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STRICT TOPOLOGIES FOR VECTOR-VALUED FUNCTIONS

ROBERT A. FONTENOT

This paper is motivated by work in two fields, the theory of strict topologies and topological measure theory. In [**1**], R. C. Buck began the study of the strict topology for the algebra $C^*(S)$ of continuous, bounded real-valued functions on a locally compact Hausdorff space S and showed that the topological vector space $C^*(S)$ with the strict topology has many of the same topological vector space properties as $C_0(S)$, the sup norm algebra of continuous real-valued functions vanishing at infinity. Buck showed that as a class, the algebras $C^*(S)$ for S locally compact and $C^*(X)$, for X compact, were very much alike. Many papers on the strict topology for $C^*(S)$, where S is locally compact, followed Buck's; e.g., see [**2**; **3**]. In [**22**], J. Wells extended some of Buck's work to $C^*(S : E)$, the bounded, continuous functions from the locally compact space S into the locally convex space E. Buck's work was then generalized to the case where X is completely regular (for scalar-valued functions); e.g., see [**5**; **6**; **17**; **18**; **20**].

In [**5**] newly defined "strict" topologies were shown to be connected with the field of topological measure theory. One of the classic papers in topological measure theory is [**21**]. Many papers followed Varadarajan's, in an attempt to answer questions raised in [**21**]; e.g. see [**8-13**].

In [**5**] and [**17**] it is shown that functional analytic, measure theoretic, and order theoretic techniques can, with skillful blending, lead to a deeper understanding of both topological measure theory and strict topologies. The author's work arose as the result of attempts to generalize topological measure theory to vector-valued measures and to extend the notion of strict topology to the spaces $C^*(X : E)$ where X is completely regular and E is a normed linear space. Some of the most interesting results, in the author's opinion, are 2.3, 3.2, 3.7, 3.12, and 3.13.

1. Preliminaries. We first need to develop some measure theory. A good reference for this is [**21**]. Let X denote a completely regular topological space. The *Baire algebra* of X, denoted $B_a{}^*(X)$ is the smallest algebra of subsets of X containing the zero-sets of functions in $C^*(X)$. We use $B_a(X)$ to denote the smallest σ-algebra containing the zero-sets. In this paper, $C^*(X)$ *always means bounded real-valued continuous functions* and *all linear spaces* considered are *real linear spaces.* A *positive Baire measure* μ on X is a finite, non-negative real

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valued, finitely-additive set function on $B_a^*(X)$ so that $A \in B_a^*(X) \Rightarrow \mu(A) = \sup\{\mu(Z) : Z \subseteq A, Z$ a zero set of $X\}$. A *Baire measure* is the difference of two positive Baire measures. The collection of all Baire measures and positive Baire measures are denoted $M(X)$ and $M^+(X)$ respectively. If m is a Baire measure, the set functions $m^+(A) = \sup\{m(B) : B \subseteq A, B \in B_a^*(X)\}$, for $A \in B_a^*(x)$, and $m^-(A) = -\inf\{m(B) : B \in B_a^*(X)$ and $B \subseteq A\}$, for $A \in B_a^*(X)$, are elements of $M^+(X)$ and $m = m^+ - m^-$. Let $|m| = m^+ + m$. Then $|m| \in M^+(X)$ and is called the *absolute value* of the Baire measure m. $M(X)$ with the norm $||m|| = m^+(X) + m^-(X)$ is a Banach space. There is an equivalent definition of $M(X)$ that is sometimes useful. Let m be a finitely-additive set function on $B_a^*(X)$. Then $m \in M(X)$ if and only if (1) $|m(A)| \leqq C$ for some $C > 0$ and all $A \in B_a^*(X)$ and (2) for any $A \in B_a^*(X)$ and $\epsilon > 0$, there is a zero-set $Z \subseteq A$ so that $|m(B)| < \epsilon$ for all $B \subseteq A \backslash Z$.

The Banach adjoint $C^*(X)'$ of $C^*(X)$ can be identified with $M(X)$. If $\Phi \in C^*(X)'$, there is a unique Baire measure $m \in M(X)$ such that $\Phi(f) = \int f dm$ for $f \in C^*(X)$. Conversely, if Φ is defined by the preceding formula for $m \in M(X)$, then $\Phi \in C^*(X)'$. Furthermore, $||\Phi|| = ||m||$. The correspondence is a vector space homomorphism and preserves order, that is, Φ is a *positive linear functional* ($\Phi(f) \geqq 0$ for $f \geqq 0$ in $C^*(X)$) if and only if $m \in M^+(X)$ [**21**, Theorem 6]).

We shall be particularly interested in three classes of measures on X. A Baire measure m is said to be *σ-additive* if $m(Z_n) \to 0$ for every sequence $\{Z_n\}_{n=1}^\infty$ of zero-sets of X such that $Z_{n+1} \subseteq Z_n$ for all n and $\bigcap_{n=1}^\infty Z_n = \phi$ (we denote this by $Z_n \downarrow \phi$). A measure $m \in M(X)$ is called *τ-additive* if $m(Z_a) \to 0$ for every net $\{Z_\alpha\}$ of zero-sets of X such that $Z_\alpha \subseteq Z_\beta$ for $\alpha \geqq \beta$ and $\bigcap_\alpha Z_\alpha = \phi$ (we denote this by $Z_\alpha \downarrow \phi$). The measure $m \in M(X)$ is called *tight* if for every $\epsilon > 0$, there exists a compact set $K_\epsilon \subseteq X$ so that $|m|_*(X \backslash K_\epsilon) < \epsilon$, where for $E \subseteq X$,

$$|m|_*(E) = \sup\{|m|(Z) : Z \text{ is a zero-set of } X \text{ and } Z \subseteq E\}.$$

If $\Phi \in C^*(X)'$, Φ is called *σ-additive* if $\Phi(f_n) \to 0$ for every sequence $\{f_n\}_{n=1}^\infty$ in $C^*(X)$ such that $f_{n+1} \leqq f_n$ for all n and $f_n \to 0$ pointwise on X (we denote this by $f_n \downarrow 0$). The functional $\Phi \in C^*(X)'$ is called *τ-additive* if $\Phi(f_\alpha) \to 0$ for every net $\{f_\alpha\} \subseteq C^*(X)$ such that $f_\alpha \leqq f_\beta$ for $\alpha \geqq \beta$ and $f_\alpha \to 0$ pointwise on X (we denote this by $f_\alpha \downarrow 0$). Finally, $\Phi \in C^*(X)'$ is called *tight* if $\Phi(f_\alpha) \to 0$ for every net $\{f_\alpha\}$ contained in the unit ball of $C^*(X)$ such that $f_\alpha \to 0$ uniformly on compact subsets of X.

