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On the roots of polynomials with log-convex coefficients

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Abstract. In this paper, we consider the family of nth degree polynomials whose coefficients form a log-convex sequence (up to binomial weights), and investigate their roots. We study, among others, the structure of the set of roots of such polynomials, showing that it is a closed convex cone in the upper half-plane, which covers its interior when n tends to infinity, and giving its precise description for every $n \in \mathbb{N}$, $n \geq 2$. Dual Steiner polynomials of star bodies are a particular case of them, and so we derive, as a consequence, further properties for their roots.

1 Introduction

The volume of a measurable set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$ and, in particular, we write $\kappa_n := \mathrm{vol}(B_n)$ for the volume of the n-dimensional Euclidean unit ball B_n.

For two convex bodies (i.e., nonempty compact and convex sets) $K, E \subset \mathbb{R}^n$ and a non-negative real number λ, the volume of the Minkowski sum $K + \lambda E$ is a polynomial of degree at most n in λ, and it is written as

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)\lambda^i. \tag{1.1}$$

This expression is called the *Steiner formula* of K and E. The coefficients $\mathrm{W}_i(K;E)$ are the *relative quermassintegrals* of K w.r.t. E, and they are a special case of the more general *mixed volumes*, for which we refer to [17, Section 5.1]. In particular, $\mathrm{W}_0(K;E) = \mathrm{vol}(K)$ and $\mathrm{W}_n(K;E) = \mathrm{vol}(E)$.

If we regard the right-hand side of (1.1) as a formal polynomial in a complex variable $z \in \mathbb{C}$, the study of its roots has been investigated in several papers [4, 6–9, 11, 12]: topology of the cone of roots, monotonicity with respect to the dimension, stability, and so on. We emphasize that most of these results are based on the characterization of

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the relative quermassintegrals via the well-known Aleksandrov–Fenchel inequalities

$$\mathrm{W}_i(K;E)^2 \geq \mathrm{W}_{i-1}(K;E)\mathrm{W}_{i+1}(K;E), \quad i = 1, \ldots, n-1$$

(see, e.g., [17, (7.66)]). This characterization problem was solved in [18] and [8]: any given set of $n+1$ non-negative real numbers $\mathrm{W}_0, \ldots, \mathrm{W}_n \geq 0$ satisfying the inequalities $\mathrm{W}_i^2 \geq \mathrm{W}_{i-1}\mathrm{W}_{i+1}$, $1 \leq i \leq n-1$, arises as the set of relative quermassintegrals of two convex bodies. Such a tuple $(\mathrm{W}_0, \ldots, \mathrm{W}_n)$ is called *log-concave*. Therefore, Steiner polynomials are precisely those ones whose coefficients, up to the combinatorial numbers, form a log-concave tuple.

As a natural counterpart to the above issue, we consider the family of those nth degree polynomials whose coefficients (up to the combinatorial numbers) form a *log-convex* sequence. From now on, $n \geq 2$.

Definition 1.1 We say that a polynomial has *log-convex* coefficients if it is of the form

$$f_{\boldsymbol{\omega}}(z) = \sum_{i=0}^{n} \binom{n}{i} \omega_i z^i$$

with $\omega_i \geq 0$ and $\boldsymbol{\omega} = (\omega_0, \omega_1, \ldots, \omega_n)$ a *log-convex* tuple, i.e., satisfying $\omega_i^2 \leq \omega_{i-1}\omega_{i+1}$ for all $1 \leq i \leq n-1$.

As in the case of the Steiner polynomials, we are interested in investigating the structure and behavior of the set of roots of these *log-convex coefficients polynomials*.

In order to state our main results, we note first that if $\boldsymbol{\omega} = (\omega_0, \omega_1, \ldots, \omega_n)$ is a log-convex tuple, then

- either $\omega_i > 0$ for all $0 \leq i \leq n$
- or $\omega_1 = \cdots = \omega_{n-1} = 0$ and $\omega_0, \omega_n \geq 0$.

Indeed, if $\omega_i = 0$ for some $i \in \{0, \ldots, n\}$, then $\omega_{i+1} = 0$ if $i \leq n-2$ because $\omega_{i+1}^2 \leq \omega_i\omega_{i+2} = 0$, and $\omega_{i-1} = 0$ if $i \geq 2$ (now $\omega_{i-1}^2 \leq \omega_{i-2}\omega_i = 0$). Thus, we can distinguish two families of log-convex finite sequences: let

$$\mathcal{L}^n = \left\{\boldsymbol{\omega} = (\omega_0, \ldots, \omega_n) \text{ log-convex} : \omega_i \geq 0,\ i = 0, \ldots, n,\ \boldsymbol{\omega} \neq (0, \ldots, 0)\right\},$$
$$\mathcal{L}^n_{>0} = \left\{\boldsymbol{\omega} = (\omega_0, \ldots, \omega_n) \in \mathcal{L}^n : \omega_i > 0,\ i = 0, \ldots, n\right\}.$$

Clearly, $\mathcal{L}^n$ can be expressed as the disjoint union

$$\mathcal{L}^n = \mathcal{L}^n_{>0} \dot{\cup} \left\{\boldsymbol{\omega} = (\omega_0, 0, \ldots, 0, \omega_n) : \omega_0, \omega_n \geq 0,\ \boldsymbol{\omega} \neq (0, \ldots, 0)\right\}. \tag{1.2}$$

From now on, we will write $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$ and $\bar{z}$ to represent the real and imaginary parts, the modulus and the complex conjugate of $z \in \mathbb{C}$, respectively. Let $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$, and for $n \geq 2$ let

$$\mathcal{R}_{\mathcal{L}}(n) = \left\{z \in \mathbb{C}^+ : f_{\boldsymbol{\omega}}(z) = 0 \text{ for some } \boldsymbol{\omega} \in \mathcal{L}^n\right\} \text{ and}$$
$$\mathcal{R}_{\mathcal{L}_{>0}}(n) = \left\{z \in \mathbb{C}^+ : f_{\boldsymbol{\omega}}(z) = 0 \text{ for some } \boldsymbol{\omega} \in \mathcal{L}^n_{>0}\right\}.$$

It is known (see [8]) that the set of roots of all log-concave coefficients (i.e., Steiner) polynomials in $\mathbb{C}^+$ is a closed convex cone containing the non-positive real axis $\mathbb{R}_{\leq 0}$. In this paper, we show that log-convex coefficients polynomials share these properties.

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Theorem 1.1 *$\mathcal{R}_{\mathcal{L}}(n)$ is a convex cone containing the non-positive real axis $\mathbb{R}_{\leq 0}$. $\mathcal{R}_{\mathcal{L}_{>0}}(n)$ is a convex cone containing the negative real axis $\mathbb{R}_{<0}$.*

As usual in the literature, we represent by $\operatorname{int} M$, $\operatorname{cl} M$, $\operatorname{bd} M$, and $\operatorname{conv} M$, the interior, closure, boundary, and convex hull of M, respectively. We also stress that, from now on, any topological issue concerning subsets of $\mathbb{C}$ must be understood with respect to the standard topology.

Theorem 1.2 *The cone $\mathcal{R}_{\mathcal{L}}(n)$ is closed. Moreover, $\operatorname{cl}\mathcal{R}_{\mathcal{L}_{>0}}(n) = \mathcal{R}_{\mathcal{L}}(n)$ and $\operatorname{int}\mathcal{R}_{\mathcal{L}_{>0}}(n) = \operatorname{int}\mathcal{R}_{\mathcal{L}}(n)$.*

These theorems will play a key role in order to get the precise description of the cones $\mathcal{R}_{\mathcal{L}}(n)$ and $\mathcal{R}_{\mathcal{L}_{>0}}(n)$. When $n = 2$, it can be directly obtained from results in [1]:

$$\mathcal{R}_{\mathcal{L}_{>0}}(2) = \{z \in \mathbb{C}^+ : \operatorname{Re}(z) < 0\}$$

(see [1, Proof of Proposition 4.2]). Here, for arbitrary $n \in \mathbb{N}$, $n \geq 3$, we describe the cones $\mathcal{R}_{\mathcal{L}_{>0}}(n)$ and $\mathcal{R}_{\mathcal{L}}(n)$, and show that they are determined by the nth roots of -1:

Theorem 1.3 *Let $n \geq 3$. Then*

$$\mathcal{R}_{\mathcal{L}_{>0}}(n) = \{a + b\,\mathrm{i} \in \mathbb{C}^+ : b > \tan(\pi/n)a\} \quad \text{and}$$
$$\mathcal{R}_{\mathcal{L}}(n) = \{a + b\,\mathrm{i} \in \mathbb{C}^+ : b \geq \tan(\pi/n)a\}.$$

A prominent subset of this family of polynomials is the one consisting of the well-known *dual Steiner polynomials* (see Section 2 for its explicit definition), which have been also studied thoroughgoingly in the last years. So, many properties of this general family of log-convex coefficients polynomials will apply to the dual Steiner ones.

The paper is organized as follows: Section 2 is devoted to a brief introduction on the dual Steiner polynomials and its connection with our general family of polynomials; the main results that we obtain in this setting are also presented. Next, in the brief Section 3, we collect the classical properties on polynomials that will be needed in the proofs of our results. In Section 4, we study the structure of the set of roots of all log-convex coefficients polynomials, showing, among others, that it is a closed convex cone in the upper half-plane (Theorems 1.1 and 1.2), monotonic with respect to n, which covers its interior when n tends to infinity. We also give its precise description for all $n \in \mathbb{N}$, $n \geq 3$ (Theorem 1.3), and derive some consequences for dual Steiner polynomials. Finally, in Section 5, we get bounds for the roots of these polynomials in terms of the coefficients. Here, we also obtain a characterization of the Euclidean ball as the only star body such that all the roots of its dual Steiner polynomial have equal real part.

2 A brief tour on dual Steiner polynomials

An outstanding extension of the classical Brunn–Minkowski theory is obtained by replacing convex bodies and the classical Minkowski addition, by another family of sets and a different additive operation: the *dual Brunn–Minkowski theory* (see, e.g., [17, Section 9.3]), introduced by Lutwak in [13, 14], and based on the radial addition

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$x\widetilde{+}y$ for $x, y \in \mathbb{R}^n$, where

$$x\widetilde{+}y = \begin{cases} x+y, & \text{if } x, y \text{ are linearly dependent,} \\ 0, & \text{otherwise.} \end{cases}$$

In general, the radial sum $K\widetilde{+}E = \{x\widetilde{+}y : x \in K,\ y \in E\}$ of two convex bodies K, E is not a convex set, but the radial sum of two *star bodies* is again a star body. In order to define star bodies, we call a nonempty set $S \subset \mathbb{R}^n$ *starshaped* (with respect to the origin) if the segment $[0, x] \subset S$ for all $x \in S$. For a compact starshaped set K its *radial function* $\rho_K : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}_{\geq 0}$ is defined by $\rho_K(u) = \max\{\rho \geq 0 : \rho u \in K\}$, where, as usual, $\mathbb{S}^{n-1}$ represents the $(n-1)$-dimensional sphere. If this function is positive and continuous then K is called a *star body*. In particular, any star body has nonempty interior and any convex body containing the origin in its interior is a star body. We denote by $\mathcal{S}_0^n$ the set of all star bodies in $\mathbb{R}^n$.

