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A Compactness Theorem for Yang-Mills Connections

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Abstract. In this paper, we consider Yang-Mills connections on a vector bundle E over a compact Riemannian manifold M of dimension $m > 4$, and we show that any set of Yang-Mills connections with the uniformly bounded $L^{\frac{m}{2}}$-norm of curvature is compact in C^∞ topology.

1 Introduction

Let M be an m-dimensional manifold with a Riemannian metric g, and E be a vector bundle over M with a compact Lie group G as its structure group. A connection A of E can be given by specifying a covariant derivative

$$D_A\colon C^\infty(E) \to C^\infty(E \otimes \Omega^1 M).$$

In the local trivialization of E, D_A is of the form $d + \alpha$ for some Lie(G)-valued 1-form α. The curvature of A is a Lie(G)-valued 2-form F_A, which is equal to D_A^2. As usual, it measures deviation from the symmetry of second derivatives. Such a connection A is Yang-Mills if it is a critical point of the Yang-Mills action. A Yang-Mills connection A satisfies the Euler-Lagrange equation $D_A^* F_A = 0$. By the second Bianchi identity, we also have $D_A F_A = 0$. The system $D_A^* F_A = 0$, $D_A F_A = 0$ is called the Yang-Mills equation and is invariant under gauge transformations.

In the analytical aspect of the Yang-Mills theory, one of the most fundamental results is K. Uhlenbeck's compactness theorem on the modulo space ([1, 2]). The modulo space of Yang-Mills connections is the quotient of the set of solutions of the Yang-Mills equation by the gauge group, which consists of all gauge transformations. It is well-known that this modulo space may not be compact. Given any sequence of Yang-Mills connections $\{A_i\}$ with a uniformly bounded L^2-norm of curvature, Uhlenbeck ([1]) (see also [3]) proved that by taking a subsequence if necessary, A_i converges, modulo gauge transformations, to a Yang-Mills connection A in the smooth topology outside a closed subset $S_b(\{A_i\})$ of Hausdorff codimension at least 4. If M is a 4-dimensional compact manifold, the blow-up locus consists of finitely many points, and the limiting connection A can be extended to be a Yang-Mills connection on the whole manifold with smaller L^2-norm of curvature [1]. With M of higher dimension, G. Tian [4] studied the geometric structures of the blow-up loci of Yang-Mills connections and introduced a natural compactification for modulo space of

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anti-self-dual instantons on higher dimensional manifolds by adding cycles with appropriate geometric structure. He also proved a removable singularity theorem for any stationary Yang-Mills connections. Particularly, this implies that the limiting connection A extends to become a smooth connection on $M \setminus S$ for a closed subset S with vanishing $(n-4)$-dimensional Hausdorff measure $H^{n-4}(S) = 0$.

In this paper we consider the compactness property of sequences of Yang-Mills connections A_i with a uniformly bounded $L^{\frac{m}{2}}$-norm of curvature. We note that the $L^{\frac{m}{2}}$-norm of curvature is conformally invariant, while the L^2-norm is not, unless $m = 4$. Our result is the following.

Main Theorem *Let E be a vector bundle over compact Riemannian manifold M of dimension $m > 4$, and $\{A_i\}$ is a sequence of smooth Yang-Mills connections on E with $\int_M |F_{A_i}|^{\frac{m}{2}}\, dV_g \leq \Lambda$; then there is a subsequence $\{A_\alpha\}$ and gauge transformations σ_α, such that $\sigma_\alpha(A_\alpha)$ converges to a smooth Yang-Mills connection A in C^∞-topology on M.*