In [**21**], it is shown that if $\Phi \in C^*(X)'$ and $m \in M(X)$ such that $\Phi(f) = \int f dm$, then Φ is σ-additive (τ-additive, tight) if and only if m is σ-additive (τ-additive, tight). Identifying functionals and the corresponding Baire measures, we denote the class of σ-additive, τ-additive, and tight functionals (σ-additive, τ-additive, and tight Baire measures) by $M_\sigma(X)$, $M_\tau(X)$, and $M_t(X)$, respectively. Note that $M_t(X) \subset M_\tau(X) \subset M_\sigma(X)$.

One of the big problems of topological measure theory is to determine when

$M_\tau(X) = M_\sigma(X)$. This problem is first mentioned in [**21**] and has been studied by many authors. Some good references for the interested reader are [**5**; **8-13**; **17**; **21**; **23**].

F. D. Sentilles and others have extended Buck's *strict topology* β [**1**] to $C^*(X)$ for X completely regular (instead of the more restrictive requirement that X be locally compact); a partial list of references is [**5**; **6**; **17**; **18**; **19**; **20**].

Connections were established between these new "strict" topologies and some aspects of topological measure theory in the work of Sentilles and that of Fremlin, Garling and Haydon. We shall describe some of the work of Sentilles as we are more familiar with his paper that with that of Fremlin *et al.*

The topology β_0 on $C^*(X)$ is defined to be the finest locally convex linear topology agreeing with the compact-open topology on norm bounded sets. Let βX denote the *Stone-Čech compactification* of X [**4**] and if $f \in C^*(X)$, let $\bar{f}$ denote the unique continuous extension of f to βX. For each compact set $Q \subseteq \beta X \backslash X$, let $C_Q(X) = \{ f \in C^*(X) : \bar{f} \equiv 0 \text{ on } Q\}$. Let β_Q be the topology on $C^*(X)$ defined by the seminorms $f \rightarrow ||fh||$ for $f \in C^*(X)$ and $h \in C_Q(X)$. Let β be the intersection of the topologies β_Q, where Q varies through all compact sets in $\beta X \backslash X$. If we instead allow Q to vary through the zero-sets (of continuous functions defined on βX) contained in $\beta X \backslash X$, the topology is called β_1. Let p denote the topology of pointwise convergence on X and $C - Op$ that of uniform convergence on compact subsets of X and $|| \quad ||$ the norm topology on $C^*(X)$. If T is any topology let $C^*(X)_T$ denote $C^*(X)$ equipped with the topology T.

Sentilles [**17**] makes a very important contribution when he calculates the adjoint spaces of $C^*(X)$ endowed with the topologies β_0, β, β_1. It is this result which allows him to use the interplay between topological measure theory techniques and functional analytic techniques to obtain a deeper understanding of both topological measure theory and his strict topologies. Sentilles shows that $C^*(X)_{\beta_0}' = M_t(X)$, $C^*(X)_{\beta_1}' = M_\sigma(X)$ and $C^*(X)_{\beta}' = M_\tau(X)$ and that $M_\sigma(X) = M_\tau(X)$ if and only if $\beta_1 = \beta$. He also proves many other interesting results which we will list as needed.

In the rest of this paper we extend many of the above mentioned results to vector-valued functions.

In what follows E will always denote a real normed linear space (in most of the results, if not all of them, E could be any locally convex space, but we feel that notation is made simpler by restricting ourselves to this case). Let X denote a completely regular topological space and let $C^*(X : E)$ denote the set of all bounded continuous functions from X to E. $C^*(X : E)$ is a real linear space.

We define the topology β_0 on $C^*(X : E)$ to be the finest locally convex linear topology agreeing with the compact-open topology on norm bounded sets. For Q a compact subset of $\beta X/X$, the topology β_Q on $C^*(X : E)$ is that topology defined by the seminorms $f \rightarrow ||hf||$, where $h \in C_Q(X)$ and $f \in C^*(X : E)$. Then β_1 and β are defined as the intersection of topologies β_Q, exactly as in the scalar case. If T is a topology, by $C^*(X : E)_T$ we mean the space $C^*(X : E)$ with the topology T.

Note that we may restrict ourselves to nonnegative functions in C_Q in defining the β_Q seminorms and the resulting topology in β_Q. Also note that if X is locally compact, then $\beta = \beta_0$ and is the topology defined for $C^*(X : E)$ by Buck in [**1**] and studied in [**22**]. That β_0 coincides with Buck's topology in the case X is locally compact follows from [**18**].

1.1 *Remark.* Let W_1 denote the topology on $C^*(X : E)$ given by the seminorms $f \to ||hf||$ for $f \in C^*(X : E)$ where h is a nonnegative real-valued function such that $\{x \in X : h(x) \geqq \epsilon\}$ is compact for all $\epsilon > 0$. Then $\beta_0 = W_1$. Sentilles notes a similar result in the case E is the reals and his proof goes through for arbitrary E. See [**17**, Theorem 2.4].

