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DUAL NAKANO POSITIVITY AND SINGULAR NAKANO POSITIVITY OF DIRECT IMAGE SHEAVES

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Abstract. Let $f : X \to Y$ be a surjective projective map, and let L be a holomorphic line bundle on X equipped with a (singular) semi-positive Hermitian metric h. In this article, by studying the canonical metric on the direct image sheaf of the twisted relative canonical bundles $K_{X/Y} \otimes L \otimes \mathscr{I}(h)$, we obtain that this metric has dual Nakano semi-positivity when h is smooth and there is no deformation by f and that this metric has locally Nakano semi-positivity in the singular sense when h is singular.

§1. Introduction

Let X be a Kähler manifold of dimension $m+n$, and let Y be a complex manifold of dimension m. We consider a proper holomorphic submersion $f : X \to Y$. The relative canonical bundle $K_{X/Y}$ corresponding to the map f is $K_{X/Y} = K_X \otimes f^*K_Y^{-1}$. There is a natural isomorphism $K_{X/Y}|_{X_t} \cong K_{X_t}$ when restricted to a generic fiber X_t of $t \in Y$. It is effective in many studies that the variation of the complex structure of each fiber X_t is reflected in the positivity of the relative canonical bundle $K_{X/Y}$. Therefore, the positivity properties of this bundle play an important role in the study of the several complex variables and complex algebraic geometry. In practice, we frequently deal with twisted versions $K_{X/Y} \otimes L$, where $L \to X$ is a holomorphic line bundle equipped with a smooth (semi)-positive Hermitian metric h. One way to research the properties of this bundle is the direct image sheaf $f_*(K_{X/Y} \otimes L)$ on Y.

The positivity of this direct image sheaf has been well studied in [1]–[3], [11], [14], [23]. In [1], Berndtsson showed that the smooth canonical Hermitian metric H on $f_*(K_{X/Y} \otimes L)$ induced by h has Nakano (semi)-positivity. First, we show that the smooth canonical Hermitian metric H has dual Nakano (semi)-positivity if complex structures of fibers have no variation, which means that we can take the Kodaira–Spencer forms to be zero. Introducing the $(n-1,n-1)$-form to determine dual Nakano positivity (see Definition 2.3), we prove it by taking over Berndtsson's method of calculation to compute the positivity of the curvature.

Theorem 1.1. *Let L be a holomorphic line bundle over a Kähler manifold X equipped with a smooth (semi)-positive Hermitian metric h, and let $f : X \to Y$ be a proper holomorphic submersion between two complex manifolds. For the Kodaira–Spencer map $\rho_t : T^{1,0}_{Y,t} \to H^{0,1}(X_t, T^{1,0}_{X_t})$, if Kodaira–Spencer forms representing classes $\rho_t(\partial/\partial t_j)$ can be taken to be zero, then the smooth canonical Hermitian metric H on $f_*(K_{X/Y} \otimes L)$ has dual Nakano semi-positivity.*

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Examples of this theorem are projection maps from the direct product of manifolds (see Corollary 3.8) and the projectivized bundle $\pi : \mathbb{P}(V) \to Y$ for an ample vector bundle $V \to Y$ when $\det V$ has a metric satisfying certain condition (see Theorem 3.6).

Second, we consider the case where the metric h on L with semi-positivity is singular, that is, h is pseudo-effective. In this case, twisting the multiplier ideal sheaf $\mathscr{I}(h)$ further to the sheaf $K_{X/Y} \otimes L$, we study the positivity of the direct image sheaf $\mathcal{E} := f_*(K_{X/Y} \otimes L \otimes \mathscr{I}(h))$, where $f : X \to Y$ is a projective surjective morphism between two connected complex manifolds. It is known that the torsion-free coherent sheaf $\mathcal{E}$ has a singular canonical Hermitian metric H induced by h, and this metric satisfies the minimal extension property and is Griffiths semi-positive (see Theorem 5.4 [2], [14], [23]).

