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HILBERT MATRIX OPERATOR ACTING BETWEEN CONFORMALLY INVARIANT SPACES

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ABSTRACT. In this article we study the action of the the Hilbert matrix operator $\mathcal{H}$ from the space of bounded analytic functions into conformally invariant Banach spaces. In particular, we describe the norm of $\mathcal{H}$ from H^∞ into BMOA and we characterize the positive Borel measures μ such that $\mathcal{H}$ is bounded from H^∞ into the conformally invariant Dirichlet space $M(\mathcal{D}_\mu)$. For particular measures μ, we also provide the norm of $\mathcal{H}$ from H^∞ into $M(\mathcal{D}_\mu)$.

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$ and let Hol($\mathbb{D}$) be the space of analytic functions in $\mathbb{D}$. With $\mathbb{T}$ we denote the unit circle, that is $\mathbb{T} = \partial\mathbb{D}$. The classical Hilbert matrix is

$$\mathcal{H} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix $\mathcal{H}$ introduces an operator on spaces of analytic functions through its action on the sequence of Taylor coefficients. For

$$f(z) = \sum_{m=0}^{\infty} a_m z^m$$

in the Hardy space H^1, $\mathcal{H}(f)$ is defined as

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{a_k}{k+n+1}\right)z^n.$$

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It is clear that $\mathcal{H}(f)$ is an analytic function on the unit disk, since Hardy's inequality [17, p. 48]

$$\sum_{m=0}^{\infty} \frac{|a_m|}{m+1} \leq \pi \|f\|_{H^1}$$

implies that the Taylor coefficients of $\mathcal{H}(f)$ are bounded.

In [15], E. Diamantopoulos and A. Siskakis initiated the study of the Hilbert matrix as an operator on Hardy spaces by using the fact that for every $z \in \mathbb{D}$, the function $\mathcal{H}(f)$ has the equivalent integral representation

$$\mathcal{H}(f)(z) = \int_0^1 \frac{f(t)}{1-tz} dt. \tag{1.1}$$

By considering $\mathcal{H}$ as the average of weighted composition operators, they proved that $\mathcal{H}$ is a bounded operator on every Hardy space H^p with $p > 1$ and they estimated its norm. Their research was further extended by M. Dostanic, M. Jevtic and D. Vukotic in [16]. Among other results, they showed that

$$\|\mathcal{H}\|_{H^p \to H^p} = \frac{\pi}{\sin(\frac{\pi}{p})},$$

for $1 < p < \infty$.

The study of the Hilbert matrix operator was subsequently extended to include the Bergman spaces of the unit disc. Diamantopoulos [14], Dostanic et al. [16] and V. Bozin and B. Karapetrovic [11] proved that $\mathcal{H}$ is bounded on the Bergaman space A^p if and only if $p > 2$. Moreover

$$\|\mathcal{H}\|_{A^p \to A^p} = \frac{\pi}{\sin\left(\frac{2\pi}{p}\right)}.$$

Following the classical case, a significant body of research has focused on generalizations of the Hilbert matrix operator, particularly regarding the characterization of their boundedness and compactness, see, for example, [8], [9] and [7] for a recent survey on the argument.

In this article, we study the action of $\mathcal{H}$ from H^∞ into conformally invariant Banach spaces. We recall that a Banach space X of analytic functions is conformally invariant if for every $f \in X$ and $\phi \in \mathrm{Aut}(\mathbb{D})$

$$\|f\|_X = \|f \circ \phi\|_X,$$

where the Mobius group $\mathrm{Aut}(\mathbb{D})$ is the set made by all the one-to-one analytic functions mapping $\mathbb{D}$ onto itself, see [2]. Main examples of conformaly invariant spaces are the BMOA space, the Q_p spaces and the classical Dirichlet space $\mathcal{D}$, while H^p with $1 \leq p < \infty$ is not.

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In this article, we consider the action of the Hilbert matrix operator from H^∞ into the space of analytic bounded mean oscillation BMOA and the conformally invariant Dirichlet spaces $M(\mathcal{D}_\mu)$.

The BMOA space consists of all the functions in the Hardy space H^2 such that

$$\|f\|_{\mathrm{BMOA}} = |f(0)| + \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^2 \log\left|\frac{1-\overline{a}z}{a-z}\right|^2 dA(z)\right)^{1/2} < \infty,$$

where, if $z = x + iy$, $dA(z) = \pi^{-1} dxdy$. With the above norm BMOA is a conformally invariant Banach space and

$$H^\infty \subsetneq \mathrm{BMOA} \subsetneq \bigcap_{0<p<\infty} H^p.$$

The norm $\|f\|_{\mathrm{BMOA}}$ can be expressed by integration on $\mathbb{T}$,

$$\|f\|_{\mathrm{BMOA}} = |f(0)| + \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{T}} |f(e^{i\theta})|^2 \frac{1-|a|}{|e^{i\theta}-a|^2}\frac{d\theta}{2\pi} - |f(a)|^2\right)^{1/2}. \tag{1.2}$$

For more information about BMOA, we refer to [19, Chapter VI] and [20].

Theorem 1.1. *The Hilbert matrix operator maps H^∞ into BMOA and its norm is $1 + \frac{\pi}{\sqrt{2}}$.*

The boundedness of $\mathcal{H}$ from H^∞ into BMOA has been first observed by B. Lanucha, M. Nowak and M. Pavlovic in [22]. Actually, it is also true that

$$\mathcal{H}(H^\infty) \subseteq \bigcap_{1<p<\infty} \Lambda\left(p, \frac{1}{p}\right) \subset \mathrm{BMOA},$$

where $\Lambda(p, \frac{1}{p})$ are the mean Lipschitz spaces (see later for definition).