We also need several results of a somewhat different character. Let A be a Banach algebra with norm $||\ \ ||$. A Banach space V, with a norm also denoted $||\ \ ||$, is called a *left A-module* if there is a mapping from AxV into V, whose value at the pair (a, v) in $A \times V$ is denoted $a \cdot v$, satisfying the conditions that $a \cdot v$ is linear in a for fixed v and linear in v for fixed a and $(ab) \cdot v = a \cdot (b \cdot v)$ for $a, b \in A$ and $v \in V$. The left A-module V is said to be *isometric* if $||a \cdot v|| \leqq ||a||\ ||v||$ for all $a \in A$ and $v \in V$. A net $\{e_\alpha\} \subseteq A$ is an *approximate identity* for A if $||e_\alpha|| \leqq 1$ for all α and

$$||e_\alpha a - a|| \underset{\alpha}{\to} 0 \quad \text{and} \quad ||ae_\alpha - a|| \underset{\alpha}{\to} 0$$

for all $a \in A$. Suppose that $\{e_\alpha\}$ is an approximate identity for A and V is a left A-module. Then V is called essential (the term is introduced in [**15**]) if $||e_\alpha \cdot v - v|| \to 0$ for every $v \in V$. The following theorem holds [**7**; **15**; **24**]:

1.2 THEOREM. *Let A be a Banach algebra having an approximate identity and V a Banach space which is an isometric left A-module. Then V is essential if and only if for all $\epsilon > 0$ and $v \in V$, there exists $a \in A$ and $w \in V$ such that $||a|| \leqq 1$, $||v - w|| < \epsilon$ and $a \cdot w = v$.*

2. The strict topologies β and β_1 for vector-valued functions. As in Section 1, let E be a real normed linear space, X a completely regular space, and $C^*(X : E)$ denote the real linear space of bounded continuous functions from X to E. When no other topology is explicitly mentioned $C^*(X : E)$ is assumed given the norm topology. We denote the norm dual of $C^*(X : E)$ by $C^*(X : E)'$. If T is any other topology on $C^*(X : E)$, $C^*(X : E)_T'$ denotes the dual of $C^*(X : E)$ with the topology T.

2.1 *Definition.* Let $\Phi \in C^*(X : E)'$. Then Φ is said to be *σ-additive* if for every sequence $\{f_n\} \subseteq C^*(X)$ such that $f_n \downarrow 0$, $\Phi(f_n g) \to$ uniformly for g in $C^*(X : E)$ of norm $\leqq 1$. Similarly, Φ is said to be τ-additive if, whenever $\{f_\alpha\}$ is a net in $C^*(X)$ such that $f_\alpha \downarrow 0$, then $\Phi(f_\alpha g) \to 0$ uniformly for g in $C^*(X : E)$ of norm $\leqq 1$.

2.2 *Remark.* The definitions in 2.1 generalize the usual ones. In order to see this, we need only show that if $\phi \in C^*(X)'$ and ϕ is a positive linear functional, then ϕ is σ-additive (τ-additive) in the sense of [**21**] implies it is σ-additive (τ-additive) in the sense of 2.1. This follows immediately from the Cauchy-Schwarz inequality for positive linear functionals [**14**, III, p. 187].

2.3 THEOREM. *Let* $\phi \in C^*(X:E)'$. *Then*
(a) ϕ *is* σ*-additive if and only if* ϕ *is* β_1 *continuous on* $C^*(X:E)$;
(b) ϕ *is* τ*-additive if and only if* ϕ *is* β *continuous on* $C^*(X:E)$.

Proof. (a) Suppose that ϕ is σ-additive. We wish to show that $\phi \in (C^*(X:E)_{\beta_1})'$. It clearly suffices to show that $\phi \in (C^*(X:E)_{\beta Q})'$ for an arbitrary zero-set Q contained in $\beta X \backslash X$. Let Q be a zero-set of βX such that $Q \subseteq \beta X \backslash X$. Since Q is a compact G_δ, $C_Q(X)$ has a countable approximate identity $\{h_n\}_{n=1}^{\infty}$ satisfying $0 \leqq h_n \leqq 1$ $\forall n$ and $1 - h_n \downarrow 0$ on X. Thus $\phi((1-e_n)g) \rightarrow 0$ uniformly for g in $C^*(X:E)'$. $C^*(X:E)'$ is a left $C_Q(X)$-module in a natural way, i.e., if $g \in C_Q(X)$ and $\phi \in C^*(X:E)'$, $g \cdot \phi(h) = \phi(gh)$ for all $h \in C^*(X:E)$. With this notation we have shown that $||e_n \cdot \phi - \phi|| \rightarrow 0$. Thus

$$\phi \in W = \{p : p \in C^*(X:E)' \text{ and } ||e_n \cdot p - p|| \rightarrow 0\}.$$

Clearly W is a Banach space and an essential left $C_Q(X)$-module in the language of Section 1. By 1.2, if $p \in W$, $p = a \cdot q$ where $a \in C_Q(X)$ and $q \in W$. Clearly then $W \subseteq (C^*(X:E)_{\beta Q})'$. Thus $\phi \in (C^*X:E)_{\beta Q})'$ for each compact zero-set $Q \subseteq \beta X \backslash X$; hence $\phi \in (C^*(X:E)_{\beta_1})'$.