It is easy to see that, for $K, E \in \mathcal{S}_0^n$ and $\lambda \geq 0$, the volume of the radial sum $K\widetilde{+}\lambda E = \{x\widetilde{+}\lambda y : x \in K,\ y \in E\}$ is also expressed as a polynomial of degree n in λ (see, e.g., [17, p. 508]), the so-called (relative) *dual Steiner formula*, which is written as

$$\text{(2.1)} \qquad \operatorname{vol}\left(K\widetilde{+}\lambda E\right) = \sum_{i=0}^{n} \binom{n}{i} \widetilde{\mathrm{W}}_i(K; E)\lambda^i.$$

The coefficients $\widetilde{\mathrm{W}}_i(K; E)$ are the (relative) *dual quermassintegrals* of K and E, and they are special cases of the dual mixed volumes, which were introduced by Lutwak in [13] (see also [17, Section 9.3]). Since star bodies have nonempty interior, it is easy to see that $\widetilde{\mathrm{W}}_i(K; E) > 0$ for all $i = 0, \ldots, n$. Dual quermassintegrals also satisfy that $\widetilde{\mathrm{W}}_0(K; E) = \operatorname{vol}(K)$ and $\widetilde{\mathrm{W}}_n(K; E) = \operatorname{vol}(E)$, and furthermore, they are homogeneous of degree $n - i$ (respectively, degree i) in the first (respectively, second) argument. When $E = B_n$, we write for short $\widetilde{\mathrm{W}}_i(K) = \widetilde{\mathrm{W}}_i(K; B_n)$.

It is well-known that for $K, E \in \mathcal{S}_0^n$,

$$\text{(2.2)} \qquad \widetilde{\mathrm{W}}_i(K; E)^2 \leq \widetilde{\mathrm{W}}_{i-1}(K; E)\widetilde{\mathrm{W}}_{i+1}(K; E), \quad 1 \leq i \leq n-1,$$

which are the "dual" counterpart to the classical Aleksandrov–Fenchel inequalities (see, e.g., [17, (9.40)]). Equality holds in (2.2) if and only if K and E are dilates.

Regarding again the right-hand side of (2.1) as a formal polynomial in a complex variable $z \in \mathbb{C}$, which we denote by

$$\widetilde{f}_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \widetilde{\mathrm{W}}_i(K; E) z^i,$$

structural properties of the set of roots of dual Steiner polynomials are investigated in [1]. Most of these results are based on a characterization of those tuples $(\omega_0, \ldots, \omega_n)$ of real numbers for which there exist $K, E \in \mathcal{S}_0^n$ with $\widetilde{\mathrm{W}}_j(K; E) = \omega_j$, $j = 0, \ldots, n$, which is also proved in [1]. This characterization is, however, much more involved than the one describing the classical quermassintegrals (see [1, Theorems 1.1 and 2.2]): now, the dual Aleksandrov–Fenchel inequalities are, in general, not enough in order to characterize dual quermassintegrals.

At this point we make a key observation. It arises from [1, Lemma 2.2] that one of the star bodies in the above mentioned characterization is always an Euclidean ball.

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Therefore, and without loss of generality, we will always take the relative star body $E = B_n$. Furthermore, those results in which the dual quermassintegrals characterization is not used are equally valid for an arbitrary $E \in \mathcal{S}_0^n$ without additional considerations, and so, for the sake of simplicity, we will always work with the dual quermassintegrals $\widetilde{\mathrm{W}}_i(K)$.

Thus, we write

$$\widetilde{\mathcal{R}}(n) = \left\{z \in \mathbb{C}^+ : \widetilde{f}_{K;B_n}(z) = 0 \text{ for some } K \in \mathcal{S}_0^n\right\}$$

to represent the set of roots of all dual Steiner polynomials in the upper half-plane, which is known to be a convex cone containing the negative real axis $\mathbb{R}_{<0}$ (see [1, Theorem 1.3]). Since the dual quermassintegrals fulfill $\widetilde{\mathrm{W}}_i(K) > 0$ for all $i = 0, \dots, n$ and every $K \in \mathcal{S}_0^n$, and they satisfy the dual Aleksandrov–Fenchel inequalities (2.2), dual Steiner polynomials are particular cases of log-convex coefficients polynomials, and thus

$$\widetilde{\mathcal{R}}(n) \subset \mathcal{R}_{\mathcal{L}_{>0}}(n). \tag{2.3}$$

But since the dual Aleksandrov–Fenchel inequalities do not characterize, in general, dual quermassintegrals, the inclusion (2.3) may be strict. However, both cones are known to coincide when $n = 2$ (see [1, Proposition 4.2]):

$$\widetilde{\mathcal{R}}(2) = \mathcal{R}_{\mathcal{L}_{>0}}(2) = \{z \in \mathbb{C}^+ : \mathrm{Re}(z) < 0\}.$$

Singular cases turn out to be dimensions $n = 2, 3$, where the dual Aleksandrov–Fenchel inequalities *do characterize* dual quermassintegrals (see [1, Proof of Proposition 4.2]) and [10, Corollary 3.1], respectively). We collect both results in the following theorem:

Theorem 2.1 *Given $\omega_0, \omega_1 > 0$, there exists a star body $K \in \mathcal{S}_0^2$ such that $\widetilde{\mathrm{W}}_i(K) = \omega_i$, $i = 0, 1$, if and only if either they verify the strict dual Aleksandrov–Fenchel inequality $\omega_1^2 < \omega_0\omega_2$, or $\omega_i = \lambda^{2-i}\kappa_2$ for some $\lambda > 0$ and $i = 0, 1$, and in this case $K = \lambda B_2$.*

Given $\omega_0, \omega_1, \omega_2 > 0$, there exists a star body $K \in \mathcal{S}_0^3$ such that $\widetilde{\mathrm{W}}_i(K) = \omega_i$, $i = 0, 1, 2$, if and only if either they verify the strict dual Aleksandrov–Fenchel inequalities $\omega_1^2 < \omega_0\omega_2$ and $\omega_2^2 < \kappa_3\omega_1$, or $\omega_i = \lambda^{3-i}\kappa_3$ for some $\lambda > 0$ and $i = 0, 1, 2$, and in this case $K = \lambda B_3$.

Remark 2.1 In dimension $n = 4$ it is easy to see that the dual Aleksandrov–Fenchel inequalities do not characterize the dual quermassintegrals. In fact, taking $\omega_0 = (4\pi^2 - 6)/(\pi^2 - 2)$, $\omega_1 = 2$ and $\omega_2 = \omega_3 = 1$, one gets

$$\omega_1^2 - \omega_0\omega_2 = \frac{-2}{\pi^2 - 2}, \quad \omega_2^2 - \omega_3\omega_1 = -1, \quad \omega_3^2 - \omega_2\kappa_4 = 1 - \frac{\pi^2}{2}.$$

However, using the characterization given in [1, Theorem 2.2] via some properties of particular Hankel matrices, one can see that the above numbers are not dual quermassintegrals of any star body: indeed, it is a straightforward computation to

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check that the Hankel matrix

$$\begin{pmatrix} \kappa_4 & \omega_3 & \omega_2 \\ \omega_3 & \omega_2 & \omega_1 \\ \omega_2 & \omega_1 & \omega_0 \end{pmatrix}$$

is not positive definite because its determinant vanishes.

Accordingly, the results satisfied by second/third-degree log-convex coefficients polynomials will have a more or less direct translation for dual Steiner polynomials of planar/three-dimensional star bodies, not so when $n \geq 4$. An example of this fact arises in the following consequence of Theorem 1.3:

Corollary 2.1 $\widetilde{\mathcal{R}}(3) = \{a + b\,\mathrm{i} \in \mathbb{C}^+ : b > \sqrt{3}\,a\}$.

Its proof can be found in Section 4. Another of our main results for dual Steiner polynomials, which cannot be derived from results for log-convex coefficients polynomials, is the following characterization of the Euclidean ball. Its proof will be collected in Section 5.2:

Theorem 2.2 *Let $K \in \mathcal{S}_0^n$ for $n \geq 3$, let γ_i, $i = 1, \dots, n$, be the roots of $\widetilde{f}_{K;B_n}(z)$ and let $a > 0$. Then $\mathrm{Re}(\gamma_i) = -a$ for all $i = 1, \dots, n$ if and only if $K = aB_n$.*

3 Background on polynomials

Since many of our results are strongly based on specific properties which are satisfied by the roots of polynomials, in order to make the reading of the manuscript easier, we devote this brief section to those results on polynomials that will be needed in the subsequent proofs.

An important well-known result establishes that the roots of a (complex) polynomial are continuous functions of its coefficients:

Theorem 3.1 [15, Theorem (1,4)] *Let*

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n = a_n \prod_{i=1}^{r} (z - z_i)^{m_i}, \qquad a_n \neq 0,$$
$$F(z) = (a_0 + \varepsilon_0) + (a_1 + \varepsilon_1)z + \cdots + (a_{n-1} + \varepsilon_{n-1})z^{n-1} + a_n z^n$$

be complex polynomials, and let $0 < r_k < \min |z_k - z_i|$ for $i = 1, \dots, k-1, k+1, \dots, r$ and all $k = 1, \dots, r$. Then there exists $\varepsilon > 0$ such that, if $|\varepsilon_i| \leq \varepsilon$ for $i = 0, \dots, n-1$, then $F(z)$ has precisely m_k roots in the disk $\{z \in \mathbb{C} : |z - z_k| < r_k\}$, $k = 1, \dots, r$.

The following results provide bounds for the modulus of the roots of a polynomial:

Proposition 3.1 [15, p. 137, Exercise 2] *Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a real polynomial with $a_i > 0$, $i = 0, \dots, n$. Then its roots lie in the ring*

$$\min_{0 \leq i \leq n-1} \frac{a_i}{a_{i+1}} \leq |z| \leq \max_{0 \leq i \leq n-1} \frac{a_i}{a_{i+1}}.$$

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Proposition 3.2 [15, p. 126, Exercise 7] *Let $g(z) = z^n + b_1 z^{n-1} + \cdots + b_n$ be a complex polynomial with roots $z_1, \dots, z_n$, and let $M = \max_{i=1,\dots,n} |z_i|$. Then*

$$M \geq \frac{1}{n} \sum_{i=1}^{n} \left| \frac{b_i}{\binom{n}{i}} \right|^{1/i}.$$

Theorem 3.2 [15, Theorem (33,3)] *Let $r \in \{1, \dots, n-1\}$. A complex polynomial of the form $g(z) = 1 + b_r z^r + \cdots + b_n z^n$ with $b_r \neq 0$ has at least r roots in the disk $\left\{z \in \mathbb{C} : |z| \leq \left(\binom{n}{r}/|b_r|\right)^{1/r}\right\}$.*

An important tool will be the well-known Lucas theorem on the location of the roots of the derivative (critical points) of a polynomial (see, e.g., [15, Theorem (6,1)]):

Theorem 3.3 (Lucas' theorem) *All the critical points of a (nonconstant) complex polynomial $f(z)$ lie in the convex hull C of the set of roots of $f(z)$. Moreover, if the roots of $f(z)$ are not collinear, then no critical point of $f(z)$ lies on the boundary of C unless it is a multiple root of $f(z)$.*

Next property provides with a relation between the imaginary parts of the roots of the derivative of a polynomial and of the polynomial itself.

Theorem 3.4 [16, Theorem 1.4.1] *Let the complex polynomial $f(z)$ of degree $n > 1$ have the roots $z_1, \dots, z_n$, and let $w_1, \dots, w_{n-1}$ be those of $f'(z)$. Then*

$$\frac{1}{n-1} \sum_{i=1}^{n-1} |\mathrm{Im}(w_i)| \leq \frac{1}{n} \sum_{i=1}^{n} |\mathrm{Im}(z_i)|.$$

The last result, known as the Davenport–Pólya theorem, deals with the convolution of two log-convex sequences.