Subsequently, we fix our attention on the conformally invariant Dirichlet space $M(\mathcal{D}_\mu)$. Let $d\mu(z)$ be a positive, Borel measure in $\mathbb{D}$. The spaces $M(\mathcal{D}_\mu)$ consists of all the functions $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{M(\mathcal{D}_\mu)} = |f(0)| + \sup_{\phi\in\mathrm{Aut}(\mathbb{D})} \left(\int_{\mathbb{D}} |f'(w)|^2 U_\mu(\phi(w)) dA(w)\right)^{1/2}, \tag{1.3}$$

where

$$U_\mu(z) = \int_{\mathbb{D}} \log\left|\frac{1-\overline{w}z}{z-w}\right|^2 d\mu(w)$$

is a superharmonic function in $\mathbb{D}$. The most famous examples of $M(\mathcal{D}_\mu)$ spaces are the Q_p spaces, see (2.2) and [3]. An equivalent expression for $\|f\|_{M(\mathcal{D}_\mu)}$ is

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$$\|f\|_{M(\mathcal{D}_\mu)} \sim |f(0)| + \sup_{\phi \in \mathrm{Aut}(\mathbb{D})} \left(\int_{\mathbb{D}} |f'(w)|^2 V_\mu(\phi(w)) dA(w) \right)^{1/2}, \tag{1.4}$$

where

$$V_\mu(z) = \int_{\mathbb{D}} (1 - |\sigma_z(w)|^2) d\mu(w).$$

We characterize the measures $d\mu(z)$ such that the Hilbert matrix operator $\mathcal{H}$ is bounded from H^∞ into $M(\mathcal{D}_\mu)$.

Theorem 1.2. *Let $d\mu(z)$ be a positive Borel measure on $\mathbb{D}$. The following conditions are equivalent:*

(i) The Hilbert matrix operator $\mathcal{H}$ sends H^∞ into $M(\mathcal{D}_\mu)$.
(ii) $\log(1 - z) \in M(\mathcal{D}_\mu)$.
(iii)

$$\sup_{\lambda \in \mathbb{T}} \int_{\mathbb{D}} \frac{V_\mu(z)}{|1 - \lambda z|^2} dA(z) < \infty.$$

(iv)

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{V_\mu(z)}{|1 - az|^2} dA(z) < \infty.$$

We highlight that the proof of Theorem 1.2 is similar to the proof of the analogous result for the Cesáro operator, see [6, Theorem 1.1]. In addition, for some measures $d\mu(z)$, we also provide the norm of $\mathcal{H}$ from H^∞ into $M(\mathcal{D}_\mu)$.

Theorem 1.3. *Let $d\mu(z)$ be a positive, radial, Borel measure on $\mathbb{D}$. If $\mathcal{H} : H^\infty \to M(\mathcal{D}_\mu)$ is bounded then*

$$\|\mathcal{H}\|_{H^\infty \to M(\mathcal{D}_\mu)} = 1 + \left(\int_{\mathbb{D}} \frac{4}{|1 - z^2|^2} U_\mu(z) dA(z) \right)^{1/2}.$$

Significantly, the measure associated to the Q_p spaces satisfies the hypothesis of the above theorem and we are able to compute the norm of the Hilbert matrix operator from H^∞ into Q_p.

The methodology developed in this article also works for the Cesáro operator

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\sum_{k=0}^{n} a_k \right) z^n, \text{ with } f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathrm{Hol}(\mathbb{D}).$$

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The analogy between the Hilbert matrix operator and the Cesáro operator comes from their matrix representations, that is

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We observe that $\mathcal{H}$ is obtained from $\mathcal{C}$ by the following formal manipulation: we erase the zeros in each column of $\mathcal{C}$ and we shift up the columns to their first non-zero entry. In rigorous terms this is equivalent to the following algebraic relation. Let $e_n(z) = z^n, n = 0, 1, 2, ...$ be the monomials, which form an orthonormal basis of H^2. We have that

$$\mathcal{C}_{\mathcal{S}}(e_n) = S^n \mathcal{H}(e_n),$$

where S is the shift operator, that is $Sf(z) = zf(z)$.

Theorem 1.4. *Let $d\mu(z)$ be a positive, radial, Borel measure on $\mathbb{D}$. If $\mathcal{C} : H^\infty \to M(\mathcal{D}_\mu)$ is bounded then*

$$\|\mathcal{C}\|_{H^\infty \to M(\mathcal{D}_\mu)} = 1 + \left(\int_{\mathbb{D}} \frac{4}{|1 - z^2|^2} U_\mu(z) dA(z) \right)^{1/2}.$$

In light of the analogy between $\mathcal{C}$ and $\mathcal{H}$, a comparative analysis of the Hilbert matrix operator results with their Cesáro operator counterparts is warranted.

The rest of the article is divided in six sections. Section 2 is devoted to preliminary material: we recall the definition of Hardy spaces and we describe the conformally invariant Dirichlet spaces $M(\mathcal{D}_\mu)$. In Section 3, we deal with BMOA and we prove Theorem 1.1. In section 4, we briefly describe the action of the Hilbert matrix operator from H^∞ into the mean Lipschitz spaces. In section 5, we describe the action of the Hilbert matrix operator from H^∞ into $M(\mathcal{D}_\mu)$ and we prove Theorem 1.2.
We prove Theorem 1.4 in Section 7. Finally, in section 6, we prove Theorem 1.3. We conclude the article with some open problems.

We use the following notation. By the expressions $f \lesssim g$, we mean that there exists a positive constant C such that

$$f \leq Cg.$$

If both $f \lesssim g$ and $f \gtrsim g$ hold, we write $f \sim g$. The capital letter C, will denote constants whose values may change every time they appear. Finally with $\delta_{m,n}$ we denote the classical Kronecker symbol.

2. Preliminary

In this preliminary section, we recall some definitions that will be used throughout the whole article.

2.1. **Hardy spaces.** Let $1 \leq p < \infty$ and $f \in \mathrm{Hol}(\mathbb{D})$. For $0 \leq r < 1$, let

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}$$

be the usual integral means of f on the circle of radius r. The Hardy space $H^p = H^p(\mathbb{D})$ consists of all the functions $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} M_p(r,f) < \infty$$

and, for $p = \infty$, H^∞ consists of the bounded analytic functions on $\mathbb{D}$, i.e. all the functions in $\mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)| < \infty.$$

See [17] for the theory of Hardy spaces. Important examples of H^∞ functions are the automorphisms, $\phi \in \mathrm{Aut}(\mathbb{D})$. We recall that every $\phi \in \mathrm{Aut}(\mathbb{D})$ can be written as

$$\phi(z) = e^{i\theta} \sigma_a(z) \text{ with } \sigma_a(z) = \frac{a - z}{1 - \bar{a}z},$$

where θ is real and $a \in \mathbb{D}$.

