^n)$ is prime, $J = pJ$, and if Q is a prime ideal with $Q \subsetneq (p)$, then $Q \subseteq J$. Less well known is the following generalization [3, Corollary 2.3].

THEOREM 2. *Let R be a commutative ring and (p) a principal prime ideal of R with $\mathrm{ht}(p) \neq 0$ and set $J = \bigcap_{n=1}^{\infty}(p^n)$. Then:*

(1) *J is prime;*
(2) *$pJ = J$; and*
(3) *any prime ideal properly contained in (p) is contained in J.*

The above theorem is false if (p) is a minimal prime ideal. For example, in $\mathbb{Z}/(4)$, $\bigcap_{n=1}^{\infty}(\bar{2}^n)$ is not prime. However, in this example condition (2) still holds. In [3, Example 2.4] it is shown that condition (2) may also fail. Briefly, let k be a field and let $R = k[X, Z, Y_1, Y_2, \ldots]$ be the polynomial ring over k in indeterminates $X, Z, Y_1, Y_2, \ldots$. Let

$$A = (X - ZY_1, X - Z^2Y_2, X - Z^3Y_3, \ldots)$$

and put $\bar{R} = R/A$. Then $(\bar{X}, \bar{Z}) = (\bar{Z})$ is a minimal principal prime ideal of $\bar{R}$; but

$$\bar{X} \in \bigcap_{n=1}^{\infty}(\bar{Z}^n) - \bar{Z}\bigcap_{n=1}^{\infty}(\bar{Z}^n).$$

Suppose that (p) is a principal prime ideal of R with $\mathrm{ht}(p) \geq 1$ and let $J = \bigcap_{n=1}^{\infty}(p^n)$. Let P be a minimal prime ideal of R. If $x \in (0 : p)$, then $xp = 0 \in P$ and $p \notin P$ gives $x \in P$. Therefore, $(0 : p) \subseteq \sqrt{0} \subseteq J$ (since J is prime by Theorem 2). Thus if R is reduced, then p is regular. In the case where $\mathrm{ht}(p) \geq 1$, it is easy to characterize the ideals between (p) and J.

THEOREM 3. *Let (p) be a principal prime ideal of a commutative ring R with $\mathrm{ht}(p) \geq 1$ and let $J = \bigcap_{n=1}^{\infty}(p^n)$. For ideals $A \subseteq B$ of R, let*

$$[A, B] = \{K \text{ is an ideal of } R \mid A \subseteq K \subseteq B\}.$$

Then the map $\theta : [J, R] \to [J, (p)]$ given by $\theta(I) = pI$ is a complete lattice isomorphism.

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PROOF. First $\theta([J, R]) \subseteq [J, (p)]$ since $pJ = J$ by Theorem 2. If $B \in [J, (p)]$, then $J \subseteq B \subseteq (p)$ gives $B = p(B : p)$ where $(B : p) \supseteq J$, so θ is onto. By the remarks of the previous paragraph, $(0 : p) \subseteq J$. Therefore, if $pA = pB$ where $J \subseteq A$, B are ideals, then $A = A + (0 : p) = B + (0 : p) = B$; thus θ is one-to-one. Certainly θ preserves arbitrary sums. Finally, we show that if $\{A_\alpha\} \subseteq [J, R]$, then

$$\theta\Big(\bigcap A_\alpha\Big) = \bigcap \theta(A_\alpha) \quad \text{or} \quad p\Big(\bigcap A_\alpha\Big) = \bigcap pA_\alpha.$$

Therefore

$$\begin{aligned}\Big(\Big(p\bigcap A_\alpha\Big) : p\Big) &= \Big(\bigcap A_\alpha\Big) + (0 : p) = \bigcap A_\alpha = \bigcap (A_\alpha + (0 : p))\\ &= \bigcap (pA_\alpha : p) = \Big(\Big(\bigcap pA_\alpha\Big) : p\Big);\end{aligned}$$

hence

$$p\bigcap A_\alpha = p\Big(\Big(p\bigcap A_\alpha\Big) : p\Big) = p\Big(\Big(\bigcap pA_\alpha\Big) : p\Big) = \bigcap pA_\alpha.$$

This concludes the proof. □

The following theorem is well known, at least for integral domains, but we could not find a reference.

