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A PROOF OF HIGGINS'S CONJECTURE

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Let $\Theta\colon G = \mathop{\prod^{*}}\limits_{\lambda\in\Lambda} G_\lambda \to B = \mathop{\prod^{*}}\limits_{\lambda\in\Lambda} B_\lambda$ be a group homomorphism between free products of groups such that $G_\lambda\Theta = B_\lambda$ for all $\lambda \in \Lambda$. Let $H \subseteq G$ be a subgroup such that $H\Theta = B$. Then $H = \mathop{\prod^{*}}\limits_{\lambda\in\Lambda} H_\lambda$ such that $H_\lambda\Theta = B_\lambda$ and $H_\lambda = \prod^{*}(H \cap G_\lambda^{\beta_{\lambda,\mu}}) * F_\lambda$ where F_λ is free.

1. INTRODUCTION

Recall that the free product of groups G_λ is the group $\mathop{\prod^{*}}\limits_{\lambda\in\Lambda} G_\lambda$ generated by the G_λ in which every relation follows from group identities. In other words, free product is the same as the coproduct in the category of groups. We use $\prod^{*}$ or $*$ to denote free products.

Let $H^x := x^{-1}Hx$ denote a conjugate of a group H. There are two main theorems about subgroups of free products of groups:

THEOREM 1.1. (Kuroš's Theorem) *Let $H \subseteq \mathop{\prod^{*}}\limits_{\lambda\in\Lambda} G_\lambda$ be a subgroup of a free product. Then H has a free decomposition $H = \mathop{\prod^{*}}\limits_{\lambda\in\Lambda, x_\lambda} (H \cap G_\lambda^{x_\lambda}) * F$ where for each λ the x_λ runs through a suitable set of representatives of double cosets $G_\lambda x H$ such that $G_\lambda H$ is represented by 1. Moreover, F is free.*

THEOREM 1.2. (Higgins's Theorem) *Let $\Theta\colon G = \mathop{\prod^{*}}\limits_{\lambda\in\Lambda} G_\lambda \to B = \mathop{\prod^{*}}\limits_{\lambda\in\Lambda} B_\lambda$ be a group homomorphism such that $G_\lambda\Theta = B_\lambda$ for all $\lambda \in \Lambda$. Let $H \subseteq G$ be a subgroup such that $H\Theta = B$. Then there are groups H_λ such that $H = \mathop{\prod^{*}}\limits_{\lambda\in\Lambda} H_\lambda$ and $H_\lambda\Theta = B_\lambda$.*

Higgins proved the above theorems in [**3**, Chapter 14] using groupoids. These proofs are similar and Higgins conjectured that they can be united to a single proof of a common generalisation of the two theorems.

However, Heath and Nickolas showed in [**1**] that there are difficulties in generalising the proof and, in particular, Ordman's proof of Higgins's conjecture in [**4**] is incorrect.

Nevertheless, we give a simple proof of Higgins's conjecture in this paper using the two theorems above.

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THEOREM 1.3. (Higgins's conjecture) *Let* $\Theta\colon G = \prod^{*}_{\lambda\in\Lambda} G_\lambda \to B = \prod^{*}_{\lambda\in\Lambda} B_\lambda$ *be a group homomorphism such that* $G_\lambda\Theta = B_\lambda$ *for all* $\lambda \in \Lambda$. *Let* $H \subseteq G$ *be a subgroup such that* $H\Theta = B$. *Then* $H = \prod^{*}_{\lambda\in\Lambda} H_\lambda$ *such that* $H_\lambda\Theta = B_\lambda$ *where* $H_\lambda = \prod^{*}_{x_\lambda}(H \cap G_\lambda^{x_\lambda}) * F_\lambda$ *such that* $x_\lambda\Theta = 1$ *for all* x_λ, *and for each* λ *the* x_λ *runs through a suitable set of representatives of double cosets* $G_\lambda x H$ *such that* $G_\lambda H$ *is represented by* 1. *Furthermore, the* F_λ *are free.*

Obviously, the H_λ in the conjecture satisfy the requirements of Higgins's Theorem. Therefore to prove the conjecture, one "only" has to decompose the H_λ provided by Higgins's Theorem. Fortunately, this is easy to do for the H_λ in Higgins's proof: the Kuroš's Theorem just provides the right decomposition. This is what we are going to do.