Theorem 3.5 (Davenport–Pólya's theorem [3]) *Let $(a_i)_{i=0}^n$ and $(b_i)_{i=0}^n$ be log-convex sequences of positive numbers. Then the sequence $(v_i)_{i=0}^n$ given by*

$$v_i = a_0 b_i + \binom{i}{1} a_1 b_{i-1} + \binom{i}{2} a_2 b_{i-2} + \cdots + a_i b_0$$

is also log-convex.

4 The sets of roots of log-convex coefficients polynomials

In this section, we prove our main results regarding the structure of the set of roots of log-convex coefficients polynomials. We start with Theorem 1.1, for which we need the following auxiliary result:

Lemma 4.1 *Let $\boldsymbol{\omega} \in \mathcal{L}^n$ (respectively, $\boldsymbol{\omega} \in \mathcal{L}^n_{>0}$). Then:*

(i) *for every $\lambda > 0$, there exists $\boldsymbol{\omega}' \in \mathcal{L}^n$ (respectively, $\boldsymbol{\omega}' \in \mathcal{L}^n_{>0}$) such that $f_{\boldsymbol{\omega}}(\lambda z) = f_{\boldsymbol{\omega}'}(z)$ and*

(ii) *for all $a > 0$, there exists $\boldsymbol{\omega}' \in \mathcal{L}^n$ (respectively, $\boldsymbol{\omega}' \in \mathcal{L}^n_{>0}$) such that $f_{\boldsymbol{\omega}}(z + a) = f_{\boldsymbol{\omega}'}(z)$.*

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Proof Let $\boldsymbol{\omega} = (\omega_0, \dots, \omega_n) \in \mathcal{L}^n$. To see (i), it suffices to consider the tuple $\boldsymbol{\omega}' = (\omega_0, \lambda\omega_1, \dots, \lambda^n \omega_n)$, and the result follows.

To prove (ii), we have to distinguish whether $\boldsymbol{\omega} \in \mathcal{L}^n_{>0}$ or $\boldsymbol{\omega} \in \mathcal{L}^n \backslash \mathcal{L}^n_{>0}$. First we assume that $\boldsymbol{\omega} \in \mathcal{L}^n_{>0}$. Taking into account that also the reversed tuple $(\omega_n, \omega_{n-1}, \dots, \omega_0) \in \mathcal{L}^n_{>0}$ and that $(1, a, a^2, \dots, a^n) \in \mathcal{L}^n_{>0}$, we construct the new tuple

$$v_i = \sum_{k=0}^{i} \binom{i}{k} a^k \omega_{n-i+k}, \quad 0 \le i \le n,$$

given by their binomial convolution, which is also a log-convex sequence as an application of Davenport–Pólya's theorem (see Theorem 3.5). So, $\boldsymbol{\omega}' = (v_n, \dots, v_0) \in \mathcal{L}^n_{>0}$, and since $\binom{n}{i}\binom{n-i}{k} = \binom{n}{i+k}\binom{i+k}{i}$, we get

$$\begin{aligned} f_{\boldsymbol{\omega}'}(z) &= \sum_{i=0}^{n} \binom{n}{i} v_{n-i} z^i = \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{k=0}^{n-i} \binom{n-i}{k} a^k \omega_{i+k} \right) z^i \\ &= \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} \binom{n}{i+k} \binom{i+k}{i} a^k \omega_{i+k} \right) z^i \\ &= \sum_{i=0}^{n} \left(\sum_{l=i}^{n} \binom{n}{l} \binom{l}{i} a^{l-i} \omega_l \right) z^i = \sum_{j=0}^{n} \binom{n}{j} \omega_j \left(\sum_{r=0}^{j} \binom{j}{r} a^{j-r} z^r \right) \\ &= \sum_{j=0}^{n} \binom{n}{j} \omega_j (z+a)^j = f_{\boldsymbol{\omega}}(z+a). \end{aligned}$$

Finally, we assume that $\boldsymbol{\omega} = (\omega_0, 0, \dots, 0, \omega_n)$ with $\omega_0, \omega_n \ge 0$, $\boldsymbol{\omega} \ne (0, \dots, 0)$ (see (1.2)). If $\omega_n = 0$, the result is trivial, and so we assume that $\omega_n > 0$. Then $f_{\boldsymbol{\omega}}(z) = \omega_0 + \omega_n z^n$ and taking

$$\boldsymbol{\omega}' = (\omega_0 + \omega_n a^n, \omega_n a^{n-1}, \omega_n a^{n-2}, \dots, \omega_n a, \omega_n) \in \mathcal{L}^n,$$

we have

$$f_{\boldsymbol{\omega}'}(z) = \left(\sum_{i=0}^{n} \binom{n}{i} a^{n-i} z^i \right) \omega_n + \omega_0 = \omega_n (z+a)^n + \omega_0 = f_{\boldsymbol{\omega}}(z+a). \qquad \blacksquare$$

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Let $\gamma \in \mathcal{R}_{\mathcal{L}_{>0}}(n)$. Then there exists $\boldsymbol{\omega} \in \mathcal{L}^n_{>0}$ such that $f_{\boldsymbol{\omega}}(\gamma) = 0$, and Lemma 4.1 (ii) ensures that, for all $a > 0$, $\gamma - a$ is a root of $f_{\boldsymbol{\omega}}(z+a) = f_{\boldsymbol{\omega}'}(z)$ for some $\boldsymbol{\omega}' \in \mathcal{L}^n_{>0}$. Therefore, $\gamma - a \in \mathcal{R}_{\mathcal{L}_{>0}}(n)$.

Furthermore, by Lemma 4.1 (i), we know that, for any $\lambda > 0$, $\lambda\gamma$ is a root of the polynomial $f_{\boldsymbol{\omega}}(z/\lambda) = f_{\boldsymbol{\omega}''}(z)$ for some $\boldsymbol{\omega}'' \in \mathcal{L}^n_{>0}$, and hence, $\lambda\gamma \in \mathcal{R}_{\mathcal{L}_{>0}}(n)$. These two properties imply that $\mathcal{R}_{\mathcal{L}_{>0}}(n)$ is a convex cone. The proof concludes by noting that the tuple $(1, 1, \dots, 1) \in \mathcal{L}^n_{>0}$, and so $-1 \in \mathcal{R}_{\mathcal{L}_{>0}}(n)$; therefore, $\mathbb{R}_{<0} \subset \mathcal{R}_{\mathcal{L}_{>0}}(n)$.

In the case of $\mathcal{L}^n$, the argument is analogous. We just have to observe that now the tuple $(0, \dots, 0, \omega_n) \in \mathcal{L}^n$ for $\omega_n > 0$, and thus $0 \in \mathcal{R}_{\mathcal{L}}(n)$. $\blacksquare$

Remark 4.1 At this point, we would like to stress the pertinence of working with polynomials of the form $f_{\boldsymbol{\omega}}(z) = \sum_{i=0}^{n} \binom{n}{i} \omega_i z^i$ instead of just considering $\sum_{i=0}^{n} \omega_i z^i$ for $\boldsymbol{\omega} \in \mathcal{L}^n_{>0}$ (or $\mathcal{L}^n$). The combinatorial numbers play a key role if

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one aims to have analogous properties to the ones of the Steiner polynomials (e.g., convexity or monotonicity). Indeed, it is an easy computation to check that the set of roots of all the second-degree polynomials $\omega_0 + \omega_1 z + \omega_2 z^2$, with $\boldsymbol{\omega} \in \mathcal{L}^2_{>0}$, is $\{a + b\,\mathrm{i} \in \mathbb{C}^+ : -b \le \sqrt{3}\,a < 0\}$, and hence convex; however, for the third-degree polynomials $\sum_{i=0}^{3} \omega_i z^i$, $\boldsymbol{\omega} \in \mathcal{L}^3_{>0}$, the set of roots is contained in the union $\{a + b\mathrm{i} \in \mathbb{C}^+ : b > \sqrt{3}|a|\} \cup \mathbb{R}_{<0}$, and hence, since it trivially contains the set $\mathbb{R}_{<0} \cup \{b\mathrm{i} : b > 0\}$, the convexity, for instance, is lost. Of course, further research can be developed for them.

Next, we investigate the topological properties of the cones $\mathcal{R}_{\mathcal{L}}(n)$ and $\mathcal{R}_{\mathcal{L}_{>0}}(n)$: we prove Theorem 1.2.

Proof of Theorem 1.2 Let $\gamma \in \operatorname{bd}\mathcal{R}_{\mathcal{L}}(n)$, and let $(\gamma_m)_{m\in\mathbb{N}} \subset \mathcal{R}_{\mathcal{L}}(n)$ be a sequence such that $\lim_{m\to\infty} \gamma_m = \gamma$. Then, for every $m \in \mathbb{N}$, there exists $\boldsymbol{\omega}^m = (\omega_0^m, \dots, \omega_n^m) \in \mathcal{L}^n$ such that $f_{\boldsymbol{\omega}^m}(\gamma_m) = 0$.

Since $f_{\boldsymbol{\omega}^m}(1) > 0$ because $\boldsymbol{\omega}^m \ne (0, \dots, 0)$, without loss of generality we may assume that $f_{\boldsymbol{\omega}^m}(1) = 1$; otherwise one might take the tuple $\boldsymbol{\omega}^m / f_{\boldsymbol{\omega}^m}(1) \in \mathcal{L}^n$. Thus, from $f_{\boldsymbol{\omega}^m}(1) = \sum_{i=0}^{n} \binom{n}{i} \omega_i^m$, we get $\omega_i^m \in [0,1]$ for every $i = 0, \dots, n$ and all $m \in \mathbb{N}$. Thus, a subsequence of each sequence $(\omega_i^m)_{m\in\mathbb{N}}$ converges to a point ω_i, $0 \le i \le n$, and without loss of generality we assume that $\lim_{m\to\infty} \omega_i^m = \omega_i$ for all $i = 0, \dots, n$.

Moreover, since $(\omega_i^m)^2 \le \omega_{i-1}^m \omega_{i+1}^m$ for all $m \in \mathbb{N}$, the same inequality holds for the limit values, i.e., $(\omega_i)^2 \le \omega_{i-1}\omega_{i+1}$, $1 \le i \le n-1$. We also note that $\boldsymbol{\omega} \ne (0, \dots, 0)$ because $f_{\boldsymbol{\omega}^m}(1) = 1$ for all $m \in \mathbb{N}$; hence $\boldsymbol{\omega} = (\omega_0, \dots, \omega_n) \in \mathcal{L}^n$. Finally, since $f_{\boldsymbol{\omega}^m}(\gamma_m) = 0$ for every $m \in \mathbb{N}$, also $f_{\boldsymbol{\omega}}(\gamma) = 0$, and so $\gamma \in \mathcal{R}_{\mathcal{L}}(n)$.

Next, we show the identities of the statement. First, we observe that the set difference $\mathcal{R}_{\mathcal{L}}(n) \setminus \mathcal{R}_{\mathcal{L}_{>0}}(n) \subset \{z \in \mathbb{C}^+ : z^n + a = 0 \text{ for some } a \ge 0\}$. Moreover, the set in the right-hand side is the (finite) union of those rays determined by the nth roots of -1 which are contained in the upper half-plane $\mathbb{C}^+$. Then, taking into account that both $\mathcal{R}_{\mathcal{L}}(n)$ and $\mathcal{R}_{\mathcal{L}_{>0}}(n)$ are convex cones (Theorem 1.1) and that they only differ, at most, in the above finite union of rays, we may conclude that $\operatorname{int}\mathcal{R}_{\mathcal{L}_{>0}}(n) = \operatorname{int}\mathcal{R}_{\mathcal{L}}(n)$ and $\operatorname{cl}\mathcal{R}_{\mathcal{L}_{>0}}(n) = \operatorname{cl}\mathcal{R}_{\mathcal{L}}(n) = \mathcal{R}_{\mathcal{L}}(n)$. ∎

Using the previous results, in the next subsection, we will determine $\mathcal{R}_{\mathcal{L}_{>0}}(n)$ and $\mathcal{R}_{\mathcal{L}}(n)$.