The positivity properties of the singular canonical Hermitian metric on this direct image sheaf metrics are crucial, given the partial resolution of the Iitaka conjecture using singular Griffiths semi-positivity (see [14, Th. 1.1]). We show that this singular canonical Hermitian metric H on $\mathcal{E}$ has a locally L^2-type Nakano semi-positivity. Let $Y(\mathcal{E}) \subseteq Y$ denote the maximal open subset where $\mathcal{E}$ is locally free, then $Z_{\mathcal{E}} := Y \setminus Y(\mathcal{E})$ is a closed analytic subset of codimension ≥ 2. Here, we define (see Definition 4.8) the set Σ_H on Y related to the unboundedness of H by

$$\Sigma_H := \{t \in Y \mid \mathcal{E}_t \subsetneq H^0(X_t, K_{X_t} \otimes L|_{X_t})\}.$$

Using the set Σ_H, we have the following.

Theorem 1.2. *If X is projective and there is an analytic set $A \subsetneq Y$ such that $\Sigma_H \subseteq A$, then H is full locally L^2-type Nakano semi-positive on $Y(\mathcal{E})$ as in Definition 4.10.*

The restriction of $\mathcal{E}$ to $Y(\mathcal{E})$ is holomorphic vector bundle, and the L^2-subsheaf of this vector bundle with respect to H is denoted by $\mathscr{E}(H) \subseteq \mathcal{E}|_{Y(\mathcal{E})}$ over $Y(\mathcal{E})$ which analogous to multiplier ideal sheaves. For the natural inclusion $j : Y(\mathcal{E}) = Y \setminus Z_{\mathcal{E}} \hookrightarrow Y$, we define the natural extended L^2-subsheaf with respect to H over Y by $\mathscr{E}_Y(H) := j_*\mathscr{E}(H)$ as in Definition 4.8.

Theorem 1.3. *If X is projective and there exists an analytic set A such that $\Sigma_H \subseteq A$, then the natural extended L^2-subsheaf $\mathscr{E}_Y(H)$ over Y is coherent.*

Finally, we consider the relationship between the minimal extension property and Nakano semi-positivity and show that if a torsion-free coherent sheaf has a metric satisfying the minimal extension property, this sheaf does not necessarily have a Nakano semi-positive metric. As a concrete example, we show that the quotient holomorphic vector bundle $(\mathbb{P}^n \times \mathbb{C}^{n+1})/\mathcal{O}_{\mathbb{P}^n}(-1)$ over $\mathbb{P}^n$ does not have a Nakano semi-positive metric and has a metric satisfying the minimal extension property.

§2. Positivity of smooth Hermitian metrics and L^2-estimates

In this section, we define various positivity for holomorphic vector bundles and investigate its equivalence condition.

Let X be a complex manifold of complex dimension n equipped with a Hermitian metric ω, and let (E,h) be a holomorphic Hermitian vector bundle of rank r over X. Let $(U,(z_1,\dots,z_n))$ be local coordinates, and let $D = D'^h + \overline{\partial}$ be the Chern connection of

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(E,h). The Chern curvature tensor $\Theta_{E,h} = D^2 = [D'^h, \overline{\partial}]$ is a $(1,1)$-form and is written as

$$\Theta_{E,h} = \sum \Theta^h_{jk} dz_j \wedge d\overline{z}_k,$$

where the coefficients $\Theta^h_{jk} = [D'^h_{z_j}, \overline{\partial}_{z_k}]$ are defined operators on U and $\overline{\partial}_{z_j} = \partial/\partial\overline{z}_j$.

