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ON THE TOPOLOGICAL ENTROPY OF TRANSITIVE MAPS OF THE INTERVAL

ETHAN M. COVEN AND MELISSA C. HIDALGO

The topological entropy of a continuous map of the interval is the supremum of the topological entropies of the piecewise linear maps associated to its finite invariant sets. We show that for transitive maps, this supremum is attained at some finite invariant set if and only if the map is piecewise monotone and the set contains the endpoints of the interval and the turning points of the map.

INTRODUCTION

This paper is concerned with finding lower bounds for the topological entropy of transitive maps of the interval, and determining when such bounds can be attained.

Topological entropy (*entropy* for short), denoted $\mathrm{ent}(\cdot)$, is a numerical conjugacy invariant of continuous maps. A continuous map is *transitive* if some point has a dense orbit. A *map of the interval* is a continuous map of a compact interval to itself.

We will determine lower bounds on entropy in terms of finite invariant sets. If $P = \{p_1 < \ldots < p_n\}$ is such a set, let f_P be the map defined on $[p_1, p_n]$ which agrees with f on P and which is linear on $[p_i, p_{i+1}]$ $(i = 1, \ldots, n-1)$. Then $\mathrm{ent}(f) = \sup\{\mathrm{ent}(f_P)\}$ where the supremum is taken over all finite invariant sets [**10**].

In section 2, we find some fairly crude bounds for the entropy of a transitive map of the interval in terms of the number and location of its fixed points.

In section 3 we show that for transitive maps of the interval, entropy bounds obtained from finite invariant sets can be attained only in very restrictive circumstances. A map of the interval is *piecewise monotone* if the ambient interval can be written as a finite union of closed subintervals with disjoint interiors on which the map is alternately strictly increasing and strictly decreasing. The *critical points* of such a map are the endpoints of these subintervals, that is, the turning points of the map and the endpoints of the ambient interval. We prove

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IBM Research Division, Almaden Research Centre, K65-802, 650 Harry Road, San Jose, CA 95120 USA

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THEOREM 3.1. *Let f be a transitive map of the interval and let P be a finite invariant set. Then* $\operatorname{ent}(f) \geqslant \operatorname{ent}(f_P)$, *with equality if and only if f is piecewise monotone and P contains the critical points of f.*

All that is new here is that the condition is necessary — see Lemma 1.1.

The reader is advised that, as far as this paper is concerned, the maxim "one picture is worth a thousand words" greatly undervalues pictures.

1. Background

For $f\colon X \to X$ and $x \in X$, $\operatorname{orb}(x)$ denotes the *orbit* of x, that is $\{f^n(x)\colon n \geqslant 0\}$. A subset E of X is *invariant (under f)* if $f(E) \subseteq E$. (The inclusion may be proper.)

We will need the following standard facts about transitivity and entropy:

(1) $f\colon X \to X$ is transitive if and only if the only closed invariant subset of X with non-empty interior is X itself;

(2) $\operatorname{ent}(f^n) = n\operatorname{ent}(f)$ $(n \geqslant 0)$;

(3) if $f\colon Y \cup Z \to Y \cup Z$ and both Y and Z are closed and invariant, then $\operatorname{ent}(f) = \max\{\operatorname{ent}(f|Y), \operatorname{ent}(f|Z)\}$.

Let f be a map of the interval and let $P = \{p_1 < \ldots < p_n\}$ be a finite subset of the ambient interval. (Here P need not be invariant.) By a *P-interval* we mean one of the $n-1$ intervals $[p_i, p_{i+1}]$, $i = 1, \ldots, n-1$. The *matrix* of P *(relative to f)* is the $(n-1)\times(n-1)$ matrix B, indexed by P-intervals, and defined by B_{IJ} is the largest non-negative integer k such that there are k subintervals $I_1, \ldots, I_k$ of I with pairwise disjoint interiors such that $f(I_i) = J$, $i = 1, \ldots, k$.