THEOREM 4. *Let (p) be a principal prime ideal in a commutative ring R. If* $\mathrm{ht}(p) \geq 1$, *then $\{(p^n)\}_{n=1}^{\infty}$ is the set of (p)-primary ideals. Suppose that* $\mathrm{ht}(p) = 0$ *and n is the least positive integer n with $(p^n)_{(p)} = 0_{(p)}$. Then $\{(p^m)\}_{m=1}^{n}$ is the set of (p)-primary ideals.*

PROOF. First, suppose that $\mathrm{ht}(p) \geq 1$, so $J = \bigcap_{n=1}^{\infty}(p^n) \subsetneq (p)$ is prime by Theorem 2. We show that (p^n) is (p)-primary for each $n \geq 1$. Since $J \subsetneq (p^n)$ is prime, we can pass to R/J and hence assume that R is an integral domain. Now $\sqrt{(p^n)} = (p)$. Suppose that $xy \in (p^n)$, but $x \notin (p)$. Suppose that $y \notin (p^n)$, say $y = ap^m$ where $0 \leq m < n$ and $a \notin (p)$. Then

$$xap^m = xy \in (p^n) \Rightarrow xa \in (p^{n-m}) \subseteq (p),$$

which is a contradiction since $x \notin (p)$ and $a \notin (p)$. Next, let Q be (p)-primary. Now $Q \not\subset J$, so there exists an $n \geq 1$ with $Q \subseteq (p^n)$ but $Q \nsubseteq (p^{n+1})$. Therefore, $Q = A(p^n)$ where $A \nsubseteq (p)$. Hence $(p^n) \subseteq Q$ so $Q = (p^n)$.

Next, suppose that $\mathrm{ht}(p) = 0$ and n is the least positive integer with $(p^n)_{(p)} = 0_{(p)}$. Then $(p) \supsetneq (p)^{(2)} \supsetneq \cdots \supsetneq (p)^{(n)}$ is the set of (p)-primary ideals where $(p)^{(s)} = (p^s)_{(p)} \cap R$. It suffices to show that $(p^s) = (p)^{(s)}$ for $1 \leq s \leq n$. Certainly we have $(p^s) \subseteq (p)^{(s)}$. If $(p)^{(s)} \subseteq (p^n)$, then $(p)^{(s)} \subseteq (p)^{(n)}$, so $s = n$ and $(p)^{(s)} = (p^n)$. Therefore, we can assume that $(p)^{(s)} \subseteq (p^t)$, but $(p)^{(s)} \nsubseteq (p^{t+1})$ for some $1 \leq t < n$, so $(p)^{(s)} \subseteq (p^t)A$ where $A \nsubseteq (p)$. Since $(p)^{(s)}$ is (p)-primary, $(p^t) \subseteq (p)^{(s)}$ and so $(p)^{(s)} = (p^t)$. Since $(p^s)_{(p)} = {(p)^{(s)}}_{(p)} = (p^t)_{(p)}$, we have $s = t$. □

We next give our main result. Recall that a DVR (a rank-one discrete valuation ring) is a principal ideal domain with exactly one nonzero prime ideal.

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THEOREM 5. *Let R be an atomic ring and let (p) be a principal prime ideal of R. Then* $\mathrm{ht}(p) \le 1$. *Moreover,* $\mathrm{ht}(p) = 1$ *if and only if p is regular and in this case $R_{(p)}$ is a DVR.*

PROOF. Suppose that p is a zero divisor. If $\mathrm{ht}(p) \ge 1$, there is a prime ideal $Q \subsetneq (p)$. Let $0 = a_1 \cdots a_n$, a product of atoms, so some $a_i \in Q$. Then $(a_i) \subseteq Q \subsetneq (p)$. Therefore, by [2, Theorem 1], p is regular; this is a contradiction. This implies $\mathrm{ht}(p) = 0$. In addition, of course, if $\mathrm{ht}(p) = 0$, then p is a zero divisor. (Note that when p is a zero divisor we have only used the fact that 0 is a product of atoms.) Next, suppose that p is not a zero divisor. As $\mathrm{ht}(p) \ge 1$, by Theorem 2, $Q = \bigcap_{n=1}^{\infty}(p^n)$ is a prime ideal and there are no prime ideals strictly between Q and (p). Let $a \in Q$ be irreducible. Then $(a) \subseteq Q \subsetneq (p)$, so $a = rp$ for some $r \in R$. Now as a is irreducible we must have $r \sim a$; say $r = r'a$. Then $a = r'pa$, so $(1 - r'p)a = 0$ and hence $(a)_{(p)} = 0_{(p)}$. Now let $b \in Q$. Since b is a product of atoms, there is an atom a with $(b) \subseteq (a) \subseteq Q$. Hence $(b)_{(p)} \subseteq (a)_{(p)} \subseteq 0_{(p)}$ and therefore $Q_{(p)} = 0_{(p)}$. Thus $R_{(p)}$ is a DVR and $\mathrm{ht}(p) = \mathrm{ht}(p)_{(p)} = 1$. □

Of course, Example 1 shows that Theorem 3 fails if R is not atomic. Also, Theorem 3 fails if we replace the principal prime p by an atom as seen by our next example.