In the proof of the Main Theorem, the main tool which will be used is the local curvature estimate of Yang-Mills connections. First, we will show that there exists a subsequence $\{A_\alpha\} \subset \{A_i\}$ (modulo gauge transformations) converging to a Yang-Mills connection A in smooth topology outside at most finite points. Secondly, we will use a removable singularity theorem which had been proved by L. M. Sibner[6] to deduce that there is a gauge transformation σ such that $\sigma(A)$ extends to be a smooth connection on M. Furthermore, by taking subsequence if necessary, we may assume that $|F_{A_i}|^{\frac{m}{2}} dV_g$ converges (as measure) weakly to $|F_A|^{\frac{m}{2}} dV_g + \sum_{j=1}^{J} \Theta_{P_j} \delta_{P_i}$ for some constants Θ_{P_j}, where we set $\Sigma = \{P_j\}_{j=1}^{J}$, $P_j \in M$ and δ_{P_i} denotes the Dirac measure. Proceeding as in [4], we will construct bubbling connections on R^m as A_α approach A. On the other hand, by the monotonicity formula of P. Price, we can prove a non-existence theorem for Yang-Mills connections which will show that if bubbling connections do not exist, then the blow up set Σ must be empty. So the subsequence A_α (modulo gauge transformations) converges to a smooth Yang-Mills connection A in C^∞ topology on M.

2 Preliminary Results

As before, M denotes a Riemannian manifold with a metric g and E is a vector bundle over M with compact structure group G. A connection A on E is defined by specifying a covariant derivative

$$D = D_A \colon C^\infty(E) \to C^\infty(E \otimes \Omega^1 M),$$

where $C^\infty(E)$ denotes the space of C^∞ sections of the bundle E. In a local trivialization $(U_\alpha, \varphi_\alpha)$ of E, the covariant derivative takes the form

$$D = d + A_\alpha, A_\alpha \colon U_\alpha \to T^* U_\alpha \otimes Lie(G)$$

where Lie(G) denotes the Lie algebra of the structure group G. Note that A_α usually has no global description on M.

For any connection A of E, its curvature form F_A is determined by $D^2\colon \Omega^0(E) \to \Omega^2(E)$. It is a tensor, usually denoted by F_A or simply F if no confusion occurs. Formally, the curvature tensor F_A can be written as

$$F_A = dA + A \wedge A,$$

which actually means that in each local trivialization $(U_\alpha, \varphi_\alpha)$,

$$F_A = dA_\alpha + A_\alpha \wedge A_\alpha. \tag{2.1}$$

The norm of F_A at any $P \in M$ is given by

$$|F_A|^2 = \sum_{i,j=1}^{n} \left\langle F_A(e_i, e_j), F_A(e_i, e_j) \right\rangle,$$

where $\{e_i\}$ is any orthonormal basis of $T_P M$, and $\langle \cdot , \cdot \rangle$ is the Killing form of the Lie algebra Lie(G).

The Yang-Mills functional of E is defined by

$$YM(A) = \frac{1}{4\pi^2} \int_M |F_A|^2 \, dV_g. \tag{2.2}$$

If A is a critical point of YM, then we say the A is a Yang-Mills connection. The Euler-Lagrange of YM is

$$D_A^* F_A = 0, \tag{2.3}$$

where D_A^* denotes the adjoint operator of D_A with respect to the Killing form of lie(G) and the Riemannian metric g on M. On the other hand, by the second Bianchi identity, we have

$$D_A F_A = 0. \tag{2.4}$$

This, together with (2.4), is called the Yang-Mills equation.

Let G be the gauge group of E, which consists of all smooth sections of the bundle $P(E) \times_{Ad} G$ associated to the adjoint representation Ad of G, where $P(E)$ denotes the principal bundle of E. Any σ in G is called a gauge transformation. Two smooth connections A_1 and A_2 of E are equivalent if there is a gauge transformation σ such that $A_2 = \sigma(A_1)$, where $\sigma(A)$ is the connection with $D_{\sigma(A)} = \sigma \cdot D_A \cdot \sigma^{-1}$. One can easily show $YM(\sigma(A)) = YM(A)$. Then, if A is a Yang-Mills connection, so is $\sigma(A)$ for any gauge transformation σ. In other words, the Yang-Mills equation is invariant under the action of the gauge group.