2.2. **Conformally invariant Dirichlet spaces.** The Möbius invariant spaces $M(\mathcal{D}_\mu)$ generated by the Dirichlet space $\mathcal{D}_\mu$ are defined in (1.3) [1]. In order to avoid that $M(\mathcal{D}_\mu)$ contains only constant functions, we always assume that

$$(1 - |z|^2) d\mu(z) \text{ is a Carleson measure on } \mathbb{D}. \tag{2.1}$$

If (2.1) does not hold, $M(\mathcal{D}_\mu)$ is called *trivial*. From [5], it is known that if $M(\mathcal{D}_\mu)$ is not trivial, then $\mathcal{D} \subseteq M(\mathcal{D}_\mu) \subseteq \mathrm{BMOA}$. Furthermore, $M(\mathcal{D}_\mu) = \mathrm{BMOA}$ if and only if $\mu(\mathbb{D}) < \infty$.

For $0 < p < \infty$, the Q_p space consists of all the functions $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{Q_p}^2 = |f(0)| + \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2)^p dA(z) \right)^{1/2} < \infty. \tag{2.2}$$

Clearly, $Q_{p_1} \subseteq Q_{p_2}$ for $0 < p_1 < p_2 < \infty$ and for $0 < p < 1$

$$Q_p = M(\mathcal{D}_{\mu_p}),$$

where

$$d\mu_p(z) = -\Delta[(1 - |z|^2)^p] dA(z), \tag{2.3}$$

[1]For the theory of superharmonic weighted Dirichlet spaces $\mathcal{D}_\mu$, we refer to [1].

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and Δ is the Laplacian [5]. We have that $Q_1 = \text{BMOA}$ and $Q_p \subsetneq \text{BMOA}$ when $0 < p < 1$. We refer to J. Xiao's monographs [23] and [24] for more results on Q_p spaces.

3. Proof of Theorem 1.1

It is well known that $\mathcal{H}$ does not map H^∞ into its self. Indeed, by (1.1), we note that

$$\mathcal{H}(1)(z) = \int_0^1 \frac{1}{1-tz}dt = \frac{1}{z}\log\left(\frac{1}{1-z}\right) \tag{3.1}$$

and this function does not belong to H^∞.

Proof of Theorem 1.1. In [22] Lanucha, Nowak and Pavlovic proved that $\mathcal{H}$ is bounded from H^∞ into BMOA. By using the same computations of Danikas and Siskakis [13, Theorem 1] and the expression (1.2) for the BMOA norm, we realize that

$$\|\mathcal{H}(1)\|_{\text{BMOA}} = 1 + \frac{\pi}{\sqrt{2}},$$

from which it follows that

$$1 + \frac{\pi}{\sqrt{2}} \leq \|\mathcal{H}\|_{H^\infty \to \text{BMOA}}. \tag{3.2}$$

In order to prove the upper bound for the norm of $\mathcal{H}$, because of (1.1), we note that

$$\mathcal{H}(f)'(z) = \int_0^1 \frac{t\,f(t)}{(1-tz)^2}dt.$$

The convergence of the integral and the analyticity of the function f guarantee that we can change the path of integration to

$$z(t) = \frac{s}{(s-1)w+1}, \qquad 0 \leq s \leq 1,$$

which describes circular arcs contained in $\mathbb{D}$. Therefore, we obtain that

$$\mathcal{H}(f)'(w) = \frac{1}{1-w}\int_0^1 \frac{s}{(s-1)w+1} f\left(\frac{s}{(s-1)w+1}\right)ds.$$

The quantity inside the integral is $\psi_s(w)f(\psi_s(w))$, where

$$\psi_s(w) = \frac{s}{(s-1)w+1}$$

maps the unit disc into itself for each $0 \leq s < 1$. Thus, in absolute value,

$$|\psi_s(w)f(\psi_t(w))| \leq \|f\|_{H^\infty}$$

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for each $w \in \mathbb{D}$ and $0 \le s \le 1$. Consequently,

$$\begin{aligned}\|\mathcal{H}(f)\|_{\mathrm{BMOA}} &= |\mathcal{H}(f)(0)| + \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} \left|\mathcal{H}(f)'(z)\right|^2 \log\left|\frac{1-\overline{a}z}{a-z}\right|^2 dA(z) \right)^{1/2} \\ &\le \left(1 + \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} \left|\frac{1}{1-z}\right|^2 \log\left|\frac{1-\overline{a}z}{a-z}\right|^2 dA(z) \right)^{1/2} \right) \|f\|_{H^\infty} \\ &= \left(1+\frac{\pi}{\sqrt{2}}\right)\|f\|_{H^\infty},\end{aligned}$$

where in the last equality we have used the fact that $\|\log(1-z)\|_{\mathrm{BMOA}} = \frac{\pi}{\sqrt{2}}$, see [12]. Due to the above inequality and (3.2), we have the conclusion. □

The proof of Theorem 1.1 provides a usefully identity for the derivative $\mathcal{H}(f)'$ when $f \in H^\infty$. Indeed

$$\mathcal{H}(f)'(z) = \frac{b(z)}{1-z}, \tag{3.3}$$

with $b \in H^\infty$ and $\|b\|_{H^\infty} \le \|f\|_{H^\infty}$.

4. Hilbert matrix operator into mean Lipschitz spaces

The inclusion $\mathcal{H}(H^\infty) \subset \mathrm{BMOA}$ can be refined to

$$\mathcal{H}(H^\infty) \subseteq \bigcap_{1<p<\infty} \Lambda\left(p, \frac{1}{p}\right).$$

For $1 < p < \infty$ and $0 < \alpha \le 1$ the mean Lipschitz space $\Lambda(p,\alpha)$ consists all the analytic functions f on the unit disc for which

$$\|f\|_{\Lambda(p,\alpha)} = |f(0)| + \sup_{0<r<1} M_p(r, f')(1-r^2)^{1-\alpha} < \infty.$$

The spaces $\Lambda(p, \frac{1}{p})$ with $1 < p < \infty$ grow in size with p, they are all subspaces of BMOA and they all contain unbounded functions such as $\log(1-z)$, see [10].