The smooth Hermitian metric h on E is said to be Griffiths (semi)-positive if for any section u of E and any vector $v \in \mathbb{C}^n$ we have

$$\sum_{1\leq j,k\leq n} (\Theta^h_{jk} u, u)_h v_j \overline{v}_k > 0 \quad (\geq 0).$$

Moreover, h is said to be Nakano (semi)-positive if for any sections u_j of E we have

$$\sum_{1\leq j,k\leq n} (\Theta^h_{jk} u_j, u_k)_h > 0 \quad (\geq 0).$$

There is a natural antilinear isometry between E^* and E, which we will denote by J. Denote the pairing between E^* and E by $\langle\cdot,\cdot\rangle$. For any local section u of E and any local section ξ of E^*, we have

$$\langle \xi, u\rangle = (u, J\xi)_h.$$

Under the natural holomorphic structure on E^*, we obtain

$$\overline{\partial}_{z_j}\xi = J^{-1} D'^h_{z_j} J\xi,$$

and the Chern connection on E^* is given by

$$D'^{h^*}_{z_j}\xi = J^{-1}\overline{\partial}_{z_j} J\xi.$$

Then, through straightforward calculations, the following equation is obtained:

$$\begin{aligned} \overline{\partial}_{z_j}\langle\xi,u\rangle &= \langle\overline{\partial}_{z_j}\xi,u\rangle + \langle\xi,\overline{\partial}_{z_j}u\rangle, \\ \partial_{z_j}\langle\xi,u\rangle &= \langle D'^{h^*}_{z_j}\xi,u\rangle + \langle\xi, D'^h_{z_j}u\rangle, \\ 0 = [\partial_{z_j},\overline{\partial}_{z_k}]\langle\xi,u\rangle &= \langle\Theta^{h^*}_{jk}\xi,u\rangle + \langle\xi,\Theta^h_{jk}u\rangle. \end{aligned}$$

For any local sections $\xi_j \in C^\infty(E^*)$ and $u_j \in C^\infty(E)$ satisfying $u_j = J\xi_j$, we have

$$\sum (\Theta^{h^*}_{jk}\xi_j, \xi_k)_{h^*} = -\sum (\Theta^h_{jk} u_k, u_j)_h,$$

and for any local sections $u, v \in C^\infty(E)$, we have

$$\overline{\partial}_{z_k}\partial_{z_j}(u,v)_h = (D'^h_{z_j}u, D'^h_{z_k}v)_h + (\overline{\partial}_{z_k} D'^h_{z_j}u, v)_h + (u, D'^h_{z_k}\overline{\partial}_{z_j}v)_h + (\overline{\partial}_{z_k}u, \overline{\partial}_{z_j}v)_h.$$

If u is holomorphic, then $-\overline{\partial}_{z_k} D'^h_{z_j} u = \Theta^h_{jk} u$. Thus, the following equation is derived:

$$\frac{\partial^2}{\partial z_j \partial\overline{z}_k}(u,v)_h = (D'^h_{z_j}u, D'^h_{z_k}v)_h - (\Theta^h_{jk}u, v)_h,$$

for any local sections $u, v \in \mathcal{O}(E)_x$. In particular, we obtain

$$\partial_{z_j}\overline{\partial}_{z_k}(u,v)_h = -(\Theta^h_{jk}u,v)_h \quad \text{at } x,$$

if $u, v \in \mathcal{O}(E)_x$ satisfying $D'^h u = D'^h v = 0$ at x.

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Let $u=(u_1,\dots,u_n)$ be an n-tuple of local holomorphic sections of E, that is, $u_j \in \mathcal{O}(E)$. We define T^h_u, an $(n-1,n-1)$-form through

$$T^h_u := \sum (u_j,u_k)_h \widehat{dz_j \wedge d\overline{z}_k},$$

where $(z_1,\dots,z_n)$ are the local coordinates on X, and $\widehat{dz_j \wedge d\overline{z}_k}$ denotes the wedge product of all dz_l and $d\overline{z}_l$ expect dz_j and $d\overline{z}_k$, multiplied by a constant of absolute value 1, that is, $idz_j \wedge d\overline{z}_k \wedge \widehat{dz_j \wedge d\overline{z}_k} = dV_{\mathbb{C}^n}$. Hence, if $D'^h u_j = 0$ at x, then we get