Let $\{\phi_t\}_{|t|<\infty}$ be a one-parameter family of diffeomorphisms of M, A_0 a fixed smooth connection of E and D its associated covariant derivative. Then for any connection A, we can define a family of connections $A^t = \phi_t^*(A)$ as follows: In [4] (or [5]) Tian proved the following formula:

$$\frac{d}{dt} YM(A^t)|_{t=0} = -\frac{1}{4\pi^2} \int_M \Big(|F_A|^2 \operatorname{div} X - 4 \sum_{i,j=1}^{m} \left\langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) \right\rangle \Big) \, dV_g. \tag{2.5}$$

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Now suppose that A is a Yang-Mills connection; then

$$0 = \int_M \Big(|F_A|^2 \operatorname{div} X - 4 \sum_{i,j=1}^{m} \big\langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) \big\rangle \Big) \, dV_g. \tag{2.6}$$

By this variation formula, one can derive the following monotonicity.

Theorem 2.1 (A Monotonicity Formula) *There exist constants r_P, a depending only on M, such that for any $0 < \rho < \gamma < r_P$, we have*

$$\begin{aligned} \gamma^{4-m} \exp a\gamma^2 \int_{B(P,\gamma)} |F_A|^2 \, dV_g - \rho^{4-m} \exp a\rho^2 \int_{B(P,\rho)} |F_A|^2 \, dV_g \\ \geq 4 \int_{B(P,\gamma)\setminus B(P,\rho)} r^{4-m} \exp(ar^2) \Big| \frac{\partial}{\partial r} \lrcorner F_A \Big|^2 \, dV_g. \end{aligned} \tag{2.7}$$

Moreover, if $M = R^m$ and g is flat, then the equality holds in (2.7) *for $\rho \in (0, \infty)$ and $a = 0$.*

In the following, we give a basic curvature estimate for Yang-Mills connections. This estimate was first derived by K. Uhlenbeck [1] (also see [4]). Since it is crucial to us here, we will outline its proof for the reader's convenience.

Theorem 2.2 *Let A be any Yang-Mills connection of a G-bundle E over M. Then there are $\epsilon = \epsilon(m)$ and $C = C(m)$, which depend only on m and M, such that for any $P \in M$ and $\rho < r_p$, whenever*

$$\int_{B(P,\rho)} |F_A|^{\frac{m}{2}} \, dV_g \leq \epsilon,$$

then

$$\begin{aligned} \sup_{B(P,\frac{\rho}{2})} |F_A|^2 &\leq \frac{C}{\rho^4} \Big(\int_{B(P,\rho)} |F_A|^{\frac{m}{2}} \, dV_g \Big)^{\frac{4}{m}} \\ &\leq \frac{C}{\rho^4} \cdot \epsilon^{\frac{4}{m}}. \end{aligned}$$

In order to compactify the modulo space of Yang-Mills connections, we need to use singular Yang-Mills connections of a certain type. An admissible Yang-Mills connection ([4]) is a smooth connection A defined outside a closed subset $S(A)$ in M, such that

(1) $H^{n-4}(S(A) \cap K) < \infty$ for any compact subset $K \subset M$, where $H^{n-4}(\cdot)$ stands for the (n-4)-dimensional Hausdorff measure;
(2) A is Yang-Mills on $M \setminus S(A)$;
(3) A satisfies $\int_{M \setminus S(A)} |F_A|^2 \, dV_g < \infty$.

Clearly, A is smooth on M if $S(A) = \varnothing$. We will call $S(A)$ the singular set of A. This is not invariant under gauge transformations. Even if $S(A) \neq \varnothing$, there may be a gauge transformation σ on $M \setminus S(A)$ such that $\sigma(A)$ extends to become a smooth connection on M.

Furthermore, an admissible Yang-Mills connection A is called stationary if A satisfies

$$0 = \int_M \Big(|F_A|^2 \operatorname{div} X - 4 \sum_{i,j=1}^{m} \big\langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) \big\rangle \Big)\, dV_g,$$

for any vector field X, where $\{e_i\}$ is any orthonormal basis of M. If A is a smooth Yang-Mills connection, this follows from the first variation formula for Yang-Mills action.