EXAMPLE 6 [5, Exercise 8, p. 114]. Let k be a field and X, Y indeterminates over k. Let $R = k[Y][\{X^nY\}_{n \ge 1}]$. Then R is a bounded factorization domain (for each nonzero nonunit $f \in R$, there exists a positive integer $N(f)$ so that if $f = f_1 \cdots f_n$ where each f_i is a nonunit, then $n \le N(f)$), even a finite factorization domain (each nonzero nonunit of R has only finitely many (irreducible) factors, up to associates), and hence satisfies the ACCP and thus is atomic. However, $Y \in R$ is irreducible (but not prime) and $\mathrm{ht}(Y) = 2$.

What can we say about chains $(p_1) \subsetneq \cdots \subsetneq (p_n)$ of principal prime ideals? Certainly we can have maximal chains of length one or two as seen by taking $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}$. Now by [2, Theorem 1, Corollary 2] any chain of principal ideals $(a_1) \subsetneq \cdots \subsetneq (a_n)$ where a_i is irreducible has $n \le 2$ and if $n = 2$, then a_2 is regular and a_1 is a zero divisor. Since a principal prime is irreducible, any (maximal) chain of principal prime ideals has length at most two and for a chain of principal prime ideals $(p_1) \subsetneq (p_2)$, p_1 is a zero divisor and p_2 is regular. However, in the case of principal prime ideals, there is a simpler proof and more can be said. For suppose that $(p_1) \subsetneq (p_2)$ is a chain of principal prime ideals in a commutative ring R. (Note that the proof only uses the fact that p_2 is irreducible.) Since $p_1 = rp_2$ for some r, $p_2 \notin (p_1)$ and (p_1) is prime, we have $(p_1) = (p_1)(p_2)$. Let M be a maximal ideal of R containing (p_2). Then, in R_M, we have $(p_1)_M = (p_1)_M(p_2)_M$, so $(p_1)_M = (0)_M$. Thus R_M is a domain and $\mathrm{ht}(p_1) = \mathrm{ht}(p_1)_M = 0$, and hence p_1 is a zero divisor. Suppose that $sp_2 = 0$. If M is a maximal ideal of R with $M \supseteq (p_2)$, then $s/1 = 0/1$ in R_M since R_M is an integral domain. If $p_2 \notin M$, then $p_2/1$ is a unit in R_M, so $s/1 = 0/1$ in R_M. Thus $s = 0$ and p_2 is regular. Finally, what can be said if (p_1) is a maximal chain of principal prime

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ideals? Certainly we can have $\mathrm{ht}(p_1) = 0$ and p_1 can be a zero divisor. But consider a two-dimensional discrete valuation domain V with prime ideals $(\pi) \supsetneq P \supsetneq 0$. Then since $P^2 \neq P$ is P-primary, $\bar{\pi}$ is a regular element of $\bar{V} = V/P^2$, $\mathrm{ht}(\bar{\pi}) = 1$ and $(\bar{\pi})$ is a maximal chain of principal prime ideals. However, if we take an ideal Q of V with $P \supsetneq Q \supsetneq P^2$, then Q is not P-primary and $(\bar{\pi})$ is a height-one principal prime ideal of V/Q with $\bar{\pi}$ a zero divisor. In a similar manner, Example 1 can easily be modified to construct maximal chains (p) of principal prime ideals with p either regular (unless $\mathrm{ht}(p) = 0$) or a zero divisor with arbitrary height and co-height. We sum up these comments in the following theorem.

THEOREM 7. *Let R be a commutative ring. Then a chain $(p_1) \subsetneq \cdots \subsetneq (p_n)$ of principal prime ideals has $n = 1$ or 2. If $(p_1) \subsetneq (p_2)$ is a chain of principal prime ideals, then* $\mathrm{ht}(p_1) = 0$*, p_1 is a zero divisor and p_2 is regular.*

The principal prime ideals that are both minimal and maximal are easy to characterize. If $(R, (p))$ is a SPIR and S is any commutative ring, then $(p) \times S$ is a principal prime ideal of $R \times S$ that is both minimal and maximal. We next show that the converse is also true.