4.1 Describing the cones $\mathcal{R}_{\mathcal{L}}(n)$ and $\mathcal{R}_{\mathcal{L}_{>0}}(n)$

Before the proof of Theorem 1.3, we need several lemmas and additional notation.

First, we introduce and study particular polynomials that will be crucial in the proof of our main result. The classical De Moivre formula states that, for any $x \in \mathbb{R}$ and $m \in \mathbb{Z}$, $\big(\cos(x) + \sin(x)\,\mathrm{i}\big)^m = \cos(mx) + \sin(mx)\,\mathrm{i}$ (see, e.g., [2, p. 5]). Then, for any $x \in (0, \pi/m]$, $m \in \mathbb{N}$ with $m \ge 3$, we have

$$0 \le \operatorname{Im}\big(\cos(mx) + \sin(mx)\,\mathrm{i}\big) = \operatorname{Im}\Big(\big(\cos(x) + \sin(x)\,\mathrm{i}\big)^m\Big)$$
$$= \sum_{\substack{i=1 \\ i \text{ odd}}}^{m} \binom{m}{i} (-1)^{(i-1)/2} \cos^{m-i}(x) \sin^i(x),$$

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and dividing by $\cos^m(x) > 0$ we get

$$\sum_{\substack{i=1\\ i \text{ odd}}}^{m} \binom{m}{i} (-1)^{(i-1)/2} \tan^i(x) \geq 0.$$

Thus, for any $m \in \mathbb{N}$, the formal polynomial in a real variable $t \in \mathbb{R}$

$$p_m(t) := \sum_{\substack{i=0\\ i \text{ even}}}^{m-1} \binom{m}{i+1} (-1)^{i/2} t^i$$

satisfies the following basic but crucial property:

Lemma 4.2 *Let $m \in \mathbb{N}$. Then $p_m(t) \geq 0$ for all $t \in [0, \tan(\pi/m)]$, and equality holds, for $m \geq 3$, if and only if $t = \tan(\pi/m)$.*

Next lemma establishes an important relation for these polynomials that will be needed later.

Lemma 4.3 *Let $m \in \mathbb{N}$. For all $t \in \mathbb{R}$,*

$$(1+t^2)p_m(t) = 2p_{m+1}(t) - p_{m+2}(t).$$

Proof From the definition of $p_k(t)$, for $k = m, m+1, m+2$, we have

$$\begin{aligned}
(1+t^2)p_m(t) &= \sum_{\substack{i=0\\ i \text{ even}}}^{m-1} \binom{m}{i+1} (-1)^{i/2} t^i + \sum_{\substack{i=0\\ i \text{ even}}}^{m-1} \binom{m}{i+1} (-1)^{i/2} t^{i+2} \\
&= \sum_{\substack{i=0\\ i \text{ even}}}^{m-1} \binom{m}{i+1} (-1)^{i/2} t^i + \sum_{\substack{i=2\\ i \text{ even}}}^{m+1} \binom{m}{i-1} (-1)^{i/2-1} t^i \\
&= m + \sum_{\substack{i=2\\ i \text{ even}}}^{m-1} \left[\binom{m}{i+1} - \binom{m}{i-1} \right] (-1)^{i/2} t^i + S_1,
\end{aligned}$$

where

$$S_1 = \begin{cases} m(-1)^{m/2-1} t^m & \text{if } m \text{ is even,} \\ (-1)^{(m-1)/2} t^{m+1} & \text{if } m \text{ is odd,} \end{cases}$$

and

$$\begin{aligned}
2p_{m+1}(t) - p_{m+2}(t) &= 2 \sum_{\substack{i=0\\ i \text{ even}}}^{m} \binom{m+1}{i+1} (-1)^{i/2} t^i - \sum_{\substack{i=0\\ i \text{ even}}}^{m+1} \binom{m+2}{i+1} (-1)^{i/2} t^i \\
&= \sum_{\substack{i=0\\ i \text{ even}}}^{m} \left[2\binom{m+1}{i+1} - \binom{m+2}{i+1} \right] (-1)^{i/2} t^i + S_2,
\end{aligned}$$

where

$$S_2 = \begin{cases} 0 & \text{if } m \text{ is even,} \\ (-1)^{(m-1)/2} t^{m+1} & \text{if } m \text{ is odd.} \end{cases}$$

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Clearly, the constant terms m and $2\binom{m+1}{1}-\binom{m+2}{1}$ coincide in the two above polynomials. Also the leading coefficients in both cases, m even and odd, are equal: $m(-1)^{m/2-1}$ and $(-1)^{(m-1)/2}$, respectively. Furthermore, since

$$\binom{m}{i+1}-\binom{m}{i-1}=\frac{(m+1)!\,(m-2i)}{(m-i+1)!(i+1)!}=2\binom{m+1}{i+1}-\binom{m+2}{i+1},$$

for all $i=2,\ldots,m-1$, we get the required identity. ∎

Next, we define a (finite) sequence of functions $c_k:\left[0,\tan\left(\frac{\pi}{k+3}\right)\right]\longrightarrow\mathbb{R}_{>0}$, $k=1,\ldots,n-3$, recursively by

$$c_k(t)=\begin{cases}\frac{1+t^2}{2} & \text{if } k=1,\\ \frac{2c_1(t)}{2-c_{k-1}(t)} & \text{if } k=2,\ldots,n-3.\end{cases}$$

In order to assure that they are well-defined, we have to see that $c_k(t)\neq 2$ for all $k=1,\ldots,n-3$ in its domain. This will be a direct consequence of the following lemma, which will be needed in the proof of Theorem 1.3.

Lemma 4.4 *For every $k=1,\ldots,n-3$ we have*

$$c_k(t)\leq\frac{3-t^2}{2}\quad\textit{for all}\quad 0\leq t\leq\tan\left(\frac{\pi}{k+3}\right).$$

Equality holds if and only if $t=\tan\left(\pi/(k+3)\right)$. In particular, $c_k(t)<2$ for $0\leq t\leq\tan\left(\pi/(k+3)\right)$ and all $k=1,\ldots,n-3$.

Proof We notice that it is enough to show that the functions $c_k(t)$ can be expressed as

$$c_k(t)=\frac{3-t^2}{2}-\frac{p_{k+3}(t)}{2p_{k+1}(t)},\quad k=1,\ldots,n-3, \tag{4.1}$$

for all $0\leq t\leq\tan\left(\pi/(k+3)\right)$. Indeed, since $p_{k+3}(t)\geq 0$ for all $0\leq t\leq\tan\left(\pi/(k+3)\right)$ with equality if and only if $t=\tan\left(\pi/(k+3)\right)$ (see Lemma 4.2) and, in this range, $p_{k+1}(t)>0$ because $\left(\tan(\pi/m)\right)_{m=2}^{\infty}$ is a decreasing sequence and so $\tan\left(\pi/(k+3)\right)<\tan\left(\pi/(k+1)\right)$, then the result follows. As usual, we are using the convention $\tan(\pi/2)=\infty$.

We prove (4.1) by induction on k. Clearly,

$$\frac{3-t^2}{2}-\frac{p_4(t)}{2p_2(t)}=\frac{3-t^2}{2}-\frac{4-4t^2}{4}=\frac{1+t^2}{2}=c_1(t).$$

So, let $k>1$ and we assume that (4.1) holds for $k-1$. Then,

$$2-c_{k-1}(t)=2-\frac{3-t^2}{2}+\frac{p_{k+2}(t)}{2p_k(t)}=\frac{1+t^2}{2}+\frac{p_{k+2}(t)}{2p_k(t)},$$

and using Lemma 4.3, we get

$$c_k(t)=\frac{2c_1(t)}{2-c_{k-1}(t)}=\frac{2(1+t^2)p_k(t)}{(1+t^2)p_k(t)+p_{k+2}(t)}=(1+t^2)\frac{p_k(t)}{p_{k+1}(t)}.$$

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The identity (4.1) is now obtained using again Lemma 4.3 twice:

$$c_k(t) = (1+t^2)\frac{p_k(t)}{p_{k+1}(t)} = \frac{4p_{k+1}(t) - 2p_{k+2}(t)}{2p_{k+1}(t)}$$
$$= \frac{4p_{k+1}(t) - p_{k+3}(t) - (1+t^2)p_{k+1}(t)}{2p_{k+1}(t)} = \frac{(3-t^2)p_{k+1}(t) - p_{k+3}(t)}{2p_{k+1}(t)}.$$

This concludes the proof. ∎

Now, we are in a position to prove our main result.

Proof of Theorem 1.3 We state the result for $\mathcal{R}_{\mathcal{L}_{>0}}(n)$; the description of the cones $\mathcal{R}_{\mathcal{L}}(n)$ is achieved just noticing that $\mathcal{R}_{\mathcal{L}}(n) = \operatorname{cl}\mathcal{R}_{\mathcal{L}_{>0}}(n)$ (see Theorem 1.2).

First, we observe that since

$$1 + \tan\left(\frac{\pi}{n}\right)\mathrm{i} \in \left\{z \in \mathbb{C}^+ : z^n + a = 0\right\} \subset \mathcal{R}_{\mathcal{L}}(n),$$

where $a = \cos^{-n}(\pi/n)$, and $\mathcal{R}_{\mathcal{L}}(n)$ is a convex cone containing $\mathbb{R}_{\leq 0}$ (Theorem 1.1), we have from Theorem 1.2 that

$$\left\{a + b\,\mathrm{i} \in \mathbb{C}^+ : b > \tan\left(\frac{\pi}{n}\right)a\right\} \subset \operatorname{int}\mathcal{R}_{\mathcal{L}}(n) \cup \mathbb{R}_{<0}$$
$$= \operatorname{int}\mathcal{R}_{\mathcal{L}_{>0}}(n) \cup \mathbb{R}_{<0} \subset \mathcal{R}_{\mathcal{L}_{>0}}(n).$$

So, we have to prove the reverse inclusion. Let $n \in \mathbb{N}$ with $n \geq 3$, and we take $a + b\,\mathrm{i} \in \mathcal{R}_{\mathcal{L}_{>0}}(n)$. We may suppose that $a > 0$, otherwise the inequality is satisfied, and hence, using Lemma 4.1 (i), we assume without loss of generality that $a = 1$. Let $\boldsymbol{\omega} = (\omega_0, \ldots, \omega_n) \in \mathcal{L}^n_{>0}$ be such that $f_{\boldsymbol{\omega}}(1 + b\,\mathrm{i}) = 0$.