Proposition 2.3 *Let $m = \dim M > 4$ and S be a discrete set in M. If A is a Yang-Mills connection on $M \setminus S$ and satisfies $\int_K |F_A|^{\frac{m}{2}}\, dV_g < \infty$ for each compact set $K \subset M$; then A is stationary and the monotonicity formula* (2.7) *still holds on M.*

Proof Denote

$$\Phi(X) = -\frac{1}{4\pi^2} \int_M \Big(|F_A|^2 \operatorname{div} X - 4 \sum_{i,j=1}^{m} \big\langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) \big\rangle \Big)\, dV_g,$$

where X is a variation vector field with compact support set and $\{e_i\}$ is an orthonormal frame of TM.

We may assume that S consists of a single point P. For $r > 0$ we take a cut-off function $\eta_r \in C_0^\infty(M)$ satisfying $0 \leq \eta_r \leq 1$, $|\nabla \eta_r| \leq \frac{2}{r}$ in M and $\eta_r(x) = 1$, if $x \in B(P, r)$; $\eta_r(x) = 0$, if $x \in M \setminus B(P, 2r)$. Since A is Yang-Mills on $M \setminus B(a, r)$ for any $r > 0$, we have $\Phi(X - \eta_r X) = 0$ for any $r > 0$. Thus, we have

$$\begin{aligned}
|\Phi(X)| = |\Phi(\eta_r X)| &\leq C \int_M |F_A|^2 (\eta_r |\nabla X| + |\nabla \eta_r||X|)\, dV_g \\
&\leq C\Big(\int_{B(P,2r)} |F_A|^2 |\nabla X|\, dV_g + \frac{1}{r} \int_{B(P,2r)} |F_A|^2 |X|\, dV_g \Big) \\
&\leq \Big(r^{m-4} \sup_M |\nabla X| + r^{m-5} \sup_M |X| \Big) \Big(\int_{B(P,2r)} |F_A|^{\frac{m}{2}}\, dV_g \Big)^{\frac{4}{m}}.
\end{aligned}$$

By conditions the right-hand side tends to 0 as $r \to 0$. Hence, we get $\Phi(X) = 0$ for any X. This shows that A is stationary on M.

Theorem 2.4 ([6]) *Let A be a Yang-Mills connection stationary on $M \setminus S$, where S is a discrete set. If $\int_K |F_A|^{\frac{m}{2}}\, dV_g < \infty$ for each compact set $K \subset M$, then there exists a gauge transformation σ such that $\sigma^*(A)$ can be extended to be a smooth Yang-Mills connection on M.*

3 Proof of the Main Theorem

Theorem 3.1 *Let $\{A_i\}$ be a sequence of smooth Yang-Mills connections on E with $\int_M |F_A|^{\frac{m}{2}}\, dv_g \le \Lambda$; then there exists a subsequence $\{\alpha\} \subset \{i\}$ and a (possibly empty) finite set $\Sigma = \{P_j\}_{j=1}^J$ of M satisfying the following:*

(1) *the subsequence A_α converge to a smooth Yang-Mills connection A in the C^∞-topology on $M \setminus \Sigma$.*

(2) *for each $j = 1, \dots, J$, there exists constants $\theta_j > 0$ such that*

$$|F_{A_\alpha}|^{\frac{m}{2}}\, dV_g \longrightarrow |F_A|^{\frac{m}{2}}\, dV_g + \sum_{j=1}^{J} \theta_j \cdot \delta_{P_j} \tag{3.1}$$

weakly in the sense of Radon measures on M.

Here δ_{P_j} denotes Dirac measure.

Proof Let ϵ be as in Theorem 2.2. We define a closed subset for each i and $r > 0$;

$$E_{i,r} = \Big\{ x \in M \;\Big|\; \int_{B_r(x)} |F_{A_i}|^{\frac{m}{2}}\, dV_g \ge \epsilon \Big\}. \tag{3.2}$$

It is obvious that $E_{i,r} \subset E_{i,R}$ for any $r \le R$. By the standard diagonal process, we can choose a subsequence $\{i_j\}$ of $\{i\}$ such that for each k, the $E_{i_j,2^{-k}}$ converge to a closed subset $E_{2^{-k}}$. Then $E_{2^{-k}} \subset E_{2^{-l}}$ for $k \ge l$.