THEOREM 8. *Let R be a commutative ring. For a principal prime ideal (p) of R, the following are equivalent.*

(1) *(p) is both a minimal prime ideal and a maximal ideal.*
(2) *(p) is a maximal ideal and some power of (p) is idempotent.*
(3) *For some $n \geq 1$, $R/(p^n)$ is a SPIR and is a direct factor of R.*

In this case, there exists a positive integer n with $(p^n) = (p^m)$ for all $m \geq n$.

PROOF. (1) $\Rightarrow$ (2). Suppose that (p) is both a minimal prime ideal and a maximal ideal. Let M be a maximal ideal of R. If $M \neq (p)$, then $R_M = (p^n)_M$ for all $n \geq 1$. If $M = (p)$, then R_M is a SPIR and so $M_M^n = 0_M$ for some $n \geq 1$ and hence $M_M^n = M_M^m$ for $m \geq n$. Thus $(p^n) = (p^m)$ locally and hence globally for all $m \geq n$.

(2) $\Rightarrow$ (3). Suppose that (p^n) is idempotent, so $(p^n) = (p^{2n})$. Let $p^n = rp^{2n}$. Then $e = rp^n$ is idempotent and $(p^n) = (e)$. Then $R = Re \bigoplus R(1-e)$ and $R(1-e) \approx R/(p^n)$ is a SPIR.

(3) $\Rightarrow$ (1). This is clear. □

Of course all the prime ideals of a commutative ring R are principal if and only if R is a principal ideal ring. For suppose that R is not a principal ideal ring. Let I be a nonprincipal ideal of R. By Zorn's lemma, I can be enlarged to an ideal P maximal with respect to not being principal. However, by [5, Exercise 10, p. 8], an ideal P maximal with respect to not being principal is prime. Thus P is principal, which is a contradiction. Since any chain of principal prime ideals has length at most two, this result can be sharpened: a ring R is a principal ideal ring if and only if the maximal ideals and minimal prime ideals are principal. What can be said of a commutative ring

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if all its maximal ideals are principal? Of course valuation domains with principal maximal ideals abound. In fact, for a valuation domain (V, M) either M is principal or $M = M^2$. Loper [6] has given an example of a Prüfer domain (an integral domain in which every nonzero finitely generated ideal is invertible) whose maximal ideals are all principal but which is not a Bezout domain (an integral domain in which every finitely generated ideal is principal).

It is easy to see that a power series ring $R[[X]]$ has a principal maximal ideal if and only if R is a field. However, if D is a G-domain with quotient field k and nonunit a with $D[1/a] = k$, then $(aX - 1)$ is a principal maximal ideal of $D[X]$ since it is the kernel of the homomorphism $D[X] \to k$ given by $X \to 1/a$ [5, Exercise 2, p. 19]. Recall that an integral domain with quotient field k is a *G-domain* if $k = D[1/a]$ for some $a \in D$. This is equivalent to $0 = M \cap D$ for some maximal ideal M of $D[X]$ or to $\bigcap\{P \mid 0 \neq P,\ \text{a prime ideal of } D\} \neq 0$. A prime ideal P of a ring R is a *G-ideal* if R/P is a G-domain, or equivalently, $P = M \cap R$ for some maximal ideal of $R[X]$. See [5] for a treatment of G-domains. We next characterize the maximal ideals of a polynomial ring $R[X]$ that are principal.

THEOREM 9. *Let R be a commutative ring and $f = a_0 + a_1X + \cdots + a_nX^n \in R[X]$.*

(1) *Suppose that $M = (f)$ is a maximal ideal of $R[X]$. Then $P = M \cap R$ is a G-ideal, $\operatorname{ht} P = 0$, and $\operatorname{ht} M = 1$. Either $P = (a_0)$ is an idempotent maximal ideal of R or a_0 is a unit.*

(2) *If $M = (f)$ is a maximal ideal of $R[X]$ with $P = (a_0)$ idempotent, then $M = (e + (1 - e)X)$ where $e \in R$ is idempotent with $(a_0) = (e)$. Conversely, if $e \in R$ is idempotent with (e) a maximal ideal of R, then $(e + (1 - e)X)$ is a maximal ideal of $R[X]$.*

(3) *Suppose that $M = (f)$ is a maximal ideal of $R[X]$ where a_0 is a unit. Then for each prime ideal Q of R with $Q \supsetneq P = M \cap R$, we have $a_1, \ldots, a_n \in Q$. Conversely, if (f) is a prime ideal of $R[X]$ with a_0 a unit and $a_1, \ldots, a_n \in Q$ for each prime ideal Q of R with $Q \supsetneq P = (f) \cap R$, then (f) is a maximal ideal of $R[X]$.*