Conversely, suppose that ϕ is β_1 continuous, $||\phi|| \leqq 1$, $\epsilon > 0$, and $\{f_n\} \subseteq C^*(X)$ such that $||f_n|| \leqq 1$ for all n and $f_n \downarrow 0$ on X. For $f \in C^*(X)$, let $\bar{f}$ denote the unique continuous extension of f to βX. Let

$$K = \bigcap_{n=1}^{\infty} \left\{t \in \beta X : \bar{f}_n(t) \geqslant \frac{\epsilon}{2}\right\}.$$

Then K is a compact nonempty subset of $\beta X \backslash X$ and K is a countable intersection of zero sets; hence K is a zero-set of βX. Since $\phi \in C^*(X:E)_{\beta_K}{}'$ there is by 1.2, a function $0 \leqq h \leqq 1$ in $C_K(X)$ and $\psi \in C^*(X:E)'$ with $||\psi|| \leqq 2$ so that $\phi = h \cdot \psi$. Thus $|\phi(f)| = |\psi(hf)| \leqq 2\,||hf||$ for all $f \in C^*(X:E)$. Let $O = \{t \in \beta X : \bar{h}(f) < \epsilon/2\}$. Then O is open, $K \subseteq O$, and $\beta X \backslash O$ is compact. Since $O \subseteq K$, there is an integer N so that $\{t \in \beta X : \bar{f}_n(t) \geqq \epsilon/2\} \subseteq O$ for $n \geqq$ N. Then if $g \in C^*(X:E)$ and $||g|| \leqq 1$,

$$|\phi(f_n g)| \leqq 2\,||hf_n g|| \leqq 2\,||hf_n|| < \epsilon$$

for $n > N$. Hence $\phi(f_n g) \rightarrow 0$ uniformly for g of norm $\leqq 1$ in $C^*(X:E)$, i.e., ϕ is σ-additive.

(b) The proof of this equivalence is similar to that given in (a) and so is omitted.

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3. The topology β_0 on $C^*(X:E)$. In this section we characterize the dual space $C^*(X:E)_{\beta_0}'$, show that $C^*(X:E)_{\beta_0}$ has the approximation property if E has the metric approximation property, and give a vector-valued measure representation for elements of $C^*(X:E)_{\beta_0}'$, generalizing a theorem in [**22**]. We also extend 2.3.

3.1 *Definition.* Let $F \in C^*(X:E)'$. Then F is said to be *tight* if $F(g_\alpha) \to 0$ for every net $\{g_\alpha\} \subset C^*(X:E)$ such that $||g_\alpha|| \leqq 1$ for all α and $g_\alpha \to 0$ uniformly on compact subsets of X.

3.2 Theorem. *Let $F \in C^*(X:E)'$. The following statements are equivalent:*
(1) $F \in C^*(X:E)_{\beta_0}'$;
(2) *F is tight;*
(3) *The real linear functional T on $C^*(X)$ defined for $f \geqq 0$ in $C^*(X)$ by the equation*

$$T(f) = \sup\{|F(g)| : ||g(x)|| \leqq f(x), \textit{ for all } x \in X,\ g \in C^*(X:E)\}$$

and extended by linearity to all of $C^*(X)$ is tight;
(4) *if $\epsilon > 0$, there exists compact $K_\epsilon \subseteq X$ so that if $f \in C^*(X:E)$ and $||f|| \leqq 1$, then $f \equiv 0$ on K_ϵ implies that $|F(f)| < \epsilon$;*
(5) *$F(f_\alpha g) \to 0$ uniformly for $g \in C^*(X:E)$ of norm $\leqq 1$, for every net $\{f_\alpha\} \subseteq C^*(X)$ such that $||f_\alpha|| \leqq 1$ for all α and $f_\alpha \to 0$ uniformly on compact subsets of X.*

Proof. (1) $\Rightarrow$ (2). Suppose F is β_0 continuous and $g_\alpha \to 0$ uniformly on compact subsets of X. Since β_0 agrees with the compact-open topology on norm bounded subsets of $C^*(X:E)$, $g_\alpha \to 0\ \beta_0$; hence $F(g_\alpha) \to 0$.

(2) $\Rightarrow$ (1). Since β_0 is defined as the finest locally convex linear topology agreeing with the compact-open topology on norm bounded sets a linear functional F on $C^*(X:E)$ is β_0 continuous if and only if its restriction to norm bounded subsets of $C^*(X:E)$ is continuous in the compact-open topology, i.e., if and only if F is tight.

(5) $\Rightarrow$ (3). For $f \geqq 0$ in $C^*(X)$ define $T(f) = \sup\{|F(g)| : ||g(x)|| \leqq f(x)$, for $x \in X\}$. We first want to show that we can extend T to a real linear functional on $C^*(X)$. In order to establish this, all we need to show is that $T(f+g) = T(f) + T(g)$, for $f, g \geqq 0$ in $C^*(X)$.

Let $h \in C^*(X:E)$ so that $||h|| \leqq f + g$. For $x \in X$ such that $f(x) + g(x) > 0$, define

$$h_1(x) = \frac{f(x)h(x)}{f(x)+g(x)} \quad \text{and} \quad h_2(x) = \frac{g(x)h(x)}{f(x)+g(x)}.$$

If $f(x) + g(x) = 0$, let $h_1(x) = h_2(x) = 0$. Note that h_1 and $h_2 \in C^*(X:E)$ and $||h_1|| \leqq f$ and $||h_2|| \leqq g$. Thus $|F(h)| = |F(h_1) + F(h_2)| \leqq T(f) + T(g)$. Taking the supremum over all such functions h, we get $T(f+g) \leqq T(f) + T(g)$. For the other inequality, let $\epsilon > 0$ and $h_1, h_2 \in C^*(X:E)$ with $||h_1|| \leqq f$,

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$||h_2|| \leqq g$ and $0 \leqq F(h_1) \leqq T(f) \leqq F(h_1) + \epsilon/2$ and $0 \leqq F(h_2) \leqq T(g) \leqq F(h_2) + \epsilon/2$. Then $T(f) + T(g) \leqq F(h_1) + F(h_2) + \epsilon = F(h_1 + h_2) + \epsilon \leqq T(f + g) + \epsilon$. Since $\epsilon > 0$ is arbitrary we get $T(f) + T(g) \leqq T(f + g)$. Hence $T(f + g) = T(f) + T(g)$ and so T extends to a linear functional on $C^*(X)$ which is bounded.

We now show T is tight. Suppose $\{f_\alpha\} \subseteq C^*(X)$, $||f_\alpha|| \leqq 1$ for all α, and $f_\alpha \to 0$ $C - Op$. We need to show $T(f_\alpha) \to 0$ so we may assume that $f_\alpha \geqq 0$ for all α.