The Perron-Frobenius Theorem [8] asserts that among the eigenvalues of maximal modulus of a non-negative matrix B, there is one, denoted $\lambda(B)$, which is non-negative. In the sequel, we will abuse notation and declare $\log 0 = 0$.

LEMMA 1.1. ([4, 6]) *Let f be a map of the interval, let P be a finite subset of the ambient interval, and let A be the matrix of P. Then* $\operatorname{ent}(f) \geqslant \log\lambda(A)$, *with equality if P is invariant and contains the endpoints of the ambient interval, and f is monotone (but not necessarily strictly monotone) on each P-interval.*

An immediate consequence of Lemma 1.1 is

LEMMA 1.2. *Let f be a map of the interval, let P be a finite invariant set, and let A be the matrix of P. Then* $\operatorname{ent}(f_P) \leqslant \log\lambda(A)$.

2. Entropy and fixed points

In this section, we find lower bounds on the entropy of a transitive map of the interval in terms of the number and location of its fixed points.

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Theorem 2.1. [5, Lemma 1.2] *Let f be a transitive map of the interval. Then* $\operatorname{ent}(f) \geqslant (\log 2)/2$.

We say that f *has an n-horseshoe* if there is a finite invariant set $P = \{p_1 < \ldots < p_{n+1}\}$ such that

$$\begin{aligned} f(p_i) &= p_1 && \text{if } i \text{ is odd} \\ f(p_i) &= p_{n+1} && \text{if } i \text{ is even,} \end{aligned}$$

or

$$\begin{aligned} f(p_i) &= p_{n+1} && \text{if } i \text{ is odd} \\ f(p_i) &= p_1 && \text{if } i \text{ is even.} \end{aligned}$$

The matrix of such a set P is the $n \times n$ matrix each of whose entries is 1. Thus $\operatorname{ent}(f_P) = \log n$.

Theorem 2.2. [3, Lemma 3.3] *Let f be a transitive map of the interval. If f has at least two fixed points, then it has a 2-horseshoe and hence* $\operatorname{ent}(f) \geqslant \log 2$.

Theorem 2.3. *Let f be a transitive map of the interval. If the endpoints of the ambient interval are fixed points of f, then f has a 3-horseshoe and hence* $\operatorname{ent}(f) \geqslant \log 3$.

Proof: It is well-known that if

($\star$) there exist $w < x < y < z$ such that $f(w), f(y) \leqslant w$ and $f(x), f(z) \geqslant z$,

then f has a 3-horseshoe.

Let the ambient interval be $[a, b]$. Suppose first that $f^{-1}(a) \neq \{a\}$. Then $f(s) = a$ for some $s > a$. Let $u = \max f[a, s]$ and suppose that this maximum is attained at r. If $u = b$, then $a < r < s < b$ satisfy ($\star$). If $u < b$, let $\max f[a, u]$ be attained at t. Since $[a, u]$ isn't invariant, $f(t) > u$ and so $t > s$. Then $a < r < s < t$ satisfy ($\star$).

(For the reader interested only in piecewise monotone maps, the proof is complete. For it follows from [1, Lemma 2] that if $f(a) = a$ and f is transitive, then f^2 is transitive, too. But if, in addition, f is piecewise monotone, then by [7, Theorem B], $f^{-1}(a) \neq a$.)

Suppose then that $f^{-1}(a) = \{a\}$. Then a is a limit of fixed points. If not and p is the smallest fixed point greater than a, then $f(x) > x$ for every $x \in (a, p)$. Let $r = \min f[p, b]$. Since $f^{-1}(a) = \{a\}$, $r > a$, and since $[p, b]$ isn't invariant, $r < p$. But then $[r, b]$ is invariant.