The relevant additional property of the H_λ in Higgins's proof is that their intersection with G_λ is contained in a conjugate of H_λ, see Lemma 3.1. This is proved by some additional arguments to Higgins's proof.

Therefore we recall briefly Higgins's proof of his theorems in Section 2 as done in [**1**]. This will make the proof of our main lemma understandable to the reader not familiar with the groupoid proofs.

In Section 3 we formulate a stronger version of Higgins's Theorem as our main lemma. Then we prove the lemma and Higgins's conjecture.

2. Groupoid method

In this section we recall briefly Higgins's proof of Theorems 1.1 and 1.2. See [**3**, Chapter 14] for full details. We follow the discussion in [**1**].

Recall that a groupoid is a category in which every morphism is invertible. Any group G can be regarded as a groupoid with one object such that the automorphism group of the object is G. If H is a subgroup of G then the *standard covering* $\gamma\colon \widetilde{G} \to G$ is a functor defined as follows. First, we define the groupoid $\widetilde{G}$. The objects are the right cosets of H in G. Morphisms of $\widetilde{G}$ are $(N, g)\colon N \to Ng$ where N is an arbitrary right coset and $g \in G$. Composition is defined by $(N, g) \circ (Ng, h) := (N, gh)$. For example, the automorphism group of the coset H in $\widetilde{G}$ is isomorphic to H via the map $(H, h) \mapsto h$. Finally, γ is given by the formula $(N, g)\gamma := g$.

We shall also think of a groupoid as an oriented graph where the vertices are the objects and the edges are the morphisms. In this sense, we shall speak of connected groupoids, trees and so on.

We shall use free product of groupoids: let G_λ be groupoids whose objects are contained in a set S. The free product of G_λ is the groupoid generated by the G_λ in which only the necessary relations hold. The objects of the free product are the objects of all the G_λ and hence is contained in S. Free product is similar to coproduct but

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some objects are identified; that is the role of S. For example, every groupoid is the free product of its connected components. If S is a one-element set, this notion is exactly the free product of groups. In the following, S is always the set of objects of the standard covering of a subgroup H of a group G.

If C is a connected groupoid and τ is a spanning tree then τ generates a *wide tree subgroupoid* T of C, that is a subgroupoid in which there is exactly one morphism between any two objects. Then C is isomorphic to the free product and the direct product of H and T, where H is the automorphism group of an object. The canonical projection $\rho_\tau = \rho\colon C \cong H * T = H \times T \to H$ is given by $(N, h) \mapsto \beta_N^{-1} \cdot h \cdot \beta_{Nh}$. Here β_N denotes the unique isomorphism $N \to H$ in T.

Let us suppose now that $G = \prod^*_{\lambda\in\Lambda} G_\lambda$ is a free product of groups. Now the idea of the groupoid proofs of Kuroš's Theorem and Higgins's Theorem is that the free decomposition of G lifts to a free decomposition

$$\widetilde{G} = \prod^*_{\lambda\in\Lambda} \widetilde{G_\lambda} \tag{2.1}$$

where $\widetilde{G_\lambda} := G_\lambda \gamma^{-1}$. Using a suitable tree τ, the projection ρ maps this decomposition to a free decomposition of H which will satisfy the theorems.

In case of Higgins's Theorem (Theorem 1.2), let $\Theta\colon G = \prod^*_{\lambda\in\Lambda} G_\lambda \to B = \prod^*_{\lambda\in\Lambda} B_\lambda$ be a homomorphism such that $G_\lambda\Theta = B_\lambda$ and $H\Theta = B$. We choose the tree τ such that the wide tree subgroupoid T generated by τ is contained in $\ker\gamma\Theta$ (the full subgroupoid of $\widetilde{G}$ consisting of morphisms mapped to identity by $\gamma\Theta$). If τ is chosen with care, we shall have

$$H = \prod^* \underbrace{\widetilde{G_\lambda}\rho}_{H_\lambda}. \tag{2.2}$$

The condition $T \subseteq \ker\gamma\Theta$ will guarantee $H_\lambda\Theta \subseteq B_\lambda$. See [**2**] or [**3**] for more details.

In case of Kuroš's Theorem, we first decompose each $\widetilde{G_\lambda}$ into its connected components $\widetilde{G_{\lambda,\mu}}$, which we further decompose to the group $K_{\lambda,\mu}$ of one of its objects and a wide tree subgroupoid generated by a tree $\tau_{\lambda,\mu}$. This leads to the free decomposition:

$$\widetilde{G} = \prod^*_{\lambda,\mu} K_{\lambda,\mu} * F(\cup\tau_{\lambda,\mu}) \tag{2.3}$$

where $F(X)$ denotes the groupoid freely generated by the morphisms in X. Note that a tree always generates a wide tree subgroupoid freely.