Put $S = \bigcap_k E_{2^{-k}}$. We first claim that S is at most a finite set. We fixed an arbitrary compact set $K \subset int(M)$. For any $\delta > 0$ sufficiently small, let $\{B_{4\delta}(x_\alpha)\}$ be any finite covering of $S \cap K$ such that $x_\alpha \in S \cap K$; $B_{2\delta}(x_\alpha) \cap B_{2\delta}(x_\beta) = \varnothing$ for $\alpha \ne \beta$. Take k big enough such that $2^{-k} < \delta$. Then for j sufficiently large, there are $y_\alpha \in E_{i_j,2^{-k}}$ such that $d(x_\alpha, y_\alpha) < \delta$. Then $\{B_{5\delta}(y_\alpha)\}$ is a finite covering of $S \cap K$ and $B_\delta(y_\alpha) \cap B_\delta(y_\beta) = \varnothing$ for $\alpha \ne \beta$. On the other hand, for each α

$$\int_{B_\delta(y_\alpha)} |F_{A_{i_j}}|^{\frac{m}{2}}\, dV_g \ge \epsilon. \tag{3.3}$$

Summing up, we get

$$I \le \frac{1}{\epsilon} \sum_{\alpha=1}^{I} \int_{B_\delta(y_\alpha)} |F_{A_{i_j}}|^{\frac{m}{2}}\, dV_g \le \frac{\Lambda}{\epsilon}. \tag{3.4}$$

This shows $\mathtt{H}^0(S \cap K) \le \Lambda/\epsilon$ where $\mathtt{H}^0$ denotes the 0-dimensional Hausdorff measure on M. Since the 0-dimensional Hausdorff measure coincides with the counting measure, $S \cap K$ is at most finite. Since K is an arbitrary compact set and the the right-hand side of the above inequality is independent of K, then S is at most finite.

Now we prove that A_{i_j} converges to outside S modulo gauge transformations. To save the notation, we assume $\{i_j\} = \{i\}$. We notice that for any $r > 0$, there is $i(r) > 0$, $k(r) > 0$, such that for any $i \geq i(r)$ and $x \in M \setminus B_r(S)$ we have:

$$\int_{B_{2^{-k}}(x)} |F_{A_i}|^{\frac{m}{2}} \, dV_g < \epsilon. \tag{3.5}$$

This is equivalent to saying that $x \in M \setminus E_{i,2^{-k}}$. By Theorem 2.2, we deduce from the above inequality that for any $x \in M \setminus B_r(S)$,

$$|F_{A_i}|(x) < C \cdot 2^{2k(r)} \cdot \epsilon^{\frac{2}{m}}.$$

It follows from Theorem 3.6 in [2] that there exists a subsequence $\{\tilde{i}\} \subset \{i\}$ and gauge transformations $\sigma(\tilde{i})$, such that $\sigma(\tilde{i})(A_{\tilde{i}})$ converge to a smooth connection A in C^1-topology on any compact subset outside S. Since A_i are Yang-Mills connections, by the standard elliptic theory, A is a Yang-Mills connection and $\sigma(\tilde{i})(A_{\tilde{i}})$ converge to A smoothly outside S. Using Fatou's lemma we have

$$\int_M |F_A|^{\frac{m}{2}} \, dV_g \leq \liminf_{\tilde{i} \to \infty} \int_M |F_{A_{\tilde{i}}}|^{\frac{m}{2}} \, dV_g \leq \Lambda. \tag{3.6}$$

By Theorem 2.4, there exists a gauge transformation σ such that $\sigma(A)$ extends to a smooth connection on M.

In the following, we always assume that the sequence A_i converges to a smooth Yang-Mills connection A in C^∞-topology outside S with $\int_M |F_A|^{\frac{m}{2}} \, dV_g \leq \Lambda$.

Define

$$\Sigma(\{A_i\}) = \bigcap_{r>0} \Big\{ x \in int(M) | \liminf_{i \to \infty} \int_{B(x,r)} |F_{A_i}|^{\frac{m}{2}} \, dV_g \geq \epsilon \Big\}. \tag{3.7}$$

Now we want to show that $\Sigma(\{A_i\})$ is contained in the above S. In fact, for any $x_0 \in M \setminus S$, if r is sufficiently small,

$$\int_{B(x_0,r)} |F_A|^{\frac{m}{2}} \, dV_g < \epsilon.$$

This implies that for i sufficiently large,

$$\int_{B(x_0,r)} |F_{A_i}|^{\frac{m}{2}} \, dV_g < \epsilon.$$

Hence, $x_0 \in M \setminus \Sigma(\{A_i\})$. This shows that $\Sigma(\{A_i\}) \subset S$.

