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WEAK COMPACTNESS IN SPACES OF VECTOR VALUED MEASURES

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We characterise relative weak compactness in $\sigma BM(\Sigma, X)$, the space of sigma-additive, X-valued measures of bounded variation, where X is a Banach space.

Let Σ be a σ-algebra of subsets of Ω and let X be a Banach space. We denote by $\sigma BM(\Sigma, X)$ the space of σ-additive vector measures $G\colon \Sigma \to X$ of bounded variation with the variation norm $|\cdot|$.

In this note we give necessary and sufficient conditions for a subset $K \subseteq \sigma BM(\Sigma, X)$ to be relatively weakly compact. This is achieved by modifying slightly the sufficient conditions given by Brooks and Dinculeanu in [**1**].

Let $\pi \subset \Sigma$ be a finite measurable partition of Ω and let

$$G_\pi = \sum_{A\in\pi} \mu(A)^{-1} G(A)\mu_A,\ G \in \sigma BM(\Sigma, X)$$

where $\mu_A(E) = \mu(A \cap E)$ for all $E \in \Sigma$.

A topology τ on $\sigma BM(\Sigma, X)$ is induced by the family

$$\{G \to x^* G(E) : x^* \in X^* \text{ and } E \in \Sigma\}$$

of linear functionals.

THEOREM. *$K \subseteq \sigma BM(\Sigma, X)$ is relatively weakly compact if and only if the following conditions are satisfied*

1) *K is bounded;*
2) *$\{|G| : G \in K\}$ is uniformly μ-continuous for some positive measure μ;*
3) *$\{G(E) : G \in K\}$ is relatively weakly compact;*
4) *G_π converges weakly quasi-uniformly on $\overline{\overline{K}}$, the closure of K in the topology τ, to G (that is given $g^* \in \sigma BM(\Sigma, X)^*, \pi \in P = \{$ finite measurable partition of $\Omega\}$ and $\xi > 0$, then there exist $\pi_1 \ldots \pi_n \in P$, all finer than π, such that*

$$\min_{i=1\ldots n} |g^* G_{\pi_i} - g^* G| < \xi,\ G \in \overline{\overline{K}}\Big).$$

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PROOF: We showed in [2] that K is relatively weakly compact if and only if the conditions (1)...(3) of the theorem, as well as the following condition:

$$g^* \in \sigma BM(\Sigma, X)^* \text{ restricted to } \overline{\overline{K}} \text{ is } \tau\text{-continuous},$$

are satisfied.

We assume without loss of generality that $K \subseteq L_1(\Sigma, \mu, X)$: one can always resort to the Stone space approach of Brooks and Dinculeanu [1].

It is clear that the martingale $(f_\pi)_P$ converges strongly to f for every integrable f. Let $g^* \in L_1(\Sigma, \mu, X)^*$ and define g^*_π by

$$g^*_\pi(f) = g^*(f_\pi).$$

Then g^*_π converges pointwise to g^*. If K is relatively weakly compact, then g^* is τ-continuous and the Arzela-Ascoli theorem implies that the convergence $g^*_\pi \to g^*$ is quasi-uniform on $\overline{\overline{K}}$, as required.

Say conversely that $g^*_\pi \to g^*$ quasi-uniformly on $\overline{\overline{K}}$. We showed in [2] that g^* has a representation of the following form.

Let $\bar{\pi}$ be a *-finite partition of $^*\Omega$ that is finer than any standard measurable partition of Ω in some nonstandard world. Then

$$g^*(f) = st(\sum_{E\in\bar{\pi}} x^* \int_E f\,d\mu).$$

where st denotes the standard part operation.

Now a simple computation shows that

$$g^*_\pi = \sum_{A\in\pi} st(\sum_{\substack{E\in\bar{\pi}\\ E\subset A}} \mu^{-1}(A)\mu(E)x^*_E \int_A f\,d\mu)\,.$$

Hence g^*_π is represented by a simple function and it follows that g^*, being a quasi-uniform limit of τ-continuous functions, is τ-continuous on $\overline{\overline{K}}$. ∎

REFERENCES

[1] J. Brooks and N. Dinculeanu, 'Weak compactness in spaces of Bochner integrable functions and applications', *Adv. in Math.* **24** (1977), 172–188.

[2] F. Wild, *Weak compactness in function spaces*, (preprint).

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