It is easy to see that $\cup\tau_{\lambda,\mu}$ is connected and hence contains a spanning tree τ. Now ρ_τ gives the free decomposition:

$$H = \prod^*_{\lambda,\mu} K_{\lambda,\mu}\rho_\tau * F(\cup\tau_{\lambda,\mu} \setminus \tau). \tag{2.4}$$

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It is easily seen that $K_{\lambda,\mu}\rho_\tau = H \cap G_\lambda^{x_{\lambda,\mu}}$ for some $x_{\lambda,\mu}$ and $F(\cup\tau_{\lambda,\mu} \setminus \tau)$ is a free group so this gives the Kuroš decomposition of H. Actually, $(H, x_{\lambda,\mu}^{-1})$ is the unique isomorphism in $F(\tau)$ between H and the object at which $K_{\lambda,\mu}$ is located. An easy argument, which we omit, yields that the $x_{\lambda,\mu}$ form a set of representatives of double cosets $G_\lambda x H$. If we have chosen $K_{\lambda,\mu}$ at the object H whenever $\widetilde{G_{\lambda,\mu}}$ contains H, then the coset $G_\lambda H$ will be represented by 1.

Higgins conjectured that both theorems can be proved using a common τ, which would lead to a common generalisation of both theorems and their proofs. In [1] it is shown that in general there is no tree τ which is contained in both $\cup\tau_{\lambda,\mu}$ and $\ker\gamma\Theta$, so such a generalisation requires significant changes to the above proofs.

3. Proof of Higgins's conjecture

First we prove that the H_λ in Theorem 1.2 has some nice properties.

Lemma 3.1. (Generalisation of Higgins's Theorem) *Suppose that a group homomorphism* $\Theta\colon G = \prod^*_{\lambda\in\Lambda} G_\lambda \to B = \prod^*_{\lambda\in\Lambda} B_\lambda$ *between free products satisfies* $G_\lambda\Theta = B_\lambda$ *for all* $\lambda \in \Lambda$. *Let* $H \subseteq G$ *be a subgroup such that* $H\Theta = B$. *Then* $H = \prod^*_{\lambda\in\Lambda} H_\lambda$ *such that for each* λ *we have* $H_\lambda\Theta = B_\lambda$, *and there are representatives* $\beta_{\lambda,\mu}$ *of double cosets* $G_\lambda x H$ *such that* $H \cap G_\lambda^{\beta_{\lambda,\mu}} \subseteq H_\lambda$ *and* $\beta_{\lambda,\mu} \in \ker\Theta$. *Moreover,* $G_\lambda H$ *can be represented by* 1 *for all* λ *simultaneously.*

Proof: We combine the ideas of Higgins's Theorem and Kuroš's Theorem from Section 2 together.

We start with the proof of Higgins's Theorem and thus obtain a free decomposition of H into the $H_\lambda = \widetilde{G_\lambda}\rho_\tau$. Now we use the proof of Kuroš's Theorem for the tree τ. We decompose $\widetilde{G_\lambda}$ into its connected components $\widetilde{G_{\lambda,\mu}}$ and from every $\widetilde{G_{\lambda,\mu}}$ we select the automorphism group $K_{\lambda,\mu}$ of an object. Thus $K_{\lambda,\mu}\rho_\tau \subseteq H_\lambda$. It is not obvious whether we obtain a free decomposition like (2.4) but we still have $K_{\lambda,\mu}\rho_\tau = H \cap G_\lambda^{\beta_{\lambda,\mu}}$ for some representatives $\beta_{\lambda,\mu}$ of double cosets $G_\lambda x H$. We also have $(H, \beta_{\lambda,\mu}^{-1}) \in F(\tau) \subseteq \ker\gamma\Theta$. Hence $\beta_{\lambda,\mu} \in \ker\Theta$. The coset $G_\lambda H$ is represented by 1 if we choose $K_{\lambda,\mu}$ at the object H when $\widetilde{G_{\lambda,\mu}}$ contains the object H.

Thus the $\beta_{\lambda,\mu}$ satisfy the theorem. ∎

This lemma together with Theorem 1.1 is enough to prove Higgins's conjecture without using groupoids.