We first show that it suffices to show that $T(f_\alpha{}^r) \to 0$ for all positive $r > 1$. Assume, for simplicity, that $||F|| \leqq 1$. Then $0 \leqq T(f) \leqq ||f||$ for all $f \geqq 0$ in $C^*(X)$. From elementary calculus, we get that

$$0 \leqslant \sup_{0 \leqslant t \leqslant 1} t - t^r \leqslant 1 - \frac{1}{r}$$

for $r > 1$. Hence $0 \leqq T(f_\alpha - f_\alpha{}^r) \leqq ||f_\alpha - f_\alpha{}^r|| \leqq 1 - 1/r$ for $r > 1$, and if $T(f_\alpha{}^r) \to 0$ for each $r > 1$, then $T(f_\alpha) \to 0$.

It remains to be shown that $T(f_\alpha{}^r) \to 0$ for each real number $r > 1$. Fix $r > 1$, let $\epsilon > 0$ and pick α_0 such that $\alpha \geqq \alpha_0$ implies $|F(f_\alpha h)| < \epsilon$ for all $h \in C^*(X : E)$ such that $||h|| \leqq 1$. If $g \in C^*(X : E)$, $\alpha \geqq \alpha_0$, and $||g|| \leqq f_\alpha{}^r$, let $h(x) = g(x)/f_\alpha(x)$ if $f_\alpha(x) \neq 0$ and $h(x) = 0$ otherwise. Then $|F(g)| = |F(f_\alpha h)| < \epsilon$. Hence $T(f_\alpha{}^r) \leqq \epsilon$ for $\alpha \geqq \alpha_0$. Thus $T(f_\alpha{}^r) \to 0$ for each $r > 1$.

(1) $\Rightarrow$ (4). Suppose F is β_0 continuous and $\epsilon > 0$. By 1.1, there is a bounded nonnegative upper semicontinuous function g which vanishes at infinity such that $|F(f)| \leqq ||gf||$ for all $f \in C^*(X : E)$. If $||f|| \leqq 1$ and $f \equiv 0$ on $K_\epsilon = \{x \in X : g(x) \geqq \epsilon\}$, then $|F(f) < \epsilon$.

(4) $\Rightarrow$ (5). Suppose that (4) holds and that $\{f_\alpha\} \subseteq C^*(X)$, $||f_\alpha|| \leqq 1$ and $f_\alpha \to 0$ $C - Op$. We want to show that $F(f_\alpha g) \to 0$ uniformly for g in $C^*(X : E)$ of norm $\leqq 1$. Clearly, we may assume that $f_\alpha \geqq 0$ for all α and that $||F|| \leqq 1$. Let $\epsilon > 0$ and let K_ϵ be the compact subset of X given by (4). Choose α_0 so that $\alpha \geqq \alpha_0$ implies $||f_\alpha||_{K_\epsilon} < \epsilon$. Let $h_\alpha = \min\{f_\alpha, \epsilon\}$. Then if $g \in C^*(X : E)$ and $||g|| \leqq 1$, $|F(f_\alpha g - h_\alpha g)| < \epsilon$ for $\alpha \geqq \alpha_0$ since $f_\alpha - h_\alpha = 0$ on K_ϵ. Thus, for $\alpha \geqq \alpha_0$,

$$|F(f_\alpha g)| \leqq |F(f_\alpha g)| + |F(h_\alpha g)| < 2\epsilon.$$

Hence $F(f_\alpha g) \to 0$ uniformly for g in the unit ball of $C^*(X : E)$.

(3) $\Rightarrow$ (1). Suppose that T is tight. Then, by Sentilles' result $M_t = C^*(X)_{\beta_0}'$ and 1.1, there exists a bounded nonnegative upper semicontinuous function h vanishing at infinity such that $||T(g)|| \leqq ||hg||$ for all $g \in C^*(X)$. Let $f \in C^*(X : E)$. Then $|F(f)| \leqq T(||f||) \leqq || \, ||f|| h|| = ||hf||$; therefore F is β_0 continuous by 1.1 again. We have shown (2) $\Rightarrow$ (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (1), so the proof of 3.2 is complete.

3.3 *Remark*. We have the following improvement of 2.3, whose proof is clear if we look at 2.3 along with the proof of (5) $\Rightarrow$ (3) in 3.2 and make the observation that (3) $\Rightarrow$ (5) in 3.2 is trivial (although we did not prove 3.2 this way).

3.4 THEOREM. *Let* $F \in C^*(X : E)'$.

(a) *The following are equivalent:* (1) *F is σ-additive;* (2) *F is β_1 continuous;* (3) *if T is defined in terms of F as in* 3.2, *T is σ-additive.*

(b) *The following are equivalent:* (1) *F is τ-additive;* (2) *F is β continuous;* (3) *if T is defined in terms of F as in* 3.2, *T is τ-additive.*

The next topic we take up is the approximation problem in $C^*(X : E)_{\beta_0}$ where we generalize a result in [**5**]. A topological vector space H is said to have the approximation property if the identity operator can be uniformly approximated by continuous finite-rank operators on all totally bounded subsets of H. If H is a normed space and the approximating finite-rank operators can be chosen with norms $\leqq 1$, then H is said to have the *metric approximation property*. The proof of the following lemma is contained in [**5**, Theorem 10].