Define $\{y_k\}$ inductively as follows: let y_0 be a fixed point in (a, b) and let $y_{k+1} = \min f[y_k, b]$. Then $\{y_k\}$ is decreasing and $y_k \to a$. For $k \geqslant 1$, $f(x_k) = y_{k+1}$ for some x_k, $y_k < x_k < y_{k-1}$. Then $x_k \to a$, too. Since a is a limit of fixed points, (a, x_1) contains a fixed point p. Let m be the smallest integer such that $x_m < p$. We may assume that there are no fixed points in (x_m, p), and hence that $f(x) < x$ for all

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$x \in (x_m, p)$. Let $\max f[a, p]$ be attained at z. Since $[a, p]$ isn't invariant, $f(z) > p$, and so $z < x_m$. Thus for some $k \geqslant m$, $x_{k+1} < z < x_k < p$, and these four points satisfy $(\star)$. ∎

3. Attainability of entropy bounds

In this section we prove

Theorem 3.2. *Let f be a transitive map of the interval and let P be a finite invariant set. Then $\operatorname{ent}(f) \geqslant \operatorname{ent}(f_P)$, with equality if and only if f is piecewise monotone and P contains the critical points of f.*

Recall that we need prove only that the condition is necessary. The following lemma shows that we may restrict our attention to maps whose square is transitive.

Lemma 3.2. *Suppose that f is transitive but f^2 is not, and let P be a finite invariant set. Then there is a map g such that g^2 is transitive and $\operatorname{ent}(g) = 2\operatorname{ent}(f)$, and a finite set Q, invariant under g, such that $\operatorname{ent}(g_Q) = 2\operatorname{ent}(f_P)$. Furthermore, g is piecewise monotone if and only if f is, and in this case, P contains the critical points of f if and only if Q contains the critical points of g.*

Proof: Let the ambient interval be $[a, b]$. By [1, Lemma 2], there is a fixed point $c \in (a, b)$ such that $f[a, c] = [c, b]$, $f[c, b] = [a, c]$, and $f^2|[a, c]$ is transitive. Let $g = f^2|[a, c]$. [1, Lemma 2] applied to g shows that g^2 is transitive. It is immediate that $\operatorname{ent}(g) = 2\operatorname{ent}(f)$ and that g is piecewise monotone if and only if f is.

We may assume that $c \in P$. If not, replace P by $P' = P \cup \{c\}$. Since P is invariant under $f_{P'}$, $\operatorname{ent}(f_P) \leqslant \operatorname{ent}(f_{P'})$. On the other hand, if $p_i < c < p_{i+1}$, then f_P has a fixed point $p \in [p_i, p_{i+1}]$. Then $P \cup \{p\}$ is a "copy" of P' which is invariant under f_P. Therefore $\operatorname{ent}(f_{P'}) \leqslant \operatorname{ent}(f_P)$.

Suppose that $P = \{p_1 < \ldots < p_n\}$ and that $c = p_k$. If f is piecewise monotone and P contains the critical points of f, then it is easy to see that $Q = [P \cup f^{-1}(P)] \cap [a, c]$ is invariant under g and that Q contains the critical points of g. Since $[p_1, p_k]$ and $[p_k, p_n]$ are invariant under f_P^2 and mapped to each other by f_P, and $g_Q = f_P^2 \mid [p_1, p_k]$, it follows that $\operatorname{ent}(g_Q) = 2\operatorname{ent}(f_P)$.

Suppose that f is not piecewise monotone, or that it is but P does not contain the critical points of f. In either case, Q will be $P \cap [a, c]$ together with a carefully chosen subset of $f^{-1}(P \cap [c, b])$. We define $Q \cap (p_i, p_{i+1})$ for $i = 1, \ldots, k-1$. If $f(p_i) = f(p_{i+1})$, let $Q \cap (p_i, p_{i+1}) = \emptyset$. If $f(p_i) < f(p_{i+1})$ and $f_P(p_i, p_{i+1}) \cap \{p_k, \ldots, p_n\} \neq \emptyset$, then this intersection is of the form $\{p_{j+1}, \ldots, p_{j+m}\}$ where $k+1 \leqslant j+1 \leqslant j+m \leqslant n-1$. In this case, let $x_0 = p_i$, and define x_r $(r = 1, \ldots, m)$ inductively by $x_r = \min\{x > x_{r-1} : f(x) = p_{j+r}\}$. Let $Q \cap (p_i, p_{i+1}) = \{x_1 < \ldots < x_m\}$. Make the obvious modification if $f(p_i) > f(p_{i+1})$. (If f is piecewise monotone and P contains

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the critical points of f, then this construction and the one in the preceding paragraph yield the same set Q.)