Proof: (Proof of Theorem 1.3) The proof consists of two steps: first we decompose H into H_λ using Lemma 3.1 and, secondly, Kuroš's Theorem will give the required decomposition of H_λ.

By Lemma 3.1, we have a decomposition $H = \prod^*_{\lambda\in\Lambda} H_\lambda$ with $H_\lambda\Theta \subseteq B_\lambda$ such that $H \cap G_\lambda^{\beta_{\lambda,\mu}}$ is contained in H_λ for some representatives $\beta_{\lambda,\mu}$ of double cosets $G_\lambda x H$ and $\beta_{\lambda,\mu}\Theta = 1$. We do not claim that these representatives give a Kuroš type decomposition; we shall modify them.

Applying Theorem 1.1 to H_λ we obtain a Kuroš decomposition:

$$H_\lambda = \prod^*_{\varepsilon,\delta}(H_\lambda \cap G_\varepsilon^\delta) * F_\lambda, \tag{3.1}$$

where F_λ is free. We claim that this decomposition is exactly the decomposition of H_λ the theorem requires. First of all, F_λ will be the free component. Now we examine the other components.

For every pair ε, δ in (3.1) δ lies in a double coset $G_\varepsilon\beta_{\varepsilon,\mu}H$ that is

$$\delta = g\beta_{\varepsilon,\mu}h \quad \text{for some } g \in G_\varepsilon \text{ and } h \in H. \tag{3.2}$$

Then we have

$$H \cap G_\varepsilon^\delta = H \cap G_\varepsilon^{\beta_{\varepsilon,\mu}h} = \left(H \cap G_\varepsilon^{\beta_{\varepsilon,\mu}}\right)^h \subseteq H_\varepsilon^h. \tag{3.3}$$

Therefore

$$H_\lambda \cap G_\varepsilon^\delta = H_\lambda \cap \left(H \cap G_\varepsilon^\delta\right) \subseteq H_\lambda \cap H_\varepsilon^h = \begin{cases} H_\lambda & \text{if } \varepsilon = \lambda \text{ and } h \in H_\lambda, \\ 1 & \text{otherwise.} \end{cases} \tag{3.4}$$

In other words, $H_\lambda \cap G_\varepsilon^\delta$ is trivial unless $\varepsilon = \lambda$ and δ comes from a double coset $G_\lambda\beta_{\lambda,\mu}H_\lambda$, and in this case $H_\lambda \cap G_\varepsilon^\delta = H \cap G_\lambda^\delta$. So if we denote by $\beta'_{\lambda,\mu}$ the representative of $G_\lambda\beta_{\lambda,\mu}H_\lambda$ occurring in (3.1) then the free decomposition of H_λ reduces to, after omitting the components which (3.4) shows trivial:

$$H_\lambda = \prod^*_{\mu}(H \cap G_\lambda^{\beta'_{\lambda,\mu}}) * F_\lambda. \tag{3.5}$$

For each λ the elements $\beta'_{\lambda,\mu}$ obviously form a set of double coset representatives and $\beta'_{\lambda,\mu}\Theta \in (G_\lambda\beta_{\lambda,\mu}H_\lambda)\Theta = B_\lambda$. Since $G_\lambda\Theta = B_\lambda$, there are elements $g_{\lambda,\mu} \in G_\lambda$ such that $g_{\lambda,\mu}\Theta = \beta'_{\lambda,\mu}\Theta$. Setting $x_{\lambda,\mu} := g_{\lambda,\mu}^{-1} \cdot \beta'_{\lambda,\mu}$, we have $x_{\lambda,\mu} \in \ker\Theta$ and $G_\lambda^{\beta'_{\lambda,\mu}} = G_\lambda^{x_{\lambda,\mu}}$, hence

$$H_\lambda = \prod^*_{\mu}(H \cap G_\lambda^{x_{\lambda,\mu}}) * F_\lambda. \tag{3.6}$$

Moreover, the elements $x_{\lambda,\mu} \in G_\lambda\beta_{\lambda,\mu}H$ form a set of representatives of double cosets $G_\lambda x H$. For the unique μ with $\beta_{\lambda,\mu} = 1$, we have $\beta'_{\lambda,\mu} = 1$, and we may choose $g_{\lambda,\mu} = 1$. This implies $x_{\lambda,\mu} = 1$ and hence 1 occurs in the double coset representatives. ∎

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