From $f_{\boldsymbol{\omega}}(z) = \omega_n(z^2 - 2z + 1 + b^2)\left(z^{n-2} + \sum_{i=0}^{n-3} a_i z^i\right)$ we obtain the following identities: if $n = 3$, we get

$$\frac{\omega_0}{\omega_3} = a_0(1+b^2), \qquad 3\,\frac{\omega_1}{\omega_3} = 1 + b^2 - 2a_0, \qquad 3\,\frac{\omega_2}{\omega_3} = a_0 - 2,$$

whereas for $n \geq 4$, we have

$$\frac{\omega_0}{\omega_n} = a_0(1+b^2), \qquad n\,\frac{\omega_1}{\omega_n} = a_1(1+b^2) - 2a_0,$$
$$\binom{n}{i}\frac{\omega_i}{\omega_n} = a_i(1+b^2) - 2a_{i-1} + a_{i-2} \quad \text{for } i = 2, \ldots, n-3,$$
$$\binom{n}{2}\frac{\omega_{n-2}}{\omega_n} = 1 + b^2 - 2a_{n-3} + a_{n-4} \quad \text{and} \quad n\,\frac{\omega_{n-1}}{\omega_n} = a_{n-3} - 2.$$

Since $\omega_i > 0$ for all $i = 0, \ldots, n$, we have $a_0 > 0$ and

(4.2)
(i) $a_1(1+b^2) > 2a_0$, (ii) $1 + b^2 + a_{n-4} > 2a_{n-3}$,
(iii) $a_i(1+b^2) + a_{i-2} > 2a_{i-1}$ for $i = 2, \ldots, n-3$,
(iv) $a_{n-3} > 2$.

Note that for $n = 3$, there are no inequalities (ii) and (iii), being $a_1 = 1$; for $n = 4$, there is no inequality (iii). In both cases, the conclusion in this argument will be obtained directly.

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We assume now that $b = \tan(\pi/n)$, and we will get a contradiction. From (ii) and (iv) in (4.2) we immediately get that

$$3 - \tan^2\left(\frac{\pi}{n}\right) < a_{n-4}. \tag{4.3}$$

Next, we consider the recursive (finite) sequence of numbers

$$c_k = \begin{cases} \frac{1+\tan^2(\frac{\pi}{n})}{2} & \text{for } k = 1, \\ \frac{2c_1}{2-c_{k-1}} & \text{for } k = 2, \dots, n-2, \end{cases}$$

which is well-defined by Lemma 4.4: note, on the one hand, that $\tan(\pi/n) \le \tan\big(\pi/(k+3)\big)$ for all $k = 1, \dots, n-3$ and, on the other hand, that $c_{n-3} < 2$, which ensures that c_{n-2} can be defined. Using (4.2)(i), we get $a_0 < c_1 a_1$, and together with (iii) for $i = 2$, we obtain

$$a_1 < c_1 a_2 + \frac{a_0}{2} < c_1 a_2 + \frac{c_1 a_1}{2}, \qquad \text{i.e.,} \qquad a_1 < \frac{2c_1}{2-c_1} a_2 = c_2 a_2$$

because $c_1 < 2$ (see Lemma 4.4). An inductive procedure yields, using (iii) for any $2 \le i \le n-3$,

$$a_{i-1} < c_1 a_i + \frac{a_{i-2}}{2} < c_1 a_i + \frac{c_{i-1} a_{i-1}}{2},$$

and so we get

$$a_{i-1} < c_i a_i \qquad \text{for all } \; i = 2, \dots, n-3$$

because $c_{i-1} < 2$ (see Lemma 4.4). Finally, the above relation for $i = n-3$ together with (4.2)(ii) gives

$$a_{n-4} < c_{n-3} a_{n-3} < c_{n-3}\left(c_1 + \frac{a_{n-4}}{2}\right),$$

and hence, since $c_{n-3} < 2$ (see Lemma 4.4), we have

$$a_{n-4} < \frac{2c_1}{2-c_{n-3}} c_{n-3} = c_{n-2} c_{n-3}.$$

From the equality case of Lemma 4.4, we get

$$c_{n-3} = \frac{3 - \tan^2\left(\frac{\pi}{n}\right)}{2} \quad \text{and} \quad c_{n-2} = \frac{2c_1}{2-c_{n-3}} = \frac{1+\tan^2\left(\frac{\pi}{n}\right)}{2 - \frac{3-\tan^2\left(\frac{\pi}{n}\right)}{2}} = 2,$$

and so

$$a_{n-4} < c_{n-2} c_{n-3} = 3 - \tan^2\left(\frac{\pi}{n}\right),$$

which contradicts (4.3). Therefore, since $\mathcal{R}_{\mathcal{L}_{>0}}(n)$ is a convex cone containing $\mathbb{R}_{<0}$, it must be $b > \tan(\pi/n)$, which shows the reverse inclusion $\mathcal{R}_{\mathcal{L}_{>0}}(n) \subset \big\{1 + b\,\mathrm{i} \in \mathbb{C}^+ : b > \tan(\pi/n)\big\}$. This concludes the proof. ∎

Note that, surprisingly, the proof of the inclusion $\mathcal{R}_{\mathcal{L}_{>0}}(n) \subset \big\{a + b\,\mathrm{i} \in \mathbb{C}^+ : b > \tan(\pi/n)a\big\}$ in Theorem 1.3 does not make use of the log-convexity property of the

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tuple $\boldsymbol{\omega} = (\omega_0, \dots, \omega_n)$: just the positivity $\omega_i > 0$, for all $i = 0, \dots, n$, is needed. This shows the following property:

Corollary 4.1 *There exists no nth degree real polynomial $a_0 + a_1 z + \cdots + a_n z^n$ with $a_i > 0$ for all $i = 0, \dots, n$, having $1 + \tan(\pi/n)\,\mathrm{i}$ as a root.*

As a direct consequence of Theorem 1.3, we get that the cones $\mathcal{R}_{\mathcal{L}_{>0}}(n)$ and $\mathcal{R}_{\mathcal{L}}(n)$ are strictly monotonic with respect to n, and also that they cover the whole upper half-plane $\mathbb{C}^+$, except $\mathbb{R}_{\geq 0}$, when n tends to infinity. For simplicity, we state the result just for $\mathcal{R}_{\mathcal{L}_{>0}}(n)$; the case of $\mathcal{R}_{\mathcal{L}}(n)$ is analogous.

Corollary 4.2 $\mathcal{R}_{\mathcal{L}_{>0}}(n) \subsetneq \mathcal{R}_{\mathcal{L}_{>0}}(n+1)$. *Moreover, for all $\gamma \in \mathbb{C}^+ \backslash \mathbb{R}_{\geq 0}$, there exists $n_0 \in \mathbb{N}$ such that $\gamma \in \mathcal{R}_{\mathcal{L}_{>0}}(n)$ for all $n \geq n_0$.*

Proof The first assertion is obvious from the description of $\mathcal{R}_{\mathcal{L}_{>0}}(n)$ given in Theorem 1.3. Finally, for a given $\gamma \in \mathbb{C}^+ \backslash \mathbb{R}_{\geq 0}$, since $\lim_{n \to \infty} \tan(\pi/n) = 0$ then there exists $n_0 \in \mathbb{N}$ such that $\gamma \in \{a + b\,\mathrm{i} \in \mathbb{C}^+ : b > \tan(\pi/n)a\} = \mathcal{R}_{\mathcal{L}_{>0}}(n)$ for all $n \geq n_0$. This concludes the proof. ∎

As another consequence of Theorem 1.3, we can prove Corollary 2.1. Indeed, since the dual Aleksandrov–Fenchel inequalities characterize the triples $(\omega_0, \omega_1, \omega_2)$ of positive numbers that can be the dual quermassintegrals $\widetilde{\mathrm{W}}_i(K)$ of some star body $K \in \mathcal{S}_0^3$ (see Theorem 2.1), with a slight extra effort we easily get that $\widetilde{\mathcal{R}}(3) = \mathcal{R}_{\mathcal{L}_{>0}}(3)$:

Proof of Corollary 2.1 The inclusion $\widetilde{\mathcal{R}}(3) \subset \mathcal{R}_{\mathcal{L}_{>0}}(3)$ is clear. In order to prove the reverse inclusion, let $\gamma \in \mathcal{R}_{\mathcal{L}_{>0}}(3)$, which we suppose not to be real, otherwise the assertion holds. Then there exists $\boldsymbol{\omega} \in \mathcal{L}_{>0}^3$ such that $f_{\boldsymbol{\omega}}(\gamma) = 0$. Note that we may assume, without loss of generality, that $\boldsymbol{\omega} = (\omega_0, \omega_1, \omega_2, \kappa_3)$. If

$$\omega_1^2 < \omega_0\omega_2 \quad \text{and} \quad \omega_2^2 < \omega_1\kappa_3,$$

then Theorem 2.1 ensures that $f_{\boldsymbol{\omega}}(z)$ is a dual Steiner polynomial of some star body, and therefore $\gamma \in \widetilde{\mathcal{R}}(3)$.

If, on the contrary, some of the above inequalities is an equality, we consider the 4-tuple

$$\boldsymbol{\omega}' = (\omega_0', \omega_1', \omega_2', \kappa_3) := \frac{\kappa_3}{\kappa_3 + \varepsilon}\big(\omega_0 + \varepsilon, \omega_1, \omega_2, \kappa_3 + \varepsilon\big)$$

for fixed $\varepsilon > 0$. Clearly $\boldsymbol{\omega}' \in \mathcal{L}_{>0}^3$, and it is straightforward that the inequalities $(\omega_1')^2 < \omega_0'\omega_2'$ and $(\omega_2')^2 < \omega_1'\kappa_3$ hold. Hence, by Theorem 2.1, there exists a star body $K \in \mathcal{S}_0^3$ such that $f_{\boldsymbol{\omega}'}(z) = \widetilde{f}_{K;B_3}(z)$ is a dual Steiner polynomial.

Let $\delta > 0$ be fixed. Since the roots of a polynomial are continuous functions of the coefficients of the polynomial (see Theorem 3.1), if $\varepsilon > 0$ is small enough, then there exists $\gamma' \in \mathbb{C}^+$ with $f_{\boldsymbol{\omega}'}(\gamma') = 0$, i.e., $\gamma' \in \widetilde{\mathcal{R}}(3)$, such that $|\gamma - \gamma'| < \delta$. This shows that $\gamma \in \mathrm{cl}\,\widetilde{\mathcal{R}}(3)$, i.e., that $\mathcal{R}_{\mathcal{L}_{>0}}(3) \subset \mathrm{cl}\,\widetilde{\mathcal{R}}(3)$. Since both $\mathcal{R}_{\mathcal{L}_{>0}}(3)$ and $\widetilde{\mathcal{R}}(3)$ are convex cones, the proof is then concluded from the inclusion $\widetilde{\mathcal{R}}(3) \subset \mathcal{R}_{\mathcal{L}_{>0}}(3)$ jointly with the fact that the upper ray of the boundary of $\mathcal{R}_{\mathcal{L}_{>0}}(3)$, i.e., $\mathrm{bd}\,\mathcal{R}_{\mathcal{L}_{>0}}(3) \backslash \mathbb{R}_{<0}$, is not included therein. ∎

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Unfortunately, since the dual Aleksandrov–Fenchel inequalities do not characterize dual quermassintegrals (cf. Remark 2.1), Theorem 1.3 does not provide us with a description for $\widetilde{\mathcal{R}}(4)$.