Let f be piecewise monotone, and suppose that P does not contain the critical points of f. We show that Q does not contain the critical points of g. First note that $c \in Q$, and if $a \notin P$, then $a \notin Q$. So we may assume that $a \in P$. Suppose that $b \notin P$. Then $f(a) \neq b$ and every member of $f^{-1}(b)$ is a turning point of g. But $f(Q) \subseteq P$, so no member of $f^{-1}(b)$ can be in Q. Thus we may assume also that $b \in P$. By examining cases, it can be seen that if f has a turning point in (a, c) which is not in P, then f (and hence g) has a turning point in (a, c) which is not in Q. (For example, if $x \in (p_i, p_{i+1})$, $f(p_i) < f(x) < f(p_{i+1})$, and f has a maximum at x, then the next largest turning point y of f is also in (p_i, p_{i+1}) and f has a minimum at y. But then $y \notin Q$.) If f has a turning point $x \in (c, b)$ which is not in P, then every $y \in f^{-1}(x)$ is a turning point of g and no such point can be in Q. $\square$

Lemma 3.3. *If $f\colon [a, b] \to [a, b]$ and f^2 is transitive, then*

($\star$) *for every interval J, there exists n such that $f^n(J) = [a, b]$,*

unless f is not piecewise monotone and one of the following holds.

(1) $f^{-1}(a) = \{a\}$ *and* $f^{-1}(b) \neq \{b\}$.
(2) $f^{-1}(b) = \{b\}$ *and* $f^{-1}(a) \neq \{a\}$.
(3) $f^{-1}\{a, b\} = \{a, b\}$.

However, if (1) holds, then for every subinterval J and every closed subinterval $K \subseteq (a, b]$, there exists n such that $f^n(J) \supseteq K$. Similar statements are true if (2) or (3) hold.

Proof: By [**7**, Theorem B], if f is piecewise monotone and f^2 is transitive, then ($\star$) holds. On the other hand, if f^2 is transitive but f is not piecewise monotone, then by [**2**, Theorem 6], for every interval J and every closed interval $K \subseteq (a, b)$, there exists n such that $f^n(J) \supseteq K$. The rest of the lemma now follows easily. $\square$

Lemma 3.4. *Let f be a map of the interval, let P be a finite invariant set, and let A be the matrix of P. Let Q be a finite subset of the ambient interval and let B be the matrix of Q relative to f^n. If $B > 0$ and either $B \geqslant A^n$ but $B \neq A^n$, or B has a proper submatrix $B' \geqslant A^n$, then $\operatorname{ent}(f) > \operatorname{ent}(f_P)$.*

Proof: By standard Perron-Frobenius arguments [**8**], $\lambda(B) > \lambda(A^n)$. Then we have

$$n \operatorname{ent}(f) = \operatorname{ent}(f^n) \geqslant \log \lambda(B) > \log \lambda(A^n) \geqslant n \log \lambda(A) \geqslant n \operatorname{ent}(f_P).$$

$\square$

Lemma 3.5. *Let $f\colon [a, b] \to [a, b]$. If f^2 is transitive but ($\star$) does not hold, then $\operatorname{ent}(f) > \operatorname{ent}(f_P)$ for every finite invariant set P.*

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PROOF: By Lemma 3.3, one of (1)–(3) holds. Suppose (1) holds.

Let A be the matrix of P relative to f_P and let $P' = P - \{a\}$ and A' the matrix of P' relative to $f_{P'}$. Then $\mathrm{ent}(f_{P'}) = \mathrm{ent}(f_P)$, for if $a \notin P$, then $P' = P$, and if $a \in P$, then A' is obtained from A by deleting the row and column corresponding to $[a, p]$, where p is the smallest member of P greater than a. (1) implies that each entry in this column is zero except for the diagonal entry. Thus $\lambda(A') = \lambda(A)$ and hence $\mathrm{ent}(f_{P'}) = \mathrm{ent}(f_P)$.