$$i\partial\overline{\partial} T^h_u = -\sum (\Theta^h_{jk} u_j, u_k)_h dV_{\mathbb{C}^n},$$

at x by the equation

$$\begin{aligned} i\partial\overline{\partial} T^h_u &= \sum (D'^h_{z_j} u_j, D'^h_{z_k} u_k)_h dV_{\mathbb{C}^n} - \sum (\Theta^h_{jk} u_j, u_k)_h dV_{\mathbb{C}^n} \\ &= ||\sum D'^h_{z_j} u_j||^2_h - \sum (\Theta^h_{jk} u_j, u_k)_h dV_{\mathbb{C}^n}. \end{aligned}$$

Proposition 2.1 (cf. [1], [24]). *We have:*

- *h is Nakano semi-positive if and only if for any $x \in X$ and any $u_j \in \mathcal{O}(E)_x$ such that $D'^h u_j = 0$ at x, the $(n-1,n-1)$-form $-T^h_u$ is plurisubharmonic at x, that is, $-i\partial\overline{\partial} T^h_u \geq 0$.*
- *h is Nakano semi-negative if and only if for any $x \in X$ and any $u_j \in \mathcal{O}(E)_x$, the $(n-1,n-1)$-form T^h_u is plurisubharmonic at x, that is, $i\partial\overline{\partial} T^h_u \geq 0$.*

We introduce another notion about Nakano-type positivity.

Definition 2.2 (cf. [26], [18]). Let X be a complex manifold of complex dimension n, and let (E,h) be a holomorphic Hermitian vector bundle of rank r over X. (E,h) is said to be *dual Nakano positive* (resp. *dual Nakano semi-positive*) if (E^*,h^*) is Nakano negative (resp. Nakano semi-negative).

Let $\xi_j \in C^\infty(E^*)$ and $u_j \in C^\infty(E)$ be r-tuples of smooth sections E^* such that $u_j = J\xi_j$. If h is dual Nakano semi-positive, then

$$0 \geq \sum (\Theta^{h^*}_{jk} \xi_j, \xi_k)_{h^*} = -\sum (\Theta^h_{jk} u_k, u_j)_h,$$

that is, $\sum (\Theta^h_{jk} u_k, u_j)_h \geq 0$. Enough to consider at each point, for any $x \in X$ and any $u_j \in C^\infty(E)_x$ if $\sum (\Theta^h_{jk} u_k, u_j)_h \geq 0$ at x then h is dual Nakano semi-positive. Hence, we have that h is dual Nakano semi-positive if and only if $\sum (\Theta^h_{jk} u_k, u_j)_h \geq 0$ at any points x, for any $u_j \in C^\infty(E)_x$.

Definition 2.3. Let $u=(u_1,\dots,u_n)$ be an n-tuple of local holomorphic sections of E, that is, $u_j \in \mathcal{O}(E)$. We define $\widetilde{T}^h_u$ as an $(n-1,n-1)$-form

$$\widetilde{T}^h_u := \sum (u_k,u_j)_h \widehat{dz_j \wedge d\overline{z}_k},$$

where $(z_1,\dots,z_n)$ are the local coordinates on X.

Proposition 2.4. *The smooth Hermitian metric h on E is dual Nakano semi-positive if and only if for any $x \in X$ and any $u_j \in \mathcal{O}(E)_x$ such that $D'^h u_j = 0$ at x, the $(n-1,n-1)$-form $-\widetilde{T}^h_u$ is plurisubharmonic at x, that is, $-i\partial\overline{\partial}\widetilde{T}^h_u \geq 0$.*

Proof. This yields the following calculation:

$$0 \geq i\partial\overline{\partial}\widetilde{T}^h_u = -\sum(\Theta^h_{jk}u_k, u_j)_h dV_{\mathbb{C}^n} = \sum(\Theta^{h^*}_{jk}\xi_j, \xi_k)_{h^*} dV_{\mathbb{C}^n},$$

where $\xi_j := J^{-1}u_j \in \mathcal{E}(E^*)_x$. ∎

By using this proposition, we can examine dual Nakano semi-positivity of h without using the dual metric h^*. Finally, we introduce Hörmander's L^2-existence theorem.