4.2 Further properties of the cones of roots

An immediate outcome of Theorem 1.3 is the fact that, when $n \geq 3$, a pure imaginary complex root always exists in $\mathcal{R}_{\mathcal{L}_{>0}}(n)$, since it is a convex cone. Indeed, for $n = 3$, there are log-convex coefficients polynomials all whose complex roots are pure imaginary. However, when $n \geq 4$, not all the roots can be of that type. More precisely, we have:

Theorem 4.1 *For $n \geq 4$, $n \neq 5$, there does not exist $\boldsymbol{\omega} \in \mathcal{L}^n$ such that all the roots of $f_{\boldsymbol{\omega}}(z)$ are imaginary pure complex numbers (excluding the real root always existing for odd degree). When $n = 5$, the only 6-tuples satisfying the above condition are $\boldsymbol{\omega} = \lambda\big(c^5, c^4/5, c^3/5, c^2/5, c/5, 1\big) \in \mathcal{L}^5$ for $\lambda > 0$ and $c > 0$, and in this case $f_{\boldsymbol{\omega}}(z) = \lambda(z+c)\left(z^2+c^2\right)^2$.*

In order to prove the theorem, we need the following auxiliary result:

Lemma 4.5 *For any $n \geq 5$, let $x_1, \dots, x_r > 0$, $r \geq 2$, be positive real numbers, such that*

$$x_1 + \cdots + x_r \leq \frac{3}{2}\frac{n-1}{n-2}.$$

Then

$$\frac{1}{x_1} + \cdots + \frac{1}{x_r} \geq \frac{3}{2}\frac{n-1}{n-2}, \tag{4.4}$$

and equality holds if and only if $n = 5$, $r = 2$ and $x_1 = x_2 = 1$.

Proof For the sake of simplicity, we write $a_n := 3(n-1)/\big(2(n-2)\big)$. Clearly $a_5 = 2$, and since a_n is a decreasing sequence in n, it can be easily seen that $a_n < 2$ for all $n > 5$, and so $a_n \leq 4/a_n$ with equality if and only if $n = 5$.

We assume that $\sum_{i=1}^r x_i \leq a_n$, and first we show the result for $r = 2$. Let $m := x_1 + x_2 \leq a_n$. Since $m - x_1 = x_2 > 0$, inequality (4.4) can be expressed as $m \geq a_n x_1(m - x_1)$, or equivalently,

$$a_n x_1^2 - a_n m x_1 + m \geq 0. \tag{4.5}$$

Using that $a_n \leq 4/a_n$, the discriminant of the second-degree polynomial (in x_1) in the left-hand side satisfies that

$$\mathrm{D} = a_n m(a_n m - 4) \leq a_n m\left(\frac{4}{a_n}a_n - 4\right) = 0.$$

Therefore, $a_n x_1^2 - a_n m x_1 + m \geq 0$, and thus (4.4) holds. Note, moreover, that if $n > 5$ then $\mathrm{D} < 0$ and (4.4) holds strictly.

Now, we assume that $r > 2$ and consider the positive numbers

$$x := \sum_{i=1}^{r-1} x_i \quad \text{and} \quad y = x_r,$$

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which satisfy $x+y\le a_n$. We observe that since $r>2$ and $1/x$ is a convex function, then

$$\text{(4.6)}\qquad \frac{1}{\sum_{i=1}^{r-1}x_i} < \frac{1}{\frac{1}{r-1}\sum_{i=1}^{r-1}x_i} \le \sum_{i=1}^{r-1}\frac{1}{r-1}\frac{1}{x_i} < \sum_{i=1}^{r-1}\frac{1}{x_i}.$$

If we assume that $\sum_{i=1}^{r}1/x_i \le a_n$ holds, (4.6) would yield

$$a_n \ge \left(\sum_{i=1}^{r-1}\frac{1}{x_i}\right)+\frac{1}{x_r} > \frac{1}{\sum_{i=1}^{r-1}x_i}+\frac{1}{x_r} = \frac{1}{x}+\frac{1}{y},$$

in contradiction to the case $r=2$ previously proved. This shows that for $r>2$, inequality (4.4) holds strictly.

For the equality case, we assume that (4.4) holds with equality. We already know that then, necessarily, $r=2$, otherwise the inequality (4.4) would be strict. Therefore, the equality in (4.4) is equivalent to the identity $a_n x_1^2 - a_n m x_1 + m = 0$ (cf. (4.5)). But in this case, they must be $n=5$ and $\mathrm{D}=0$, otherwise (4.5) would hold strictly. Finally, notice that $\mathrm{D}=0$ (when $n=5$) occurs if and only if $m=2$. Thus,

$$0 = a_5x_1^2 - a_5mx_1 + m = 2x_1^2 - 4x_1 + 2 = 2(x_1-1)^2$$

if and only if $x_1=1$ and so $x_2=1$. ∎

Now, we are in a position to show Theorem 4.1.

Proof of Theorem 4.1 First, we note that if $\boldsymbol{\omega}=(\omega_0,0,\dots,0,\omega_n)\in\mathcal{L}^n\backslash\mathcal{L}^n_{>0}$ then $f_{\boldsymbol{\omega}}(z)=\omega_0+\omega_n z^n$, all whose roots cannot be imaginary pure complex numbers (aside from the real root if n is odd) for any values of $\omega_0,\omega_n\in\mathbb{R}_{\ge0}$. So, we will consider only tuples in $\mathcal{L}^n_{>0}$.

Let n be even, and we assume there exists $\boldsymbol{\omega}=(\omega_0,\dots,\omega_n)\in\mathcal{L}^n_{>0}$ such that all the roots of $f_{\boldsymbol{\omega}}(z)$ are $\{\pm b_j\,\mathrm{i},\ j=1,\dots,n/2\}$, with $b_j\in\mathbb{R}_{>0}$. Then

$$f_{\boldsymbol{\omega}}(z)=\omega_n\prod_{j=1}^{n/2}(z^2+b_j^2),$$

which would imply, in particular, that $\omega_{2i+1}=0$ for all $i=0,\dots,(n-2)/2$. Thus $\omega_i=0$ for every $i=1,\dots,n-1$, and so $f_{\boldsymbol{\omega}}(z)$ would be of the form $f_{\boldsymbol{\omega}}(z)=\omega_n(z^n+a)$, a contradiction.

Now let n be odd. Let $\boldsymbol{\omega}=(\omega_0,\dots,\omega_n)\in\mathcal{L}^n_{>0}$ be such that the roots of $f_{\boldsymbol{\omega}}(z)$ are $\{-c,\pm b_j\,\mathrm{i},\ j=1,\dots,(n-1)/2\}$, with $c,b_j\in\mathbb{R}_{>0}$. From

$$f_{\boldsymbol{\omega}}(z)=\omega_n(z+c)\prod_{j=1}^{(n-1)/2}(z^2+b_j^2),$$

we get

$$c\sum_{j=1}^{(n-1)/2}b_j^2=\binom{n}{3}\frac{\omega_{n-3}}{\omega_n},\qquad \sum_{j=1}^{(n-1)/2}b_j^2=\binom{n}{2}\frac{\omega_{n-2}}{\omega_n},\qquad c=n\frac{\omega_{n-1}}{\omega_n},$$
$$\prod_{j=1}^{(n-1)/2}b_j^2=n\frac{\omega_1}{\omega_n},\qquad c\sum_{j=1}^{(n-1)/2}\prod_{i\ne j}b_i^2=\binom{n}{2}\frac{\omega_2}{\omega_n},\qquad \sum_{j=1}^{(n-1)/2}\prod_{i\ne j}b_i^2=\binom{n}{3}\frac{\omega_3}{\omega_n}.$$

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Since $\boldsymbol{\omega}$ is log-convex, it must be, in particular, $\omega_{n-2}^2 \le \omega_{n-3}\omega_{n-1}$ and $\omega_2^2 \le \omega_1\omega_3$, which are equivalent to the relations

$$\frac{2}{n-1}\frac{1}{c^2}\sum_{j=1}^{(n-1)/2} b_j^2 \le \frac{3}{n-2} \quad \text{and} \quad \frac{2}{n-1}c^2\sum_{j=1}^{(n-1)/2}\prod_{i\ne j} b_i^2 \le \frac{3}{n-2}\prod_{j=1}^{(n-1)/2} b_j^2,$$

i.e.,

$$\sum_{j=1}^{(n-1)/2}\frac{b_j^2}{c^2} \le \frac{3}{2}\frac{n-1}{n-2} \quad \text{and} \quad \sum_{j=1}^{(n-1)/2}\frac{c^2}{b_j^2} \le \frac{3}{2}\frac{n-1}{n-2}, \tag{4.7}$$

respectively. Lemma 4.5 ensures that, when $n > 5$, the above two inequalities cannot hold simultaneously, which gives the desired contradiction.

If $n = 5$, then (4.7) becomes

$$\frac{b_1^2}{c^2} + \frac{b_2^2}{c^2} \le 2 \quad \text{and} \quad \frac{c^2}{b_1^2} + \frac{c^2}{b_2^2} \le 2,$$

and Lemma 4.5 ensures that both inequalities hold simultaneously if and only if $b_1 = b_2 = c$. Therefore, $f_{\boldsymbol{\omega}}(z) = \omega_5(z + c)\left(z^2 + c^2\right)^2$, which concludes the proof. ■

For dual Steiner polynomials, the property provided by Theorem 4.1 is slightly more restrictive.

Corollary 4.3 *For $n \ge 4$, there does not exist a star body $K \in \mathcal{S}_0^n$ such that all the roots of $\widetilde{f}_{K;B_n}(z)$ are imaginary pure complex numbers (excluding the real root always existing in odd dimension).*

Proof Since all dual Steiner polynomials are log-convex coefficients polynomials, Theorem 4.1 ensures that when $n \ge 4$, $n \ne 5$, there does not exist $K \in \mathcal{S}_0^n$ such that all the roots of $\widetilde{f}_{K;B_n}(z)$ are imaginary pure complex numbers.

So, we set $n = 5$. Theorem 4.1 ensures that the only possible log-convex coefficients polynomial, all whose roots are pure complex numbers (excluding the existing real root), is of the form $\lambda(z + c)\left(z^2 + c^2\right)^2$, for $\lambda, c > 0$. However, it cannot be a dual Steiner polynomial for any star body $K \in \mathcal{S}_0^5$. Indeed, if this was the case, it should be

$$\widetilde{\mathrm{W}}_0(K) = \lambda c^5, \quad \widetilde{\mathrm{W}}_i(K) = \lambda\frac{c^{5-i}}{5} \text{ for } i = 1, \dots, 4, \quad \widetilde{\mathrm{W}}_5(K) = \lambda$$

for some $K \in \mathcal{S}_0^5$, which would verify the dual Aleksandrov–Fenchel inequalities (2.2). But since, for instance,

$$\widetilde{\mathrm{W}}_2(K)^2 - \widetilde{\mathrm{W}}_1(K)\widetilde{\mathrm{W}}_3(K) = \frac{c^6}{25}\lambda^2 - \frac{c^4}{5}\lambda\frac{c^2}{5}\lambda = 0,$$

the equality case in (2.2) would imply that K is a suitable dilation of B_5, and so, all dual Aleksandrov–Fenchel inequalities should hold with equality. This is however not the case, because $\widetilde{\mathrm{W}}_4(K)^2 - \widetilde{\mathrm{W}}_3(K)\widetilde{\mathrm{W}}_5(K) = -4\lambda^2c^2/25 < 0$. It concludes the proof. ■

With respect to real roots, a comparable result to Theorem 4.1 can be obtained: not all the roots can be real numbers, unless they coincide. For completeness, we include its proof, following the argument of [1, Proposition 4.4], where it was shown that in

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the case of the more restrictive family of dual Steiner polynomials, not all their roots can be real unless they all are equal. For the proof we need the following notation: for complex numbers $z_1,\dots,z_r\in\mathbb{C}$ let

$$\mathrm{s}_i(z_1,\dots,z_r)=\sum_{\substack{J\subset\{1,\dots,r\}\\ \#J=i}}\prod_{j\in J}z_j$$

denote the ith elementary symmetric function of $z_1,\dots,z_r$, $i=1,\dots,r$, with $\mathrm{s}_0(z_1,\dots,z_r)=1$.