Theorem 4.1. *The Hilbert matrix operator $\mathcal{H}$ is bounded from H^∞ into $\Lambda(p, 1/p)$ and*

$$\|\mathcal{H}\|_{H^\infty \to \Lambda(p,\frac{1}{p})} \le 1 + \|\log(1-z)\|_{\Lambda(p,\frac{1}{p})}.$$

Proof. The proof is similar to Theorem 1.1. By [21, Theorem 1.7], we have that $\log(1-z) \in \Lambda(p, \frac{1}{p})$ for $1 < p < \infty$ with

$$\|\log(1-z)\|_{\Lambda(p,\frac{1}{p})} = \sup_{0<r<1} (1-r)^{1-1/p} \left(\int_{\mathbb{T}} \left| \frac{1}{1-re^{-i\theta}} \right|^p d\theta \right)^{1/p}.$$

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By (3.3), we have that

$$(1-r^2)^{1-1/p} M_p(r, \mathcal{H}(f)') \leq (1-r^2)^{1-1/p} \left(\int_{\mathbb{T}} \left| \frac{1}{1-re^{-i\theta}} \right|^p d\theta \right)^{1/p} \|f\|_{H^\infty}$$
$$\leq \| \log(1-z)\|_{\Lambda(p,\frac{1}{p})} \|f\|_{H^\infty}.$$

Consequently

$$\|\mathcal{H}(f)\|_{\Lambda(p,\frac{1}{p})} \leq \left(1 + \| \log(1-z)\|_{\Lambda(p,\frac{1}{p})}\right) \|f\|_{H^\infty}.$$

□

We point out that the exact value of the $\|\mathcal{H}\|_{H^\infty \to \Lambda(p,\frac{1}{p})}$ is not provided by Theorem 4.1. However, we are able to compute it when $p = 2$.

Proposition 4.2. *The Hilbert matrix operator $\mathcal{H}$ is bounded from H^∞ into $\Lambda(2, 1/2)$ and*

$$\|\mathcal{H}\|_{H^\infty \to \Lambda(2,\frac{1}{2})} = 1 + \| \log(1-z)\|_{\Lambda(2,\frac{1}{2})} = 2.$$

Proof. Since in Theorem 4.1 we have already shown the upper bound, we have to provide only the estimate from below. Since $\mathcal{H}(1)(1) = 1$ and

$$S\left(\mathcal{H}(1)'(z)\right) = -\frac{1}{z} \log\left(\frac{1}{1-z}\right) + \frac{1}{1-z} = f_1(z) + f_2(z),$$

we have

$$\begin{aligned}
\|\mathcal{H}\|_{H^\infty \to \Lambda(2,\frac{1}{2})} &\geq \|\mathcal{H}(1)\|_{\Lambda(2,\frac{1}{2})} \\
&= 1 + \sup_{0<r<1} M_2(r, \mathcal{H}(1)')(1-r^2)^{1/2} \\
&\geq 1 + \lim_{r\to 1} M_2(r, S\,\mathcal{H}(1)')(1-r^2)^{1/2} \\
&\geq 1 + \lim_{r\to 1} |M_2(r, f_1) - M_2(r, f_2)|(1-r^2)^{1/2} \\
&= 1 + \lim_{r\to 1} M_2(r, (\log(1-z))')(1-r^2)^{1/2} = 1 + \| \log(1-z)\|_{\Lambda(2,\frac{1}{2})}.
\end{aligned}$$

Since $\| \log(1-z)\|_{\Lambda(2,\frac{1}{2})} = 1$, conclusion follows.

□

5. Hilbert matrix operator into conformally invariant Dirichlet spaces

We start by proving Theorem 1.2.

Proof of Theorem 1.2. The proof of Theorem 1.2 is similar to [6, Theorem 1.2]. Nevertheless, for completeness, we include it here.

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$(i) \Rightarrow (ii)$. Let $\mathcal{H}(H^\infty) \subseteq M(\mathcal{D}_\mu)$. Since $\mathcal{H}(1) \in \mathcal{H}(H^\infty)$, we have that $M(\mathcal{D}_\mu)$ is not trivial and [5, Theorem 3.3] implies that

$$\sup_{w\in\mathbb{D}} V_\mu(w) < \infty.$$

In order to prove that $\log(1-z)$ is in $M(D_\mu)$, according to (1.4), it is enough to show that

$$\sup_{\phi\in\mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \left|\frac{1}{1-z}\right|^2 V_\mu(\phi(z))dA(z) < \infty. \tag{5.1}$$

Indeed

$$\begin{aligned}\infty &> \sup_{\phi\in\mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \left|\mathcal{H}(1)'(z)\right|^2 V_\mu(\phi(z))dA(z)\\ &= \sup_{\phi\in\mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \left|\frac{1}{z}\frac{1}{1-z} - \frac{1}{z^2}\log\frac{1}{1-z}\right|^2 V_\mu(\phi(z))dA(z).\end{aligned}$$

Since

$$\sup_{\phi\in\mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \left|\frac{1}{z^2}\log\frac{1}{1-z}\right|^2 V_\mu(\phi(z))dA(z) \leq 16\sup_{w\in\mathbb{D}} V_\mu(w)\int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \left|\log\frac{1}{1-z}\right|^2 dA(z) < \infty,$$

(5.1) holds.

$(ii) \Rightarrow (iii)$. Let $\log(1-z) \in M(\mathcal{D}_\mu)$. By using (1.4) and a change of variables, we have that

$$\sup_{a\in\mathbb{D},\lambda\in\mathbb{T}} \int_{\mathbb{D}} \left|\frac{1}{1-\lambda\sigma_a(z)}\right|^2 \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4} V_\mu(z)dA(z) < \infty.$$

Taking $a=0$ in the above condition, we have

$$\sup_{\lambda\in\mathbb{T}} \int_{\mathbb{D}} \frac{V_\mu(z)}{|1-\lambda z|^2} dA(z) < \infty.$$

$(iii) \Leftrightarrow (iv)$. We set $(1-|z|^2)d\nu(z) = V_\mu(z)dA(z)$ and apply [4, Lemma 2.2].