THEOREM 2.5 (cf. [9, Chap. VIII, Th. 6.1]). *Let $(X,\widehat{\omega})$ be a complete Kähler manifold, let ω be another Kähler metric which is not necessarily complete, and let (E,h) be a holomorphic vector bundle satisfying $A_{h,\omega} := [i\Theta_{E,h}, \Lambda_\omega] \geq 0$ on $\bigwedge^{n,q} T^*_X \otimes E$. Then, for any $\overline{\partial}$-closed $f \in L^2_{n,q}(X,E,h,\omega)$, there exists $u \in L^2_{n,q-1}(X,E,h,\omega)$ satisfies $\overline{\partial}u = f$ and*

$$\int_X |u|^2_{h,\omega} dV_\omega \leq \int_X \langle A^{-1}_{h,\omega} f, f\rangle_{h,\omega} dV_\omega,$$

where we assume that the right-hand side is finite.

§3. Dual Nakano positivity of direct image sheaves

3.1 Smooth canonical Hermitian metric of direct image sheaves

Let X be a Kähler manifold of dimension $m+n$, and let Y be a complex manifold of dimension m. We consider a proper holomorphic submersion $f : X \to Y$. The *relative canonical bundle* $K_{X/Y}$ corresponding to the map f is

$$K_{X/Y} = K_X \otimes f^* K_Y^{-1}.$$

When restricted to a generic fiber X_t of t, we get $K_{X/Y}|_{X_t} \cong K_{X_t}$.

Let L be a holomorphic line bundle over X equipped with a smooth semi-positive Hermitian metric h, that is, $i\Theta_{L,h} \geq 0$. In this subsection, we discuss the complex structure of the direct image sheaf $f_*(K_{X/Y} \otimes L)$ on Y and the smooth canonical Hermitian metric H of this sheaf induced by h (cf. [1]). Fixed a point $t \in Y$, any section $u \in H^0(X_t, K_{X_t} \otimes L|_{X_t})$ extends in the sense that there is a holomorphic section

$$U \in H^0(f^{-1}(\Omega), K_X \otimes L|_{f^{-1}(\Omega)}) \cong H^0(\Omega, K_Y \otimes f_*(K_{X/Y} \otimes L))$$

such that $U|_{X_t} = u \wedge dt$ for some neighborhood Ω of t from the Ohsawa–Takegoshi L^2-extension theorem (cf. [22]) and Kähler-ness of X. Here, we abusively denote by dt the inverse image of a local generator $dt_1 \wedge \cdots \wedge dt_m$ of K_Y. In [1], it was claimed that the total space

$$F := \bigcup_{t\in Y} H^0(X_t, K_{X_t} \otimes L|_{X_t})$$

has a natural structure of holomorphic vector bundle of rank $r := h^0(X_t, K_{X_t} \otimes L_{X_t})$ over Y and coincides with the direct image $f_*(K_{X/Y} \otimes L)$. Therefore, the space of local *smooth* sections of $F|_\Omega$ are simply the sections of the bundle $K_{X/Y} \otimes L|_{f^{-1}(\Omega)}$ whose restriction to each fiber of f is holomorphic.

The vector bundle $F = f_*(K_{X/Y} \otimes L)$ admits a natural *complex structure* as follows. Let u be a local section of E, then u is holomorphic if

$$\overline{\partial}u \wedge dt = 0.$$

This is equivalent to saying that the section $u \wedge dt$ of $K_X \otimes L$ is holomorphic.