3.5 LEMMA. *Let X be a completely regular space, C a compact subset of X, K a compact subset of $C^*(X)_{\beta_0}$ and $\epsilon > 0$. Then there exists a finite partition of unity* (see 3.6) *$\{g_i\}_{i=1}^n$ on X and points $\{c_i| \ 1 \leqq i \leqq n\}$ in C so that if P is the linear operator on $C^*(X)$ defined by the equation*

$$Pf(x) = \sum_{i=1}^{n} g_i(x)f(c_i),$$

then P is β_0 continuous, $||P|| \leqq 1$, P is of finite rank and $\sup_{x\in C}||Pf(x) - f(x)|| < \epsilon$ *for $f \in K$.*

3.6 *Definition.* Let X be a completely regular space and $\{f_\alpha\} \subseteq C^*(X)$ such that $0 \leqq f_\alpha \leqq 1$ for each α. The family $\{f_\alpha\}$ is called a *partition of unity* on X if the supports of the f_α form a locally finite cover of X and $\sum f_\alpha = 1$ on X. If there is a covering A of X so that the support of f_α is a subset of α for each $\alpha \in A$, then $\{f_\alpha\}$ is called a *partition of unity subordinate to A.*

3.7 THEOREM. *Let E be a normed linear space with the metric approximation property and X a completely regular Hausdorff space. Then $C^*(X : E)_{\beta_0}$ has the approximation property.*

Proof. Let $\epsilon > 0$, J a β_0-totally bounded subset of $C^*(X : E)$ and h a non-negative bounded upper semicontinuous function on X which vanishes at infinity such that $||h|| \leqq 1$. Since J is norm bounded, let us assume that J is a subset of the unit ball in $C^*(X : E)$.

Let $C = \{x : h(x) \geqq \epsilon/2\}$. Then C is compact. Note that $D = \{f(x) : f \in J, x \in C\}$ is a totally bounded subset of E. Hence there is a finite rank operator T on E, with $||T|| \leqq 1$, such that $||T(d) - d|| < \epsilon/2$ for all $d \in D$. Since T is finite-rank, there is a finite set $\{\varphi_i : 1 \leqq i \leqq n\} \subseteq E'$ and a finite set $\{e_i : 1 \leqq i \leqq n\} \subseteq E$ so that $||e_i|| \leqq 1$, $1 \leqq i \leqq n$, and $T(e) = \sum_{i=1}^n \varphi_i(e)e_i$ for $e \in E$. Thus

$$T(f(x)) = \sum_{i=1}^{n} (\varphi_i \circ f)(x)e_i$$

for $f \in C^*(X : E)$. Since the set $\{\varphi_i \circ f : 1 \leqq i \leqq n, f \in J\}$ is a β_0 totally bounded subset of $C^*(X)$, there is, by 3.5, a finite-rank operator P on $C^*(X)$ with $||P|| \leqq 1$, such that P is continuous for the β_0 topology on $C^*(X)$ and such that $||P(\varphi_i \circ f) - (\varphi_i \circ f)||_C < \epsilon/2n$ for $f \in J$ and $1 \leqq i \leqq n$. Furthermore, we may assume P is given by a formula such as that in 3.5. Let S be the linear operator on $C^*(X : E)$ defined for $f \in C^*(X : E)$ by the equation

$$Sf(x) = \sum_{i=1}^{n} P(\varphi_i \circ f)(x)e_i \quad \text{for all } x \in X.$$

Note that S is β_0 continuous and of finite rank. In order to compute $||S||$, we write P more explicitly. As in 3.5, let $\{g_j | 1 \leqq j \leqq m\}$ be a partition of unity on X and $\{c_j : 1 \leqq j \leqq m\} \subseteq C$ so that $Pf(x) = \sum_{j=1}^{m} g_j(x)f(c_j)$ for all $f \in C^*(X)$. If $f \in C^*(X : E)$ and $x \in X$, then

$$\begin{aligned} Sf(x) = \sum_{i=1}^{n} P(\varphi_i \circ f)(x)e_i &= \sum_{i=1}^{n} \sum_{j=1}^{m} g_j(x)(\varphi_i \circ f(c_j))e_i \\ &= \sum_{j=1}^{m} g_j(x) \left(\sum_{i=1}^{n} \varphi_i \circ f(c_j)e_i \right) = \sum_{j=1}^{m} g_j(x)T(f(c_j)). \end{aligned}$$

Thus

$$||Sf(x)|| \leqslant \max_{1 \leqslant j \leqslant m} ||T(f(c_j))|| \leqslant ||f||$$

and so $||S|| \leqq 1$.

What remains to be shown is that $||h(Sf - f)|| < \epsilon$ for $f \in Q$, If $x \in X \backslash C$, $h(x) < \epsilon/2$ so that

$$||h(x)(Sf(x) - f(x))|| < (\epsilon/2)(2||f||) < \epsilon.$$

If $x \in C$, then

$$\begin{aligned} ||Sf(x) - f(x)|| \leqslant &\left\| \sum_{i=1}^{n} P(\varphi_i \circ f)(x)e_i - \sum_{i=1}^{n} \varphi_i \circ f(x)e_i \right\| \\ &+ ||T(f(x)) - f(x)|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus, if $x \in C$, $||h(x)(Sf - f)(x)|| < \epsilon$ since $||h|| \leqq 1$. Hence $||h(Sf - f)|| < \epsilon$ for all $f \in Q$ and the proof is complete.

Our last results in Section 3 have to do with a vector measure representation for tight linear functionals on $C^*(X : E)$.

3.8 *Definition.* Let X be a completely regular space and E a normed linear space. By $M(X : E')$ we denote the set of all set functions m defined on $B_a^*(X)$, with range in E', which satisfy the following two conditions: (a) the measure $m(\cdot)e$, defined for $e \in E$ by $m(\cdot)e(A) = m(A)(e)$, $A \in B_a^*(X)$, belongs to $M(X)$; (b) there exists $C > 0$ so that

$$\sum_{i=1}^{n} ||m(A_i)|| < C$$

for every partition of X into sets $A_i \in B_a^*(X)$. Let $M_\sigma(X : E')$, $M_t(X : E')$ and $M_\tau(X : E')$ denote the set of $m \in M(X : E')$, so that for each $e \in E$, $m(\cdot)e \in M_\sigma(X)$, $M_t(X)$, and $M_\tau(X)$, respectively.