$(iv) \Rightarrow (i)$. Let $f \in H^\infty$. By using (3.3), we note that

$$\begin{aligned}&\int_{\mathbb{D}} \left|\mathcal{H}(f)'(z)\right|^2 V_\mu(\phi(z))dA(z)\\ &\quad\leq \int_{\mathbb{D}} \left|\frac{1}{1-z}\right|^2 V_\mu(\phi(z))dA(z)\,\|f\|_\infty^2\\ &\quad= \int_{\mathbb{D}} \left|\frac{1}{1-\phi(z)}\right|^2 \frac{\left(1-|a|^2\right)^2}{|1-\overline{a}z|^4} V_\mu(z)dA(z)\,\|f\|_\infty^2,\end{aligned}$$

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where we have chosen

$$\phi(z)=\lambda\frac{a-z}{1-\overline{a}z},\ \lambda\in\mathbb{T},a\in\mathbb{D}.$$

Consequently, $\mathcal{H}$ is bounded from H^∞ into $M(\mathcal{D}_\mu)$ if

$$\begin{aligned}I&=\sup_{a\in\mathbb{D},\lambda\in\mathbb{T}}\int_{\mathbb{D}}\left|\frac{1}{1-\lambda\sigma_a(z)}\right|^2\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}V_\mu(z)dA(z)\\&=\sup_{a\in\mathbb{D},\lambda\in\mathbb{T}}\frac{(1-|a|^2)^2}{|1-\lambda a|^2}\int_{\mathbb{D}}\left|\frac{1}{1+\frac{\lambda-\overline{a}}{1-\lambda a}z}\right|^2\frac{1}{|1-\overline{a}z|^2}V_\mu(z)dA(z)<\infty.\end{aligned}$$

We set $\eta=\frac{\lambda-\overline{a}}{1-\lambda a}$, $|\eta|=1$. Through a change of variables, we get

$$\begin{aligned}I&=\sup_{a\in\mathbb{D},\lambda\in\mathbb{T}}\frac{(1-|a|^2)^2}{|1-\lambda a|^2}\int_{\mathbb{D}}\left|\frac{1}{1+\zeta}\right|^2\frac{1}{|1-\overline{a\eta}\zeta|^2}V_\mu(\overline{\eta}\zeta)dA(\zeta)\\&=\sup_{a\in\mathbb{D},\lambda\in\mathbb{T}}\int_{\mathbb{D}}\left|\frac{1}{1+\zeta}+\frac{\overline{a\eta}\zeta}{1-\overline{a\eta}\zeta}\right|^2V_\mu(\overline{\eta}\zeta)dA(\zeta)\\&\le C\left(\sup_{\eta\in\mathbb{T}}\int_{\mathbb{D}}\left|\frac{1}{1+\zeta}\right|^2V_\mu(\overline{\eta}\zeta)dA(\zeta)+\sup_{a\in\mathbb{D},\lambda\in\mathbb{T}}\int_{\mathbb{D}}\left|\frac{1}{1-\overline{a\eta}\zeta}\right|^2V_\mu(\overline{\eta}\zeta)dA(\zeta)\right)\\&\le C\sup_{\eta\in\mathbb{T}}\int_{\mathbb{D}}\left|\frac{1}{1+\eta z}\right|^2V_\mu(z)dA(z)+\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|\frac{1}{1-\overline{a}z}\right|^2V_\mu(z)dA(z).\end{aligned}$$

By using conditions (iii) and (iv), the proof is complete.

□

Remark 5.1. Let

$$d\mu_a(z)=\Delta\left(\frac{1}{\left(\log\frac{e^{1+a}}{1-|z|^2}\right)^a}\right)dA(z)$$

with $z\in\mathbb{D}$, $a\in(0,\infty)$ and Δ the classical Laplace operator. The proof [6, Theorem 1.2] gives that

(i) If $0<a\le 1$, then $M(\mathcal{D}_{\mu_a})$ is not trivial and $\mathcal{H}(H^\infty)\not\subseteq M(\mathcal{D}_{\mu_a})$.
(ii) If $a>1$, then $\mathcal{H}(H^\infty)\not\subseteq M(\mathcal{D}_{\mu_a})\not\subseteq\bigcap_{0<p<\infty}Q_p$.

Consequently, there are measures μ for which $\mathcal{H}(H^\infty)\not\subseteq M(\mathcal{D}_\mu)$.

6. Norm of the Hilbert matrix operator into conformally invariant Dirichlet spaces

For particular measures μ, it is possible to compute $\|\mathcal{H}\|_{H^\infty\to M(\mathcal{D}_\mu)}$. We need two preliminary lemmas.

Lemma 6.1. *Let $d\mu(z)$ be a radial measure. Then $U_\mu(z)dA(z)$ is also a radial measure.*

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Proof. Let $\lambda \in \mathbb{T}$. We note that

$$U_\mu(\lambda z) = \int_{\mathbb{D}} \log\left|\frac{1-\overline{w}\lambda z}{\lambda z - w}\right| d\mu(w) = \int_{\mathbb{D}} \log\left|\frac{1-\overline{w\overline{\lambda}}z}{z-\overline{\lambda}w}\right| d\mu(w)$$
$$= \int_{\mathbb{D}} \log\left|\frac{1-\overline{w\overline{\lambda}}z}{z-\overline{\lambda}w}\right| d\mu(\overline{\lambda}w) = U_\mu(z).$$

Therefore

$$U_\mu(z)dA(z) = U_\mu(r)\, rdr\frac{d\theta}{2\pi},$$

from which the Lemma follows.

□

Lemma 6.2. *Let $d\mu(z)$ be a positive, radial measure such that $\log(1-z) \in M(\mathcal{D}_\mu)$. Then*

$$\sum_{n=1}^{\infty} \widehat{U_\mu}_{2n+1} < \infty,$$

where

$$\widehat{U_\mu}_{2n+1} = \int_0^1 r^{2n+1} U_\mu(r) dr.$$

Proof. We note that

$$\sum_{n=1}^{\infty} \widehat{U_\mu}_{2n+1} = \sum_{n=1}^{\infty} \int_0^1 r^{2n} U_\mu(r) rdr = \sum_{n=0}^{\infty} \int_0^1 r^{2n} U_\mu(r) rdr - \widehat{U_\mu}_1$$
$$\leq \int_0^1 \frac{1}{1-r^2} U_\mu(r) rdr = \int_0^1 \int_0^{2\pi} \frac{1}{1-e^{i\theta}r}\frac{1}{1-e^{-i\theta}r}\frac{d\theta}{2\pi} U_\mu(r) rdr$$
$$= \int_{\mathbb{D}} \left|\frac{1}{1-z}\right|^2 U_\mu(z) dA(z) \leq \|\log(1-z)\|^2_{M(\mathcal{D}_\mu)}.$$

Since $\log(1-z) \in M(\mathcal{D}_\mu)$, the conclusion follows.