Note that u is holomorphic, that is, $\overline{\partial}u\wedge dt=0$, which means that $\overline{\partial}u$ can be written

$$\overline{\partial}u=\sum\eta^j\wedge dt_j,$$

with η^j smooth forms of bidegree $(n-1,1)$. Here, the following relationship is known (see [1]) between η^j and the Kodaira–Spencer map $\rho_t:T^{1,0}_{Y,t}\to H^{0,1}(X_t,T^{1,0}_{X_t})$:

$$\eta^j=\theta_j\rfloor u,$$

on each fiber where the classes $\rho_t(\partial/\partial t_j)$ can be represented by Kodaira–Spencer forms θ_j, that is, $\{\theta_j\}\in\rho_t(\partial/\partial t_j)$.

The smooth Hermitian metric h of L induces the smooth *canonical Hermitian metric* H of F as follows. Let u,v be two local sections of F. We denote by (u_t) the family of L-twisted holomorphic $(n,0)$-forms on fibers K_{X_t} induced by u. The restriction of u_t to X_t is unique and denoted simply as u. Then the canonical Hermitian metric H is defined by

$$(u,v)_H(t):=\int_{X_t}c_nu_t\wedge\overline{v}_te^{-\varphi}=\int_{X_t}c_nu\wedge\overline{v}e^{-\varphi},$$

where $h=e^{-\varphi}$ on locally and $c_n=i^{n^2}$. This metric is smooth by Ehresmann's fibration theorem and compactness of each fiber. This inner product of H is a function of t, and it will be convenient to write this function as

$$(u,v)_H=f_*(c_nu\wedge\overline{v}e^{-\varphi}),$$

where u and v are forms on X that represent the sections. Here f_* denotes the direct image of form defined by

$$\int_Y f_*(\alpha)\wedge\beta=\int_X\alpha\wedge f^*(\beta),$$

if α is a form on X and β is a form on Y.

3.2 Berndtsson calculation and Nakano positivity

Let $(t_1,\dots,t_m)$ be a local coordinate whose center is fixed point $y\in Y$. Let u_j be an m-tuple of local holomorphic sections to F that satisfy $D'^Hu_j=0$ at y, that is, $t=0$. Represent the u_j by smooth forms on X and put

$$\hat{u}:=\sum u_j\wedge\widehat{dt_j},$$

then we get

$$T^H_u=c_Nf_*(\hat{u}\wedge\overline{\hat{u}}e^{-\varphi}),$$

where $N=n+m-1$ and $\widehat{dt_j}$ is the wedge product of all differentials dt_k except dt_j such that $dt_j\wedge\widehat{dt_j}=dt=dt_1\wedge\cdots\wedge dt_m$.

Using the following proposition, Berndtsson computed $i\partial\overline{\partial}T^H_u$ at fixed points.

PROPOSITION 3.1 (cf. [1, Prop. 4.2]). *Let u be a section of F over an open set U containing the origin such that $\overline{\partial}u=0$ in U, that is, holomorphic, and $D'^Hu=0$ at $t=0$. Then u can be represented by a smooth $(n,0)$-form, still denoted u such that*

$$\overline{\partial}u=\sum\eta^k\wedge dt_k,$$

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where η^k is primitive on X_0, that is, satisfies $\eta^k \wedge \omega = 0$ on X_0, and furthermore

$$\partial^{\varphi} u \wedge \widehat{dt_j} = 0,$$

at $t=0$ for all j. Here, $\partial^{\varphi} \cdot = e^{\varphi}\partial(e^{-\varphi}\cdot)$.

Let $u_j \in \mathcal{O}(F)$ such that $D'^H u_j = 0$ at $t=0$, then we have

$$i\partial\overline{\partial} T^H_u = -c_N f_*(\hat{u}\wedge\overline{\hat{u}}\wedge i\partial\overline{\partial}\varphi e^{-\varphi}) - \Big(\int_{X_0} |\eta|^2 e^{-\varphi} dV_z\Big) dV_t,$$

at $t=0$, where $\overline{\partial} u_j = \sum \eta^k_j \wedge dt_k$ and $\eta = \sum \eta^j_j$.