3.9 PROPOSITION. *Let $m \in M(X : E')$ and $A \in B_a^*(X)$. Let*

$$|m|(A) = \sup \left\{ \sum_{i=1}^{n} ||m(A_i)|| : \{A_i\} \subseteq B_a^*(X) \text{ is a partition of } A \right\}.$$

Then $|m| \in M(X)$. If $m \in M_\sigma(X : E')(M_t(X : E'))$, then $|m| \in M_\sigma(X)$ $(M_t(X))$.

Proof. The proof of the first assertion is straightforward.

Suppose that $m \in M_\sigma(X : E')$, $A \in B_a(X)$ and $e \in E$. From [**21**, Theorem 18] there is a unique countably additive (regular) measure m_e on $B_a(X)$ which extends $m(\cdot)e$. Let $m'(A) = m_e(A)$ for $A \in B_a(X)$. By regularity and uniqueness of extension $m' \in M(X : E')$ and $|m| = |m'|$ on $B_a^*(X)$. Also m' is countably additive in norm, i.e., if $\{A_n\}_{n=1}^{\infty}$ is a disjoint collection in $B_a(X)$, then

$$\left\| m'\left(\bigcup_{n=1}^{\infty} A_n\right) - \sum_{n=1}^{p} m'(A_n) \right\| \underset{p}{\rightarrow} 0.$$

Hence, by modifying standard arguments such as [**16**, Theorem 6.2], $|m'|$ is countably additive. Hence $|m|$ is σ-additive.

Next suppose that $m \in M_t(X : E')$. We show that $|m| \in M_t(X)$. First, we need one additional bit of terminology. A set $U \subseteq X$ is called a *cozero* set of X if $X\backslash U$ is a zero set of X, i.e., if there is a function $f \in C^*(X)$ such that $f^{-1}(0) = X\backslash U$. The following observation will be used in the rest of this proof: A measure $\mu \in M(X)$ belongs to $M_t(X)$ if and only if for all $\epsilon > 0$, there exists $K \subseteq X$ (depending on ϵ) such that K is compact and if W is a cozero set of X which contains K, then $|\mu|(X) < |\mu(W) + \epsilon$. The proof of this observation is straightforward and is omitted.

We now proceed to prove that $m \in M_t(X : E')$ implies that $|m| \in M_t(X)$. Let $\epsilon > 0$ and choose a finite disjoint collection of zero sets $\{Z_i\}_{i=1}^{n} \subseteq X$ and a set of points $\{e_i\}_{i=1}^{n} \subseteq E$, $||e_i|| \leqq 1$, such that

$$|m|(X) < \frac{\epsilon}{3} + \sum_{i=1}^{n} |m(Z_i)(e_i)|.$$

For $1 \leqq i \leqq n$, let m_i denote the Baire measure $m(\cdot)e_i$ and let p_i denote the total variation of m_i.

According to the observation made earlier in this proof, choose for each $1 \leqq i \leqq n$ a compact set $H_i \subseteq X$ such that if W is a cozero set of X containing

H_i, then $p_i(X) < \epsilon/6n + p_i(W)$. Let $K_i = H_i \cap Z_i$ and note that if W is a cozero set of X containing K_i, then $p_i(Z_i) < \epsilon/6n + p_i(W)$ for all $1 \leqq i \leqq n$. Using regularity of p_i, $1 \leqq i \leqq n$, and the fact that the sets $\{Z_i\}_{i=1}^n$ are disjoint zero-sets, choose *disjoint* cozero sets O_i so that $Z_i \subseteq O_i$ and such that $0 < p_i(O_i) < \epsilon/6n + p_i(Z_i)$ for all $1 \leqq i \leqq n$.

Let $K = \bigcup_{i=1}^n K_i$ and W be a cozero set of X such that $K \subseteq W$. For $1 \leqq i \leqq n$, let $V_i = O_i \cap W$. Also let $V = \bigcup_{i=1}^n V_i$.

Note that $p_i(V_i) \leqq p_i(O_i) \leqq p_i(Z_i) + \epsilon/6n \leqq p_i(V_i) + \epsilon/3n$ for $1 \leqq i \leqq n$. Hence $|m_i(O_i \backslash V_i)| \leqq p_i(O_i \backslash V_i) < \epsilon/3n$; thus $|m_i(O_i)| \leqq |m_i(V_i)| + \epsilon/3n$ for $1 \leqq i \leqq n$. Also, since $p_i(O_i \backslash Z_i) < \epsilon/6n$, $|m_i(O_i) - m_i(Z_i)| = |m_i(O_i \backslash Z_i)| < \epsilon/3n$; hence $|m_i(Z_i)| < \epsilon/3n + |m_i(O_i)|$ for $1 \leqq i \leqq n$. Thus

$$|m|(X) < \sum_{i=1}^n |m_i(Z_i)| + \frac{\epsilon}{3} \leqslant \frac{\epsilon}{3} + \frac{\epsilon}{3} + \sum_{i=1}^n |m_i(O_i)| \leqslant \epsilon$$
$$+ \sum_{i=1}^n |m_i(V_i)| \leqslant \epsilon + |m|(V) \leqslant \epsilon + |m|(W).$$

In summary, $|m|(X) \leqq |m|(W) + \epsilon$; hence $|m| \in M_t(X)$ by the observation made earlier in this proof.

3.10 *Definition*. Let $m \in M(X : E')$ and $f \in C^*(X : E)$. The integral of f with respect to m, denoted

$$\int_X f dm,$$

is the real number R if for $\epsilon > 0$ there is a finite partition $P(\epsilon)$ of X into elements of $B_a{}^*(X)$ so that

$$\left| \sum_{i=1}^n m(A_i)(f(x_i)) - R \right| < \epsilon$$

if $\{A_i\}_{i=1}^n \subseteq B_a{}^*(X)$ is any partition of X refining $P(\epsilon)$ and $\{x_i\}_{i=1}$ is any choice of points such that $x_i \in A_i$ for $1 \leqq i \leqq n$.