Let $c = \min f[p, b]$. Since (1) holds, $c > a$, and since $[p, b]$ isn't invariant, $c < p$. Let $Q = (P - \{a\}) \cup \{c\}$. By Lemma 3.3, there exists n such that $f^n(J) \supseteq [c, b]$ for every Q-interval J. Let B be the matrix of Q relative to f^n. Then $B > 0$ and its submatrix B', obtained by deleting the row and column corresponding to $[c, p]$, satisfies $B' \geqslant (A')^n$. By Lemma 3.4, $\mathrm{ent}(f) > \mathrm{ent}(f_{P'})$.

Similar arguments work if (2) or (3) hold. □

PROOF OF THEOREM 3.1: As noted above, we need prove only that if f is transitive, P is a finite invariant set, and $\mathrm{ent}(f) = \mathrm{ent}(f_P)$, then f is piecewise monotone and P contains the critical points of f.

Let the ambient interval be $[a, b]$. By Lemma 3.2, we may assume that f^2 is transitive, and so by Lemma 3.5, $(\star)$ holds.

Suppose that $a \notin P$. Let $Q = P \cup \{a\}$. By $(\star)$, there exists n such that $f^n(J) = [a, b]$ for every Q-interval J. Let B be the matrix of Q relative to f^n. Then $B > 0$ and B contains a submatrix $B' \geqslant A^n$. By Lemma 3.4, $\mathrm{ent}(f) > \mathrm{ent}(f_P)$. Therefore $a \in P$. Similarly, $b \in P$.

Suppose that $a, b \in P$, but that f is not one-to-one on some P-interval $[p, q]$. There exist $u < v$ in $[p, q]$ with $f(u) = f(v)$. Since f is transitive, it is not constant on $[u, v]$. Therefore $f[u, v]$ is a non-degenerate interval, and hence either $f(u) < \max f[u, v]$ or $f(u) > \min f[u, v]$. Without loss of generality, assume the former, and let this maximum be attained at w.

It follows from a result of Sarkovskij [9] — the Birkhoff centre of a map of the interval is the closure of its periodic points — that the periodic points of f are dense. Let $x \in (u, v)$ be a periodic point such that $w \notin \mathrm{orb}(x)$ and there exists a point $y \in (u, v)$ such that $x \neq y$, $f(x) = f(y)$, and no member of $\mathrm{orb}(x)$ lies between x and y. Let $Q = P \cup \mathrm{orb}(x) \cup \{y\}$, and let C be the matrix of Q relative to f_Q. By $(\star)$, there exists n such that $f^n(J) = [a, b]$ for every Q-interval J. Let B be the matrix of Q relative to f^n. Then $B > 0$, $B \geqslant C^n$, and C^n has a row (corresponding to the Q-interval whose endpoints are x and y) consisting of zeros. By Lemma 3.4, $\mathrm{ent}(f) > \mathrm{ent}(f_Q)$.

Since P is invariant under f_Q, it follows from Lemmas 1.1 and 1.2 that $\mathrm{ent}(f_Q) \geqslant \mathrm{ent}(f_P)$, and hence $\mathrm{ent}(f) > \mathrm{ent}(f_P)$. This is a contradiction, and so

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f is strictly monotone on every P-interval. Therefore f is piecewise monotone and P contains all the turning points of f. $\square$

An easy consequence of Theorems 2.2 and 3.1 is: if f is transitive and has at least three fixed points, then $\text{ent}(f) > \log 2$. On the other hand, there are transitive maps with arbitrarily many fixed points and entropy arbitrarily close to $\log 2$.

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Department of Mathematics
Wesleyan University
Middletown, CT 06457
United States of America

Department of Mathematics
University of Hartford
West Hartford, CT 06117
United States of America

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