□

When $d\mu(z)$ is a radial measure, (1.3) becomes

$$\|f\|_{M(\mathcal{D}_\mu)} = |f(0)| + \sup_{a\in\mathbb{D},\lambda\in\mathbb{T}} \left(\int_{\mathbb{D}} |f'(w)|^2 U_\mu(\lambda\sigma_a(w)) dA(w)\right)^{1/2}$$
$$= |f(0)| + \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f'(w)|^2 U_\mu(\sigma_a(w)) dA(w)\right)^{1/2}.$$

Proposition 6.3. *Let $d\mu(z)$ be a positive, radial, Borel measure on $\mathbb{D}$. Then*

$$\|\log(1-z)\|^2_{M(\mathcal{D}_\mu)} = \int_{\mathbb{D}} \left|\frac{2}{1-z^2}\right|^2 U_\mu(z) dA(z).$$

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Proof. Since the measure $d\mu(z)$ is radial, we observe that

$$\begin{aligned}\|\log(1-z)\|_{M(\mathcal{D}_\mu)} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|\frac{1}{1-z}\right|^2 U_\mu(\sigma_a(z))dA(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|\frac{1}{1-\sigma_a(z)}\right|^2 |\sigma_a'(z)|^2 U_\mu(z)dA(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^2}{|1-a+(1-\overline{a})z|^2}\frac{1}{|1-\overline{a}z|^2}U_\mu(z)dA(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|\frac{1-\overline{a}}{1-a+(1-\overline{a})z}+\frac{\overline{a}}{1-\overline{a}z}\right|^2 U_\mu(z)dA(z).\end{aligned}$$

In particular, by using the power series expansion, we have that

$$\begin{aligned}\|\log(1-z)\|_{M(\mathcal{D}_\mu)} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|\sum_{n=0}^{\infty}\left((-1)^n\left(\frac{1-\overline{a}}{1-a}\right)^{n+1}+\overline{a}^{n+1}\right)z^n\right|^2 U_\mu(z)dA(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|\sum_{n=0}^{\infty}\left(-\left(\frac{1-\overline{a}}{1-a}\right)^{n+1}+\overline{-a}^{n+1}\right)(-z)^n\right|^2 U_\mu(z)dA(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|\sum_{n=1}^{\infty}\left(-\left(\frac{1-\overline{a}}{1-a}\right)^{n}+\overline{-a}^{n}\right)(-z)^{n-1}\right|^2 U_\mu(z)dA(z).\end{aligned}$$

Since $d\mu(z)$ is radial and

$$\int_0^{2\pi} e^{(n-1)i(\theta+\pi)}e^{-(m-1)i(\theta+\pi)}\frac{d\theta}{2\pi} = \delta_{n,m}, \quad n,m\in\mathbb{N},$$

we have that

$$\begin{aligned}\|\log(1-z)\|_{M(\mathcal{D}_\mu)} &= \sup_{a\in\mathbb{D}}\int_0^1\sum_{n=1}^{\infty}\left|-\left(\frac{1-\overline{a}}{1-a}\right)^n-\overline{a}^n\right|^2 r^{2n+1}U_\mu(r)dr\\ &= \sup_{a\in\mathbb{D}}\sum_{n=1}^{\infty}\left|-\left(\frac{1-\overline{a}}{1-a}\right)^n-\overline{a}^n\right|^2 \widehat{U_\mu}_{2n+1}.\end{aligned}$$

Hence,

$$\|\log(1-z)\|_{M(\mathcal{D}_\mu)} = \sup_{a\in\mathbb{D}}\sum_{n=1}^{\infty}\widehat{U_\mu}_{2n+1}\left(1+|a|^{2n}+2(-1)^{n+1}\operatorname{Re}\left(\frac{1-\overline{a}}{1-a}a\right)^n\right).$$

With the change of variable

$$w = \frac{1-\overline{a}}{1-a}a,$$

where $|w| = |a|$ for $a\in\overline{\mathbb{D}}\setminus\{1\}$ and $|w|\to 1$ as $a\to 1$, we note that the function

$$B(w) = \sum_{n=1}^{\infty}\widehat{U_\mu}_{2n+1}2(-1)^{n+1}w^n$$

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is analytic in the unit disc. Moreover, since $B(w)$ is continuous in $\overline{\mathbb{D}}$ due to Lemma 6.2, the maximum principle for harmonic functions tells us that

$$\sup_{w\in\mathbb{D}} \operatorname{Re} B(w) = \sup_{w\in\mathbb{T}} \operatorname{Re} B(w).$$

Consequently,

$$\begin{aligned}\sup_{w\in\mathbb{T}} \operatorname{Re} B(w) &= -2 \inf_{w\in\mathbb{T}} \operatorname{Re}\left(\sum_{n=1}^{\infty} \widehat{U_\mu}_{2n+1}(-1)^n w^n\right)\\ &= -2 \inf_{w\in\mathbb{T}} \operatorname{Re} \int_0^1 \sum_{n=1}^{\infty}(-wr^2)^n U_\mu(r) r dr\\ &= 2\widehat{U_\mu}_1 - 2 \inf_{w\in\mathbb{T}} \operatorname{Re} \int_0^1 \frac{1}{1+wr^2} U_\mu(r) r dr\\ &= 2\left(\widehat{U_\mu}_1 - \int_0^1 \frac{1}{1+r^2} U_\mu(r) r dr\right) = \operatorname{Re} B(1).\end{aligned}$$

In particular, due to Lemma 6.2, we know that

$$\begin{aligned}\|\log(1-z)\|_{M(\mathcal{D}_\mu)} &= \lim_{a\to 1} \int_{\mathbb{D}} \left|\frac{1}{1-z}\right|^2 U_\mu(\sigma_a(z)) dA(z)\\ &= \lim_{a\to 1, a\in\mathbb{R}} \int_{\mathbb{D}} \left|\frac{1}{1-z}\right|^2 U_\mu(\sigma_a(z)) dA(z)\,.\end{aligned}$$