Suppose $x_0 \in S \setminus \Sigma(\{A_i\})$; then there is an $r_0 > 0$ such that

$$\int_{B(x_0,r_0)} |F_{n_i}|^{\frac{m}{2}} \, dV_g < \epsilon$$

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for some subsequence $n_i \to \infty$. By Theorem 2.2,

$$\sup_{x \in B(x_0, \frac{1}{2} r_0)} |F_{n_i}| \le C_0 \cdot r_0^2 \cdot \epsilon^{\frac{2}{m}}$$

for some constant $C_0 = C_0(m, M)$ and all n_i. This implies that A is a limit of some subsequence of $\{A_{n_i}\}$ (modulo gauge transformations) in $B(x_0, \frac{1}{2} r_0)$ in the C^∞ topology. Then, there exists a subsequence $\{A_\alpha\} \subset \{A_i\}$ and a finite set $\Sigma = \Sigma(A_\alpha)$ such that A_α(modulo gauge transformations) converges to A in the C^∞ topology on $M \setminus \Sigma$.

Consider the Radon measure $\mu_\alpha = |F_\alpha|^{\frac{m}{2}} dV_g$. By taking a subsequence if necessary, we may assume that $\mu_\alpha \to \mu$ weakly on M as Radon measures. Let us write (by Fatou's lemma)

$$\mu = |F_A|^{\frac{m}{2}} dV_g + \nu \tag{3.8}$$

for some nonnegative Radon measure ν on M. Since $\{A_\alpha\}$ converges to A in the C^∞ topology on $M \setminus \Sigma$, the support of measure ν is contained in the discrete set Σ. Thus, we have $\nu = \sum_{j=1}^J \theta_j \delta_{P_j}$ for some $\theta_j \ge 0$ where we set $\Sigma = \Sigma(\{A_\alpha\}) = \{P_j\}_{j=1}^J$.

We show each θ_j is positive. Fix any P_j. For arbitrarily small $r > 0$, we take a cut-off function $\eta_r \in C_0^\infty$ satisfying $0 \le \eta_r \le 1$ in M and $\eta_r(x) = 1$ if $x \in B(P_j, r)$; $\eta_r(x) = 0$ if $x \in M \setminus B(P_j, 2r)$. By definition of Σ we have

$$\epsilon \le \liminf_{\alpha \to \infty} \int_{B(P_j, r)} |F_{A_\alpha}|^\alpha \, dV_g \le \lim_{\alpha \to \infty} \int_M \eta_r |F_{A_\alpha}|^{\frac{m}{2}} \, dV_g \le \theta_j + \int_{B(P_j, 2r)} |F_A|^{\frac{m}{2}} \, dV_g. \tag{3.9}$$

Letting $r \to 0$, we obtain $\theta_j \ge \epsilon > 0$. This completes the proof. ■

Theorem 3.2 *Let $\{A_\alpha\}$, Σ be as in Theorem* 3.1 *and $P \in \Sigma$. Then there are linear transformations $\sigma_\alpha \colon T_P M \to T_P M$ such that a subsequence of $\sigma_\alpha^* \exp_P^* A_\alpha$ converges smoothly to a Yang-Mills connection B on $(T_P M, g_{P,0})$; and satisfying $F_B \ne 0$ and $\int_{T_P M} |F_B|^{\frac{m}{2}} \, dx \le \theta_P$; where θ_P is determined in Theorem* 3.1.