Proposition 4.1 *Let $\boldsymbol{\omega}\in\mathcal{L}^n$. If $\boldsymbol{\omega}=(\omega_0,\dots,\omega_n)\in\mathcal{L}^n_{>0}$, then all the roots of $f_{\boldsymbol{\omega}}(z)$ are real if and only if $\boldsymbol{\omega}=\omega_n(a^n,a^{n-1},\dots,a,1)$ for some $a>0$, i.e., if and only if $f_{\boldsymbol{\omega}}(z)=\omega_n(z+a)^n$; therefore, all the roots are equal.*

If $\boldsymbol{\omega}=(\omega_0,0,\dots,0,\omega_n)\in\mathcal{L}^n$, then all the roots are real if and only if $\omega_0=0$, i.e., if and only if all the roots are equal 0.

Proof If $\boldsymbol{\omega}=(\omega_0,0,\dots,0,\omega_n)\in\mathcal{L}^n$, then $f_{\boldsymbol{\omega}}(z)=\omega_0+\omega_n z^n$, and the thesis follows trivially.

So we assume that $\boldsymbol{\omega}=(\omega_0,\dots,\omega_n)\in\mathcal{L}^n_{>0}$, and let $\gamma_1,\dots,\gamma_n\in\mathbb{R}_{<0}$ be the roots of $f_{\boldsymbol{\omega}}(z)$. Then, its coefficients can be expressed in terms of the elementary symmetric functions of the roots, namely,

$$\mathrm{s}_i(\gamma_1,\dots,\gamma_n)=(-1)^i\binom{n}{i}\frac{\omega_{n-i}}{\omega_n},\qquad i=0,\dots,n.$$

We know that the elementary symmetric functions satisfy the Newton inequalities (see, e.g., [5, Theorem 51]), i.e.,

$$\left(\frac{\mathrm{s}_i(\gamma_1,\dots,\gamma_n)}{\binom{n}{i}}\right)^2\geq\frac{\mathrm{s}_{i-1}(\gamma_1,\dots,\gamma_n)}{\binom{n}{i-1}}\frac{\mathrm{s}_{i+1}(\gamma_1,\dots,\gamma_n)}{\binom{n}{i+1}},\qquad i=1,\dots,n-1,$$

and thus we get $\omega_{n-i}^2\geq\omega_{n-i+1}\omega_{n-i-1}$, for all $i=1,\dots,n-1$. Since $\boldsymbol{\omega}$ is a log-convex tuple, we must have the equalities $\omega_{n-i}^2=\omega_{n-i+1}\omega_{n-i-1}$ for every $i=1,\dots,n-1$, and hence, setting $a=\omega_{n-1}/\omega_n$, we have that $\omega_{n-i}=a^i\omega_n$ for all $i=0,\dots,n$. Therefore, $f_{\boldsymbol{\omega}}(z)=\omega_n(z+a)^n$, as required. ■

5 Bounds for the roots of log-convex coefficients and dual Steiner polynomials

In this section, we investigate additional properties for the roots of dual Steiner polynomials, providing, among others, bounds for them and a new characterization of the Euclidean ball. Again, some of these results will be obtained as consequences of the corresponding ones for log-convex coefficients polynomials. We start with some bounds for the moduli of the roots of these general polynomials.

Proposition 5.1 *Let $\boldsymbol{\omega}=(\omega_0,\dots,\omega_n)\in\mathcal{L}^n_{>0}$ and let γ_i, $i=1,\dots,n$, be the roots of the polynomial $f_{\boldsymbol{\omega}}(z)$. Then,*

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(i) *The roots are bounded by*

$$\frac{\omega_{n-1}}{n\omega_n} \le |\gamma_i| \le \frac{n\omega_0}{\omega_1}.$$

(ii) *Moreover,*

$$\frac{\omega_{n-1}}{\omega_n} \le \max_{1\le i\le n} |\gamma_i| \quad \textit{and} \quad \min_{1\le i\le n} |\gamma_i| \le \frac{\omega_0}{\omega_1}.$$

Proof In order to prove (i), we follow the ideas of [6, Proposition 2.1]. Using Proposition 3.1, we just have to find the minimum and maximum of the quotient $\binom{n}{i}\omega_i/\left(\binom{n}{i+1}\omega_{i+1}\right)$, $i = 0,\dots,n-1$. The log-convexity of the tuple $(\omega_0,\dots,\omega_n)$, and since $\omega_i > 0$ for all $i = 0,\dots,n$, implies that

$$\text{(5.1)} \qquad \frac{\omega_0}{\omega_1} \ge \frac{\omega_1}{\omega_2} \ge \cdots \ge \frac{\omega_{n-1}}{\omega_n},$$

whereas $\binom{n}{i}/\binom{n}{i+1}$ is increasing. So we get

$$\frac{\omega_{n-1}}{n\omega_n} \le \frac{\binom{n}{i}\omega_i}{\binom{n}{i+1}\omega_{i+1}} \le \frac{n\omega_0}{\omega_1}$$

for $i = 0,\dots,n-1$.

Next, we use Proposition 3.2: for $f_{\boldsymbol{\omega}}(z)$ it is $b_i = \binom{n}{n-i}\omega_{n-i}/\omega_n$, and hence the biggest (with regard to modulus) root of $f_{\boldsymbol{\omega}}(z)$ satisfies

$$\max_{1\le i\le n} |\gamma_i| \ge \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\omega_{n-i}}{\omega_n}\right)^{1/i}.$$

Now, from (5.1) and taking geometric means, we get that

$$\left(\frac{\omega_{n-i}}{\omega_{n-i+1}}\frac{\omega_{n-i+1}}{\omega_{n-i+2}}\cdots\frac{\omega_{n-2}}{\omega_{n-1}}\right)^{1/(i-1)} \ge \frac{\omega_{n-1}}{\omega_n}, \qquad i = 1,\dots,n,$$

and thus $\omega_{n-i} \ge \omega_{n-1}^{i}/\omega_n^{i-1}$ for all $i = 1,\dots,n$. Therefore,

$$\max_{1\le i\le n} |\gamma_i| \ge \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\omega_n^{1/i}}\frac{\omega_{n-1}}{\omega_n^{(i-1)/i}} = \frac{\omega_{n-1}}{\omega_n}.$$

Finally, to show the second bound in (ii), we note, on the one hand, that γ_i are also roots of the polynomial $f_{\boldsymbol{\omega}'}(z)$ for $\boldsymbol{\omega}' = (1, \omega_1/\omega_0,\dots,\omega_n/\omega_0) \in \mathcal{L}^n_{>0}$. On the other hand, Theorem 3.2 for $r = 1$ ensures that

$$\min_{1\le i\le n} |\gamma_i| \le \frac{\omega_0}{\omega_1}.$$

This concludes the proof. ■

Next, we get bounds for the real and the imaginary parts of the roots.

Proposition 5.2 *Let $\boldsymbol{\omega} = (\omega_0,\dots,\omega_n) \in \mathcal{L}^n_{>0}$ and let γ_i, $i = 1,\dots,n$, be the roots of the polynomial $f_{\boldsymbol{\omega}}(z)$. Then, the following properties hold:*

(i) *There exists $i_0 \in \{1,\dots,n\}$ such that $\left|\mathrm{Re}(\gamma_{i_0})\right| \le \omega_0/\omega_1$.*

(ii) $\sum_{i=0}^{n}\left|\mathrm{Re}(\gamma_i)\right| \ge n\omega_{n-1}/\omega_n$, *and equality holds if* $\mathrm{Re}(\gamma_i) \le 0$ *for all* $i = 1,\dots,n$.

(iii) $\max_{1\le i\le n}\left|\mathrm{Im}(\gamma_i)\right| \ge (1/\omega_n)\sqrt{\omega_n\omega_{n-2} - \omega_{n-1}^2}$.

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Proof Since $f_{\boldsymbol{\omega}}(z) = \omega_n \prod_{i=1}^n (z - \gamma_i)$ we have

$$(-1)^n \omega_0 = \omega_n \prod_{i=1}^n \gamma_i \quad \text{and} \quad (-1)^{n-1} n\omega_1 = \omega_n \sum_{i=1}^n \prod_{j\neq i} \gamma_j.$$

Thus, we get

$$-n\frac{\omega_1}{\omega_0} = \sum_{i=1}^n \frac{1}{\gamma_i} = \sum_{i=1}^n \operatorname{Re}\left(\frac{1}{\gamma_i}\right).$$

Therefore, there exists a root γ_{i_0} such that $\operatorname{Re}(1/\gamma_{i_0}) \leq -\omega_1/\omega_0$, i.e., such that $|\operatorname{Re}(1/\gamma_{i_0})| \geq \omega_1/\omega_0$, and hence $|\operatorname{Re}(\gamma_{i_0})| \leq \omega_0/\omega_1$. It proves (i).

Next, since

$$\omega_n \sum_{i=1}^n \gamma_i = -n\omega_{n-1},$$

we have

$$n\frac{\omega_{n-1}}{\omega_n} = \left|\sum_{i=1}^n \gamma_i\right| = \left|\sum_{i=1}^n \operatorname{Re}(\gamma_i)\right| \leq \sum_{i=1}^n |\operatorname{Re}(\gamma_i)|.$$

Furthermore, if all the roots have non-positive real part, we even have

$$\sum_{i=1}^n |\operatorname{Re}(\gamma_i)| = \left|\sum_{i=1}^n \operatorname{Re}(\gamma_i)\right|,$$

which shows (ii).

Finally, we prove (iii). Denoting by $\gamma_i^{(j)}$, $j = 1, \ldots, n-2$ and $i = 1, \ldots, n-j$, the roots of the jth derivative $f_{\boldsymbol{\omega}}^{(j)}(z)$, Theorem 3.4 yields

$$\begin{aligned}\sum_{i=1}^n |\operatorname{Im}(\gamma_i)| &\geq \frac{n}{n-1}\sum_{i=1}^{n-1} \left|\operatorname{Im}\left(\gamma_i^{(1)}\right)\right| \geq \frac{n}{n-1}\frac{n-1}{n-2}\sum_{i=1}^{n-2} \left|\operatorname{Im}\left(\gamma_i^{(2)}\right)\right| \\ &\geq \frac{n}{n-1}\frac{n-1}{n-2}\cdots\frac{3}{2}\sum_{i=1}^{2} \left|\operatorname{Im}\left(\gamma_i^{(n-2)}\right)\right| = \frac{n}{2}\sum_{i=1}^{2} \left|\operatorname{Im}\left(\gamma_i^{(n-2)}\right)\right|.\end{aligned}$$

Since $f_{\boldsymbol{\omega}}^{(n-2)}(z) = (n!/2)\left(\omega_{n-2} + 2\omega_{n-1} z + \omega_n z^2\right)$, the roots of $f_{\boldsymbol{\omega}}^{(n-2)}(z)$ are given by

$$\gamma_1^{(n-2)}, \gamma_2^{(n-2)} = \frac{-\omega_{n-1}}{\omega_n} \pm \frac{\sqrt{\omega_n\omega_{n-2} - \omega_{n-1}^2}}{\omega_n}\,\mathrm{i},$$

and then we get

$$\sum_{i=1}^n |\operatorname{Im}(\gamma_i)| \geq n\frac{\sqrt{\omega_n\omega_{n-2} - \omega_{n-1}^2}}{\omega_n}.$$

Therefore,

$$n\max_{1\leq i\leq n} |\operatorname{Im}(\gamma_i)| \geq \sum_{i=1}^n |\operatorname{Im}(\gamma_i)| \geq n\frac{\sqrt{\omega_n\omega_{n-2} - \omega_{n-1}^2}}{\omega_n}.$$ ∎

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5.1 Bounds for the roots of dual Steiner polynomials

Using Propositions 5.1 and 5.2, we directly get bounds for the moduli and real parts of the roots of dual Steiner polynomials depending on the dual quermassintegrals. We are also interested in obtaining bounds in terms of additional functionals related to star bodies.