Consequently

$$\begin{aligned}\|\log(1-z)\|_{M(\mathcal{D}_\mu)} &= \lim_{a\to 1, a\in\mathbb{R}} \int_{\mathbb{D}} \left|\frac{1}{1-z}\right|^2 U_\mu(\sigma_a(z)) dA(z)\\ &= \lim_{a\to 1, a\in\mathbb{R}} \int_{\mathbb{D}} \left|\frac{1}{1-\sigma_a(z)}\right|^2 |\sigma_a'(z)|^2 U_\mu(z) dA(z)\\ &= \lim_{a\to 1, a\in\mathbb{R}} \int_{\mathbb{D}} \left|\frac{1+a}{(1+z)(1-az)}\right|^2 U_\mu(z) dA(z)\\ &= \int_{\mathbb{D}} \left|\frac{2}{1-z^2}\right|^2 U_\mu(z) dA(z)\,.\end{aligned}$$

□

By using Proposition 6.3, we are now able to prove Theorem 1.3.

Proof of Theorem 1.3. As we saw earlier (3.3), we have that

$$(1-z)\mathcal{H}(f)'(z) = b(z),$$

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where $b \in H^\infty$ and $\|b\|_{H^\infty} \leq \|f\|_{H^\infty}$. Consequently,

$$\begin{aligned}\|\mathcal{H}(f)\|_{M(\mathcal{D}_\mu)} &= |\mathcal{H}(f)(0)| + \sup_{\phi\in \operatorname{Aut}(\mathbb{D})}\left(\int_{\mathbb{D}} |\mathcal{H}(f)'(w)|^2 U_\mu(\phi(w))dA(w)\right)^{1/2}\\ &= \left|\int_0^1 f(t)dt\right| + \sup_{\phi\in \operatorname{Aut}(\mathbb{D})}\left(\int_{\mathbb{D}} \left|\frac{b(w)}{1-w}\right|^2 U_\mu(\phi(w))dA(w)\right)^{1/2}\\ &\leq (1+\|\log(1-z)\|_{M(\mathcal{D}_\mu)})\|f\|_{H^\infty}.\end{aligned}$$

By Proposition 6.3, we have that

$$\|\mathcal{H}\|_{H^\infty\to M(\mathcal{D}_\mu)} \leq 1 + \left(\int_{\mathbb{D}} \frac{4}{|1-z^2|^2} U_\mu(z)dA(z)\right)^{1/2}. \tag{6.1}$$

Once again, we recall that

$$z\left(\mathcal{H}(1)\right)'(z) = -\frac{1}{z}\log\left(\frac{1}{1-z}\right) + \frac{1}{1-z} = f_1(z) + f_2(z). \tag{6.2}$$

We observe that the function $h(z) = (1-z)f_1(z)$ is in H^∞ with

$$\lim_{z\to 1} h(z) = 0.$$

Let $0 < a < 1$ and set

$$\begin{aligned}I^a = \int_{\mathbb{D}} |f_1(z)|^2 U_\mu(\sigma_a(z))dA(z) &= \int_{\mathbb{D}} \left|\frac{h(z)}{(1-z)}\right|^2 U_\mu(\sigma_a(z))dA(z)\\ &= \int_{\mathbb{D}} \left|\frac{h(\sigma_a(z))}{1-\sigma_a(z)}\right|^2 |\sigma_a'(z)|^2 U_\mu(z)dA(z)\\ &= \int_{\mathbb{D}} g_a(z)dA(z),\end{aligned}$$

where

$$g_a(z) = |h(\sigma_a(z))|^2 \frac{(1+a)^2}{|(1-az)(1+z)|^2} U_\mu(z).$$

For $z \in \mathbb{D}$ fixed observe that $\lim_{a\to 1}\sigma_a(z) = 1$. This implies that for each fixed $z \in \mathbb{D}$, $g_a(z) \to 0$ as $a \to 1$. Since $h \in H^\infty$ with norm M, we have

$$g_a(z) \leq k_a(z) = M^2 \frac{(1+a)^2}{|(1-az)(1+z)|^2} U_\mu(z).$$

Moreover

$$k_a(z) \to \frac{4M^2}{|1-z^2|^2} U_\mu(z),$$

as $a \to 1$ and, by repeating the proof of Proposition 6.3,

$$\int_{\mathbb{D}} k_a(z)dA(z) \to \int_{\mathbb{D}} \frac{4M^2}{|1-z^2|^2} U_\mu(z)dA(z) < \infty.$$

Therefore by dominated convergence [18, p.59; Ex.20], we have that

$$\lim_{a\to 1} I^a = 0. \tag{6.3}$$

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According to Proposition 6.3, we have that

$$\begin{aligned}\|\mathcal{H}(1)\|_{M(\mathcal{D}_\mu)} &= |\mathcal{H}(1)(0)| + \sup_{\phi\in\mathrm{Aut}(\mathbb{D})}\left(\int_{\mathbb{D}} |\mathcal{H}(1)'(w)|^2 U_\mu(\phi(w))dA(w)\right)^{1/2}\\ &\geq 1 + \sup_{a\in\mathbb{D}}\left(\int_{\mathbb{D}} |z\mathcal{H}(1)'(w)|^2 U_\mu(\sigma_a(w))dA(w)\right)^{1/2}\\ &\geq 1 + \lim_{\mathbb{R}\ni a\to 1}\left(\int_{\mathbb{D}} |f_1(w)+f_2(w)|^2 U_\mu(\sigma_a(w))dA(w)\right)^{1/2}.\end{aligned}$$

By using (6.3), the last lim inf becomes

$$\begin{aligned}&\lim_{\mathbb{R}\ni a\to 1}\left(\int_{\mathbb{D}} |f_1(w)+f_2(w)|^2 U_\mu(\sigma_a(w))dA(w)\right)^{1/2}\\ &\quad\geq \lim_{\mathbb{R}\ni a\to 1}\left|\left(\int_{\mathbb{D}} |f_1(w)|^2 U_\mu(\sigma_a(w))dA(w)\right)^{1/2} - \left(\int_{\mathbb{D}} |f_2(w)|^2 U_\mu(\sigma_a(w))dA(w)\right)^{1/2}\right|\\ &\quad= \lim_{\mathbb{R}\ni a\to 1}\left(\int_{\mathbb{D}} \left|\frac{1}{1-w}\right|^2 U_\mu(\sigma_a(w))dA(w)\right)^{1/2}.\end{aligned}$$