Proof We take a normal coordinate neighborhood $B(P, 2R)$ of P and a normal coordinate system x of M centered at P. Choose $R > 0$ small enough so that $\Sigma \cap B(P, 2R) = \{P\}$. Let $B(x, r)$ be the open ball in the x-coordinates with center x and radius r and let $B(r) = B(0, r)$. Defining the concentration function

$$Y_\alpha(t) = \sup_{y \in B(R)} \int_{B_y(t)} |F_\alpha|^{\frac{m}{2}} \, dV_g \tag{3.10}$$

for any $0 \le t < R$. Each function Y_α is continuous and non-decreasing in t, and $Y_\alpha(0) = 0$. By the definition of Σ

$$Y_\alpha(R) \ge \int_{B(R)} |F_\alpha|^{\frac{m}{2}} \, dV_g \ge \frac{7\epsilon}{8} \tag{3.11}$$

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holds for sufficiently large α. Here, the constant ϵ is taken as in Theorem 2.2. By continuity of Y_α, there exist $0 < r_\alpha < R$ and $x_\alpha \in \overline{B(R)}$ such that

$$Y_\alpha(r_\alpha) = \int_{B(\exp_P(x_\alpha), r_\alpha)} |F_\alpha|^{\frac{m}{2}} \, dV_g = \frac{\epsilon}{2}.$$

Since the P is a unique point in $\Sigma \cap B(P, 2R)$, we obtain $r_\alpha \to 0$, $x_\alpha \to P$, as $\alpha \to \infty$. Defining linear transformations $\sigma_\alpha(x) = x_\alpha - r_\alpha \cdot x$ on $T_P M$, let $U(\alpha) = B(\frac{x_\alpha}{r_\alpha}, \frac{2R}{r_\alpha}) \subset T_P M$. It is easy to see that $B(2R) = \sigma_\alpha(U(\alpha))$. Since x_α lies in $B(\frac{R}{2})$ for sufficiently large α, we have $B(\frac{R}{r_\alpha}) \subset U(\alpha)$, which leads to $U(\alpha) \to T_P M$ as $\alpha \to \infty$.

We set $B_\alpha = \sigma_\alpha^* \exp_P^*(A_\alpha)$. We can easily see B_α is a Yang-Mills connection on $(U(\alpha), g_\alpha)$, where the metric $g_\alpha = r_\alpha^{-2} \sigma_\alpha^* \exp_P^* g$. Note that the based manifolds $(T_P M, g_\alpha)$ converge to $(T_P M, g_{P,0}) \cong R^m$ as $\alpha \to \infty$. By the definition of B_α, x_α, r_α, we have

$$\int_{U(\alpha)} |F_{B_\alpha}|^{\frac{m}{2}} \, dV_{g_\alpha} = \int_{B(P,2R)} |F_{A_\alpha}|^{\frac{m}{2}} \, dV_g \leq \Lambda.$$

$$Y_\alpha(r_\alpha) = \int_{B(1)} |F_{B_\alpha}|^{\frac{m}{2}} \, dV_{g_\alpha} = \sup_{z \in \sigma_\alpha^{-1}(B(R))} \int_{B(z,1)} |F_{B_\alpha}|^{\frac{m}{2}} \, dV_{g_\alpha} = \frac{\epsilon}{2}. \tag{3.12}$$

The constant ϵ in Theorem 2.2 may depend on the metric in general, but by the definition of g_α we are able to take the constant ϵ independent of α. In fact, the positive numbers ϵ and C in Theorem 2.2 ([1]) depend only on the bound of sectional curvature of metrics. Since $g_\alpha \to g_{P,0}$ in C^∞ topology as $\alpha \to \infty$, we can conclude that the sectional curvature of g_α are uniformly bounded on $B(1)$, so we can take the constants ϵ and C independent of α. Using Theorem 2.2, we have

$$\sup_{B(z,\frac{1}{2})} |F_{B_\alpha}| \leq C_1 \epsilon^{\frac{2}{m}}$$

for any $z \in \sigma_\alpha^{-1}(B(R))$, here C_1 is a constant independent of α. Note that