3.11 Lemma. *Let $f \in C^*(X : E)$ and $m \in M(X : E')$. Then $\int_X f dm$ exists and*

$$\left| \int_X f dm \right| \leqslant \int_X ||f|| d|m|.$$

3.12 Proposition. *Let $m \in M(X : E')$ and*

$$F(f) = \int_X f dm$$

for $f \in C^*(X : E)$. Then $F \in C^*(X : E)'$ and $||F|| = |m|(X)$. If $m \in M_\sigma(X : E')$ or $m \in M_t(X : E')$, then F is σ-additive or tight, respectively.

Proof. Apply 3.11 and 3.9 plus Sentilles' results [**17**], for all assertions but the equality $||F|| = |m|(X)$. From 3.11 it is clear that $||F|| \leqq |m|(X)$.

For the reverse inequality, it suffices to show that $\sum_{i=1}^n m(Z_i)(e_i) \leqq ||F|| + \epsilon$ for every $\epsilon > 0$, finite set $\{e_i\}_{i=1}^n$ contained in the unit ball of E, and disjoint collection $\{Z_i\}_{i=1}^n$ of zero-sets such that $m(Z_i)(e_i) \geqq 0$ for $1 \leqq i \leqq n$.

Suppose that $\{Z_i\}_{i=1}^n$ and $\{e_i\}_{i=1}^n$ are sets as above and $\epsilon > 0$. For $\mu \in M(X)$, let $|\mu|$ denote the total variation of μ. Choose disjoint cozero-sets $\{D_i\}$, $1 \leqq i \leqq n$, so :hat $Z_i \subseteq D_i$ and $|m(\cdot)e_i|(D_i \backslash Z_i) < \epsilon/n$, and functions $\{f_i : 1 \leqq i \leqq n\} \subseteq C^*(X)$ such that $0 \leqq f_i \leqq 1$, $1 \leqq i \leqq n$, $f_i \equiv 1$ on Z_i and $f_i \equiv 0$ on $X \backslash D_i$. For $f \in C^*(X)$ and $e \in E$, let $f \otimes e(x) = f(x)e$, for all $x \in X$. Note that $f \otimes e \in C^*(X : E)$. Then

$$\sum_{i=1}^n m(Z_i)(e_i) = \sum_{i=1}^n \int_{Z_i} f_i \otimes e_i \, dm \leqslant \left| \sum_{i=1}^n \int_{D_i} f_i \otimes e_i \, dm \right| + \epsilon = \left| F\left(\sum_{i=1}^n f_i \otimes e_i \right) \right| + \epsilon \leqslant ||F|| + \epsilon.$$

Hence $|m|(X) \leqq ||F||$ as claimed.

3.13 THEOREM. *Suppose F is a tight linear functional on $C(X : E)$. Then there exists $m \in M_t(X : E')$ so that*

$$F(f) = \int_X f dm \quad \textit{for all } f \in C^*(X : E).$$

Proof. For $g \in C^*(X)$ and $e \in E$, let $g \otimes e(x) = g(x)e$ for $x \in X$. Let $C^*(X) \otimes E$ denote the linear subspace of $C^*(X : E)$ spanned by all functions $g \otimes e$ for $g \in C^*(X)$ and $e \in E$. By using partitions of unity, we see that $C^*(X) \otimes E$ is β_0-dense in $C^*(X : E)$.

For $e \in E$, let $F_e(f) = F(f \otimes e)$ for all $f \in C^*(X)$. Since $F \in C^*(X : E)_{\beta_0}'$, $F_e \in C^*(X)_{\beta_0}'$, so by Sentilles' results [**17**], there is a unique $m_e \in M_t(X)$ so that

$$F_e(f) = \int_X f dm_e \quad \text{for } f \in C^*(X).$$

Note that $||m_e|| = ||F_e|| \leqq ||F|| \, ||e||$ [**21**, Theorem 6]. For $A \in B_a{}^*(X)$, let $m(A)(e) = m_e(A)$. Note that $m(A) \in E'$ for all $A \in B_a{}^*(X)$ since $m_{e_1} + {}_{e_2} = m_{e_1} + m_{e_2}$ for any pair (e_1, e_2) in $E \times E$. This last statement follows from the uniqueness guaranteed by the work in [**17**] or [**21**].

We may show that $\sum_{n=1}^p ||m(A_n)|| \leqq ||F||$, for every partition $\{A_n\}_{n=1}^p$ of X into Baire sets, exactly as in the last part of 3.12. Hence $m \in M_t(X : E')$. Note that $F(f) = \int_X f dm$, for all $f \in C^*(X) \otimes E$. By 3.12 and β_0-denseness of $C^*(X) \otimes E$,

$$F(f) = \int_X f dm$$

holds for all $f \in C^*(X : E)$.

3.14 *Question.* What can be said about $C^*(X : E)_{\beta_1}'$ and $M_\sigma(X : E')$ or about $C^*(X : E)_\beta'$ and $M_\tau(X : E')$? We wish to acknowledge our gratitude to Professor Robert Wheeler for some helpful suggestions here, as well as for pointing out an error in earlier proof of 3.9. Note that if $m \in M_\tau(X : \mathrm{E}')$

implies $|m| \in M_\tau(X)$ and F is defined as in 3.12, then F is τ-additive. Also if $C^*(X) \otimes E$ (see 3.13) is β_1-dense in $C^*(X : E)$ we would be able to extend 3.13 by adding that if F is a σ-additive linear functional on $C^*(X : E)$ then there exists $M \in M_\sigma(X : E')$ such that $F(f) = \int f dm$ for all f in $C^*(X : E)$. Similarly if it can be shown that $C^*(X) \otimes E$ is β-dense in $C^*(X : E)$ then the proof of 3.13 need be modified very little to show that τ-additive linear functionals on $C^*(X : E)$ are "represented" by elements of $M_\tau(X : E')$.

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Oakland University,
Rochester, Michigan