PROOF. (1) Since P is the contraction of a maximal ideal of $R[X]$, it is a G-ideal. Suppose that P is not minimal, say $P_0 \subsetneq P$. Pass to $\bar{R} = R/P_0$. Therefore, $(\bar{f})$ is a maximal ideal of the integral domain $\bar{R}[X]$ and $(\bar{f}) \cap \bar{R} = \bar{P} \cap \bar{R} \neq \bar{0}$. But $(\bar{f})$ maximal gives $\deg \bar{f} \geq 1$ so $(\bar{f}) \cap \bar{R} = \bar{0}$, which is a contradiction. Since $\operatorname{ht} P = 0$, we have $\operatorname{ht}(f) = 1$ since in a polynomial ring a chain of three prime ideals can not have the same contraction to R. Since M is maximal, $(f, X) = (M, X) = (f)$ or $R[X]$. First, suppose that $(f, X) = (f)$. Then $(f) = (a_0, X)$, so $P = (f) \cap R = (a_0, X) \cap R = (a_0)$. Note that, since (a_0, X) is a maximal ideal of $R[X]$, (a_0) must be a maximal ideal of R. Now $a_0R[X] \subseteq (f)$ and $a_0R[X]$ is prime, so

$$a_0R[X] = a_0R[X](f) = a_0(a_0, X) = (a_0^2, a_0X).$$

Hence $(a_0) = (a_0^2, a_0X) \cap R = (a_0^2)$. Next, suppose that $(f, X) = R[X]$. Then $R[X] = (f, X) = (a_0, X)$, so a_0 is a unit.

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(2) Suppose $P = (a_0)$ is idempotent, so $(a_0) = (e)$ for some idempotent $e \in R$. Then

$$(f) = (a_0, X) = (e, X) = (e + (1 - e)X).$$

Conversely, if (e) is a maximal ideal of R with e idempotent, then (e, X) is a maximal ideal of $R[X]$ and $(e, X) = (e + (1 - e)X)$.

(3) Suppose that $M = (f)$ is a maximal ideal of $R[X]$ with a_0 a unit. Let $Q \supsetneq P = (f) \cap R$ be a prime ideal of R. Since $Q[X] \nsubseteq (f)$, we have $(f) + Q[X] = R[X]$. Therefore, $(\bar{f}) = \bar{R}[X]$ for $\bar{R} = R/Q$. Thus $\bar{a_1}, \dots, \bar{a_n} = \bar{0}$ in $\bar{R}$ or $a_1, \dots, a_n \in Q$. (Of course, this condition is vacuous if there are no such prime ideals Q.) Conversely, suppose that (f) is a prime ideal of $R[X]$ where a_0 is a unit and $a_1, \dots, a_n \in Q$ for each prime ideal $Q \supsetneq P = (f) \cap R$. Suppose that (f) is not a maximal ideal; say $(f) \subsetneq N$, a maximal ideal. Then $N \cap R \supsetneq P$ since $(f) \cap R = P[X] \cap R = P$, so $a_1, \dots, a_n \in N \cap R \subseteq N$. Thus $a_0 \in N$ where a_0 is a unit, which is a contradiction. □

We end by characterizing the commutative rings in which all minimal prime ideals are principal.

THEOREM 10. *For a commutative ring R the following conditions are equivalent.*

(1) *All the minimal prime ideals of R are principal.*
(2) *$\sqrt{0}$ is a finite intersection of principal prime ideals.*
(3) *$\sqrt{0}$ is a finite product of principal prime ideals.*
(4) *0 is a finite product of principal prime ideals.*

Thus in this case R has only finitely many minimal prime ideals.

PROOF. (1) ⇒ (2). By [1, Theorem], if all the minimal prime ideals are finitely generated, R has only finitely many minimal prime ideals. Since $\sqrt{0}$ is the intersection of the minimal prime ideals of R, the result follows.

(2) ⇒ (3). Suppose that $\sqrt{0} = (p_1) \cap \cdots \cap (p_n)$, a finite intersection of principal prime ideals. We can assume that each (p_i) is a minimal prime ideal. Let $x \in (p_1) \cap \cdots \cap (p_n)$. Then $x = rp_1$ for some $r \in R$, so $rp_1 \in (p_2) \cap \cdots \cap (p_n)$ gives $r \in (p_2) \cap \cdots \cap (p_n)$. Continuing we get $x = r' p_1 \cdots p_n$ for some $r' \in R$. Thus

$$(p_1) \cap \cdots \cap (p_n) \subseteq (p_1) \cdots (p_n) \subseteq (p_1) \cap \cdots \cap (p_n).$$

(3) ⇒ (4). This is clear.

(4) ⇒ (1). This is clear. □

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