We define the inner and outer radii, $\mathrm{r}(K)$ and $\mathrm{R}(K)$, of a star body K as

$$\mathrm{r}(K) = \max\{r > 0 : rB_n \subset K\}, \qquad \mathrm{R}(K) = \min\{R > 0 : K \subset RB_n\},$$

and we will use the inequalities

$$\mathrm{r}(K)\widetilde{\mathrm{W}}_{i+1}(K) \le \widetilde{\mathrm{W}}_i(K) \le \mathrm{R}(K)\widetilde{\mathrm{W}}_{i+1}(K) \tag{5.2}$$

for $i = 0, \dots, n-1$: since $\mathrm{r}(K)B_n \subset K$ and $K \subset \mathrm{R}(K)B_n$, the above inequalities are a direct consequence of the monotonicity of the dual mixed volumes. We stress that the functionals $\mathrm{r}(K)$ and $\mathrm{R}(K)$ are different from the classical inradius and circumradius, because here the balls are taken to be centered at the origin; indeed, since dual quermassintegrals are not translation invariant, inequalities (5.2) would be in general not true for the classical inradius and circumradius.

We start considering the two-dimensional case. By the dual Aleksandrov–Fenchel inequalities (2.2), the roots of the polynomial $\widetilde{f}_{K;B_2}(z)$, namely,

$$\gamma_i = -\frac{\widetilde{\mathrm{W}}_1(K)}{\pi} \pm \frac{\sqrt{\pi \mathrm{vol}(K) - \widetilde{\mathrm{W}}_1(K)^2}}{\pi}\,\mathrm{i}, \quad i = 1, 2,$$

are always nonreal complex numbers (unless $K = \lambda B_2$), with modulus $|\gamma_i| = \sqrt{\mathrm{vol}(K)/\pi}$; hence, using (5.2), we get the bounds

$$-\mathrm{R}(K) \le \mathrm{Re}(\gamma_i) \le -\mathrm{r}(K) \quad \text{and} \quad \mathrm{r}(K) \le |\gamma_i| \le \mathrm{R}(K).$$

In arbitrary dimension, we can get the following bounds for the moduli and real parts of the roots. They are obtained as direct consequences of Propositions 5.1 and 5.2, and using (5.2):

Proposition 5.3 *Let $K \in \mathcal{S}_0^n$ and let γ_i, $i = 1, \dots, n$, be the roots of the dual Steiner polynomial $\widetilde{f}_{K;B_n}(z)$. Then the following properties hold:*

(i) *The roots are bounded by*

$$\frac{\mathrm{r}(K)}{n} \le |\gamma_i| \le n\mathrm{R}(K).$$

Furthermore,

$$\mathrm{r}(K) \le \max_{1\le i\le n} |\gamma_i| \quad \textit{and} \quad \min_{1\le i\le n} |\gamma_i| \le \mathrm{R}(K).$$

(ii) *There exists $i_0 \in \{1, \dots, n\}$ such that $|\mathrm{Re}(\gamma_{i_0})| \le \mathrm{R}(K)$.*

(iii) *Moreover, if $\mathrm{Re}(\gamma_i) \le 0$ for all $i = 1, \dots, n$ then $\sum_{i=0}^n |\mathrm{Re}(\gamma_i)| \le n\mathrm{R}(K)$.*

(iv) $\sum_{i=0}^n |\mathrm{Re}(\gamma_i)| \ge n\mathrm{r}(K)$.

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Remark 5.1 Following the argument of the proof of (i) in Proposition 5.2, we can also write the inequality

$$-n\frac{1}{\mathrm{r}(K)} \le -n\frac{\widetilde{\mathrm{W}}_1(K)}{\mathrm{vol}(K)} = \sum_{i=1}^{n} \mathrm{Re}\left(\frac{1}{\gamma_i}\right).$$

Thus, one may conclude the existence of a root γ_j with $\mathrm{Re}(1/\gamma_j) \ge -1/\mathrm{r}(K)$. This would provide a bound for the real part of the root $\widetilde{\gamma}_j = 1/\gamma_j$ of the dual Steiner polynomial $\widetilde{f}_{B_n;K}(z)$. Indeed, since $\widetilde{\mathrm{W}}_i(K) = \widetilde{\mathrm{W}}_{n-i}(B_n;K)$ we have $\widetilde{f}_{K;B_n}(z) = z^n \widetilde{f}_{B_n;K}(1/z)$, and thus, the roots of $\widetilde{f}_{B_n;K}(z)$ are precisely $1/\gamma_i$, $i = 1, \dots, n$.

5.2 A characterization of the ball via the roots of the dual Steiner polynomials

As we have seen, for any dual Steiner polynomial there always exists a root γ with real part lying in the interval $\left[-\mathrm{R}(K), \mathrm{R}(K)\right]$ (cf. Proposition 5.3 (ii)), and indeed, there are star bodies such that all the real parts of the roots of $\widetilde{f}_{K;B_n}(z)$ lie in the above interval, as the following example shows.

Example 5.1 We consider the positive numbers $(2,1,1)$. Since they satisfy the dual Aleksandrov–Fenchel inequalities, Theorem 2.1 ensures the existence of $K \in \mathcal{S}_0^3$ such that $\mathrm{vol}(K) = 2$ and $\widetilde{\mathrm{W}}_1(K) = \widetilde{\mathrm{W}}_2(K) = 1$. Then, by numerical computations, one can check that the real parts of the roots of the dual Steiner polynomial $\widetilde{f}_{K;B_3}(z) = 2 + 3z + 3z^2 + (4\pi/3)z^3$ lie in the interval

$$(-2,2) = \left(-\frac{\mathrm{vol}(K)}{\widetilde{\mathrm{W}}_1(K)}, \frac{\mathrm{vol}(K)}{\widetilde{\mathrm{W}}_1(K)}\right) \subset \left(-\mathrm{R}(K), \mathrm{R}(K)\right),$$

where the last inclusion follows from (5.2).

However, the interval $\left(-\mathrm{R}(K), \mathrm{R}(K)\right)$ cannot be reduced to the one determined by the inner radius, as Corollary 5.1 shows. This will be an easy consequence of the characterization of the Euclidean ball given in Theorem 2.2, which we prove next.

Proof of Theorem 2.2 If $K = aB_n$, then $\widetilde{f}_{K;B_n}(z) = \kappa_n(z+a)^n$, and hence it has an n-fold real root, namely, $\gamma_1 = \cdots = \gamma_n = -a$. So we assume that $\mathrm{Re}(\gamma_i) = -a$ for $i = 1, \dots, n$, and we prove the assertion by induction on the dimension.

Let $n = 3$, and we suppose that K is not a ball. Let $\gamma_1 = -a$, $\gamma_2 = -a + b\mathrm{i}$ and $\gamma_3 = -a - b\mathrm{i}$, where $b > 0$ because $K \ne aB_3$. Then $\widetilde{f}_{K;B_3}(z) = \kappa_3\left(z^3 + 3az^2 + (3a^2 + b^2)z + a(a^2 + b^2)\right)$, and hence

$$\widetilde{\mathrm{W}}_2(K) = \kappa_3 a, \quad \widetilde{\mathrm{W}}_1(K) = \frac{\kappa_3}{3}(3a^2 + b^2), \quad \mathrm{vol}(K) = \kappa_3 a(a^2 + b^2). \tag{5.3}$$

Since K is not a ball, the dual Aleksandrov–Fenchel inequalities must hold strictly; in particular one has $\widetilde{\mathrm{W}}_1(K)^2 < \mathrm{vol}(K)\widetilde{\mathrm{W}}_2(K)$, or equivalently, using (5.3), we have

$$\frac{\kappa_3^2}{9} b^2(b^2 - 3a^2) < 0.$$

Since $b \ne 0$, the above inequality holds if and only if $b < \sqrt{3}\,a$. But this is a contradiction, because, as we have proved in Corollary 2.1, if $-a + b\mathrm{i}$ is a root of a

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three-dimensional dual Steiner polynomial then $b > \sqrt{3}\,a$. Therefore, $K = aB_3$, which concludes the proof in the case $n = 3$.

Now let $n > 3$, and we assume that the assertion is true in dimension $n-1$. Again, we suppose that K is not a ball. It is known that the derivative of $\widetilde{f}_{K;B_n}(z)$ is also a dual Steiner polynomial (see [1, Proposition 4.1]). Hence, there exists $K' \in \mathcal{S}_0^{n-1}$ such that

$$\begin{aligned}\widetilde{f}'_{K;B_n}(z) &= \sum_{i=1}^{n}\binom{n}{i} i\widetilde{\mathrm{W}}_i(K)z^{i-1} = \sum_{i=0}^{n-1}\binom{n-1}{i} n\widetilde{\mathrm{W}}_{i+1}(K)z^i\\ &= \frac{n\kappa_n}{\kappa_{n-1}}\sum_{i=0}^{n-1}\binom{n-1}{i}\widetilde{\mathrm{W}}_i(K')z^i = \frac{n\kappa_n}{\kappa_{n-1}}\widetilde{f}_{K';B_{n-1}}(z),\end{aligned}$$

where $\widetilde{\mathrm{W}}_i(K') = (\kappa_{n-1}/\kappa_n)\widetilde{\mathrm{W}}_{i+1}(K)$, $i = 0,\dots,n-1$. On the one hand, since K is not a ball, the dual Aleksandrov–Fenchel inequalities (2.2) hold strictly for the dual quermassintegrals $\widetilde{\mathrm{W}}_i(K)$, and therefore, the same occurs for $\widetilde{\mathrm{W}}_i(K')$; so, K' is not an $(n-1)$-dimensional ball.

On the other hand, denoting by γ'_i, $i = 1,\dots,n-1$, the roots of $\widetilde{f}'_{K;B_n}(z)$, Lucas' theorem (see Theorem 3.3) ensures that

$$\gamma'_1,\dots,\gamma'_{n-1} \in \operatorname{conv}\{\gamma_1,\dots,\gamma_n\},$$

and hence $\operatorname{Re}(\gamma'_i) = -a$ for all $i = 1,\dots,n-1$. Since we are assuming that the assertion is true in dimension $n-1$, we get the desired contradiction. This concludes the proof. ■

Corollary 5.1 *Let $K \in \mathcal{S}_0^n$ and let γ_i, $i = 1,\dots,n$, be the roots of the dual Steiner polynomial $\widetilde{f}_{K;B_n}(z)$. If $\operatorname{Re}(\gamma_i) \in [-\mathrm{r}(K),\mathrm{r}(K))$ for all $i = 1,\dots,n$, then $\gamma_i = -\mathrm{r}(K)$ for all $i = 1,\dots,n$, and hence $K = \mathrm{r}(K)B_n$.*

Proof We assume there exists $j \in \{1,\dots,n\}$ such that $\operatorname{Re}(\gamma_j) > -\mathrm{r}(K)$. Then $\sum_{i=1}^{n}|\operatorname{Re}(\gamma_i)| < n\mathrm{r}(K)$, which is not possible (see Proposition 5.3 (iv)). Therefore $\operatorname{Re}(\gamma_i) = -\mathrm{r}(K)$ for all $i = 1,\dots,n$ and Theorem 2.2 gives the result. ■

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