This implies that

$$\begin{aligned}\|\mathcal{H}(1)\|_{M(\mathcal{D}_\mu)} &\geq 1 + \lim_{\mathbb{R}\ni a\to 1}\left(\int_{\mathbb{D}} \left|\frac{1}{1-w}\right|^2 U_\mu(\sigma_a(w))dA(w)\right)^{1/2}.\\ &= 1 + \lim_{\mathbb{R}\ni a\to 1}\left(\int_{\mathbb{D}} \left|\frac{1}{1-\sigma_a(w)}\right|^2 |\sigma_a'(w)|^2 U_\mu((w))dA(w)\right)^{1/2}\\ &\geq 1 + \left(\int_{\mathbb{D}} \frac{4}{|1-w^2|^2} U_\mu(w)dA(w)\right)^{1/2}.\end{aligned}$$

□

A significant application of Theorem 1.3 to Q_p spaces with $\|\cdot\|_{M(\mathcal{D}_{\mu_p})}$ provides the norm of $\mathcal{H}$ in this situation.

Corollary 6.4. *The norm of $\mathcal{H} : H^\infty \to Q_p$ is equal to*

$$\|\mathcal{H}\|_{H^\infty\to Q_p} = 1 + \|\log(1-z)\|_{M(\mathcal{D}_{\mu_p})}.$$

Proof. We apply Theorem 1.3. We recall that, if $0 < p < 1$, then $Q_p = M(\mathcal{D}_{\mu_p})$ where, according to (2.3),

$$d\mu_p(z) = -\Delta[(1-|z|^2)^p]dA(z) = 4p\left(1-|z|^2 p\right)(1-|z|^2)^{p-2}dA(z).$$

□

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Remark 6.5. We note that, if we consider Q_p with the norm $\|\cdot\|_{Q_p}$ defined as in (2.2), with exactly the same reasoning of Theorem 1.3 we obtain that

$$\|\mathcal{H}\|_{H^\infty \to Q_p} = 1 + \|\log(1-z)\|_{Q_p}.$$

Indeed $(1-|z|^2)^p dA(z)$ is a radial measure.

7. The norm of he Cesaro operator

We use the following identity

$$\int_{\mathbb{D}} \left(\int_0^{2\pi} |f(e^{i\theta})|^2 P_\zeta(e^{i\theta}) \frac{d\theta}{2\pi} - |f(\zeta)|^2 \right) d\mu(\zeta) = \int_{\mathbb{D}} |f'(z)|^2 U_\mu(z)\, dA(z)$$

where μ is a positive Borel measure such that $(1-|z|^2)d\mu(z)$ is finite.

Proof of Theorem 1.4. Firstly observe that $\mathcal{C}(1)(z) = -\frac{1}{z}\log(1-z)$. As for the Hilbert matrix operator, this implies that

$$\|\mathcal{C}\|_{H^\infty \to M(\mathcal{D}_\mu)} \geq 1 + \left(\int_{\mathbb{D}} \frac{4}{|1-z^2|^2} U_\mu(z) dA(z) \right)^{1/2}.$$

Let $f \in H^\infty$. Set $h(z) = z\,\mathcal{C}(f)(z)$, then $(1-z)h'(z) = f(z)$. We have that

$$||\mathcal{C}f||_{M(D_\mu)} = |\mathcal{C}(f)(0)| + \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |\mathcal{C}(f)'(z)|^2 U_\mu(\sigma_a(z)) dA(z) \right)^{1/2},$$

and that $|\mathcal{C}(f)(0)| \leq ||f||_{H^\infty}$. For the integral we work as follows. Let $\mathcal{C}f = g$; then

$$\begin{aligned}
&\int_{\mathbb{D}} |g'(z)|^2 U_\mu(\sigma_a(z)) dA(z) = \int_{\mathbb{D}} |(g\circ\sigma_a)'(z)|^2 U_\mu(z) dA(z) \\
&= \int_{\mathbb{D}} \left(\int_0^{2\pi} |g(\sigma_a(e^{i\theta}))|^2 P_\zeta(e^{i\theta}) \frac{d\theta}{2\pi} - |g(\sigma_a(\zeta))|^2 \right) d\mu(\zeta) \\
&= \int_{\mathbb{D}} \left(\int_0^{2\pi} |h(\sigma_a(e^{i\theta}))|^2 P_\zeta(e^{i\theta}) \frac{d\theta}{2\pi} - |h(\sigma_a(\zeta))|^2 + |h(\sigma_a(\zeta))|^2 - |g(\sigma_a(\zeta))|^2 \right) d\mu(\zeta) \\
&= \int_{\mathbb{D}} |h'(z)|^2 U_\mu(\sigma_a(z)) dA(z) - \int_{\mathbb{D}} (1-|\sigma_a(\zeta)|^2) |g(\sigma_a(\zeta))|^2 d\mu(\zeta) \\
&\leq \int_{\mathbb{D}} |h'(z)|^2 U_\mu(\sigma_a(z)) dA(z).
\end{aligned}$$

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Since $h'(z)(1-z) = f(z)$ and due to Proposition 6.3, we have

$$||\mathcal{C}f||_{M(D_\mu)} \leq ||f||_{H^\infty} + \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} \left| \frac{f(z)}{1-z} \right|^2 U_\mu(\sigma_a(z)) dA(z) \right)^{1/2}$$
$$\leq \left(1 + \left(\int_{\mathbb{D}} \frac{4}{|1-z^2|^2} U_\mu(z) dA(z) \right)^{1/2} \right) ||f||_{H^\infty}.$$

□

8. Open questions

We note that for every radial measure μ that satisfies (2.1),

$$\|\mathcal{H}\|_{H^\infty \to M(\mathcal{D}_\mu)} = 1 + \left(\int_{\mathbb{D}} \frac{4}{|1-z^2|^2} U_\mu(z) dA(z) \right)^{1/2}.$$

However we are not able to compute the exact value of $\|\mathcal{H}\|_{H^\infty \to M(\mathcal{D}_\mu)}$ in general.

Question 8.1. What is the exact norm of $\mathcal{H}$ from H^∞ to $M(\mathcal{D}_\mu)$ for non-radial μ?

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