$$\sigma_\alpha^{-1}(B(R)) \to T_P M$$

as $\alpha \to \infty$. It follows from Theorem 3.6 in [2] that there exists a subsequence $\{\beta\} \subset \{\alpha\}$ and gauge transformations $\tau(\beta)$, such that $\tau(\beta)(B_\beta)$ converge to a smooth connection B in C^1-topology on any compact subset of $T_P M$. Since B_α is a Yang-Mills connection, and g_α converges to the flat metric $g_{P,0}$ on $T_P M$, by the standard elliptic theory, B is a Yang-Mills connection on $(T_P M, g_{P,0})$ and $\tau(\beta)(B_\beta)$ converge to B smoothly. Passing to the limit in (3.12), we have

$$\int_{B(1)} |F_B|^{\frac{m}{2}} \, dx = \frac{\epsilon}{2}.$$

This shows that $F_B \neq 0$. By Fatou's lemma, we have

$$\int_{T_P M} |F_B|^{\frac{m}{2}} \, dx \leq \liminf_{\beta \to \infty} \int_{U(\beta)} |F_{B_\beta}|^{\frac{m}{2}} \, dV_{g_\beta}, \leq \theta_P + \int_{B(P,2R)} |A|^{\frac{m}{2}} \, dV_g.$$

Letting $R \to 0$, we have

$$\int_{T_P M} |F_B|^{\frac{m}{2}}\, dx \le \theta_P.$$

This completes the proof. ■

Theorem 3.3 *If B is a Yang-Mills connection on R^m ($m \ge 5$) and satisfying*

$$\int_{R^m} |F_B|^{\frac{m}{2}}\, dx < \infty,$$

then $F_B \equiv 0$.

Proof Suppose to the contrary that $F_B \ne 0$. Then, there exists $r > 0$ such that

$$\Delta = r^{2-m} \cdot \int_{B(r)} |F_B|^2\, dx > 0.$$

From the monotonicity formula we have

$$\Delta \le t^{2-m} \int_{B(t)} |F_B|^2\, dx$$

for any $t \ge r$. Thus, we have

$$\Delta \le t^{2-m} \Big(\int_{B(s)} |F_B|^2\, dx + \int_{B(t) \setminus B(s)} |F_B|^2\, dx \Big), \tag{3.13}$$

for any $s \le t$. Using the Hölder inequality we obtain

$$\Delta \le t^{2-m} \int_{B(s)} |F_B|^2\, dx + c(m) \Big(\int_{R^m \setminus B(s)} |F_B|^{\frac{m}{2}}\, x \Big)^{\frac{4}{m}}. \tag{3.14}$$

Since $\int_{R^m} |F_B|^{\frac{m}{2}}\, dx < \infty$, we may take s large enough to satisfy

$$c(m) \Big(\int_{R^m \setminus B(s)} |F_B|^{\frac{m}{2}}\, x \Big)^{\frac{4}{m}} \le \frac{\Delta}{4}.$$

Fixing such s, we may take $t > s$ large enough to satisfy

$$t^{2-m} \int_{B(s)} |F_B|^2\, dx \le \frac{\Delta}{4}.$$

Thus, we have $0 < \Delta \le \frac{\Delta}{4} + \frac{\Delta}{4} = \frac{\Delta}{2}$, which makes a contradiction. This completes the proof. ■

From Theorem 3.2 and Theorem 3.3 we obtain that the finite subset Σ in Theorem 3.1 is empty. Then, the subsequence A_α (modulo gauge transformations) converges to a smooth Yang-Mills connection A in the C^∞-topology on M. This completes the proof of Main Theorem.

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References

[1] K. K.Uhlenbeck, *Removable singularities in Yang-Mills fields.* Comm. Math. Phys. **83**(1982), 11–29.
[2] ———, *Connections with L^p bounds on curvature.* Comm. Math. Phys. **83**(1982), 31–42.
[3] H. Nakajima, *Compactness of the moduli space of Yang-Mills connections in higher dimensions.* J. Math. Soc. Japan **40**(1988), 383–392.
[4] G. Tian, *Gauge theory and calibrated geometry.* Ann. Math. **151**(2000), 193–208.
[5] P. Price, *A monotonicity formula for Yang-Mills fields.* Manuscripta Math. **43**(1983), 131–166.
[6] L. M. Sibner, *The isolated point singularity problem for the coupled Yang-Mills eqaution in higher dimensions.* Math. Ann. **271**(1985), 125–131.

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