From this calculation and Proposition 2.1, the following theorem is obtained.

THEOREM 3.2 (cf. [1, Th. 1.2]). *If L has a smooth (semi)-positive Hermitian metric, then the smooth canonical Hermitian metric H on $F = f_*(K_{X/Y}\otimes L)$ is Nakano (semi)-positive.*

3.3 Calculation of $\widetilde{T}^H_u$ for the canonical Hermitian metric on $f_*(K_{X/Y}\otimes L)$

Represent the u_j by smooth forms on X and put

$$\tilde{u} := \sum \overline{u}_j \wedge \widehat{dt_j},$$

then we have the equality

$$\widetilde{T}^H_u = (-1)^n c_N f_*(\tilde{u}\wedge\overline{\tilde{u}} e^{-\varphi}),$$

where using $ic_N = (-1)^N(-1)^{nm} c_n c_m$ and $ic_m(-1)^m \widehat{dt_j}\wedge\widehat{d\overline{t}_k} = \widehat{dt_j\wedge d\overline{t}_k}$.

In this subsection, we show the following proposition.

PROPOSITION 3.3. *Let $u_j \in \mathcal{O}(F)$ such that $D'^H u_j = 0$ at $t=0$, then we have that*

$$i\partial\overline{\partial}\widetilde{T}^H_u = -c_N f_*(\hat{v}\wedge\overline{\hat{v}}\wedge i\partial\overline{\partial}\varphi e^{-\varphi}) + c_n\Big(\int_{X_0}\sum \eta^j_k \wedge \overline{\eta}^k_j e^{-\varphi}\Big) dV_t$$

at $t=0$, where $u_j = U_j dz$, $\hat{v} = \sum \overline{U}_j \wedge dz \wedge \widehat{dt_j}$ and $\overline{\partial} u_j = \sum \eta^k_j \wedge dt_k$. Here $c_N f_*(\hat{v}\wedge\overline{\hat{v}}\wedge i\partial\overline{\partial}\varphi e^{-\varphi}) \geq 0$ if φ is plurisubharmonic.

In particular, if η^j_k is primitive on X_0, then we can write $\eta^j_k = \sum \eta_{jkl}\widehat{dz_l}\wedge d\overline{z}_l$ and get

$$c_n\Big(\int_{X_0}\sum \eta^j_k\wedge\overline{\eta}^k_j e^{-\varphi}\Big) dV_t = -\Big(\int_{X_0}\sum \eta_{jkl}\overline{\eta}_{kjl} e^{-\varphi} dV_z\Big) dV_t.$$

Proof. By $\partial^{\varphi}\cdot = e^{\varphi}\partial(e^{-\varphi}\cdot)$, we get

$$(-1)^n \partial\widetilde{T}^H_u = c_N f_*(\partial\tilde{u}\wedge\overline{\tilde{u}} e^{-\varphi}) + (-1)^N c_N f_*(\tilde{u}\wedge\partial^{\varphi}\overline{\tilde{u}} e^{-\varphi}).$$

From the equation

$$\partial\tilde{u} = \sum \partial\overline{u}_j\wedge\widehat{dt_j} = \sum \overline{\overline{\partial} u_j}\wedge\widehat{dt_j} = \sum \overline{\eta}^l_j \wedge d\overline{t}_l\wedge\widehat{dt_j},$$

the form

$$\begin{aligned}\partial\tilde{u}\wedge\overline{\tilde{u}} &= \sum \overline{\eta}^l_j\wedge d\overline{t}_l\wedge\widehat{dt_j}\wedge u_k\wedge\widehat{d\overline{t}_k}\\ &= (-1)^N\sum\overline{\eta}^l_j\wedge\widehat{dt_j}\wedge u_k\wedge d\overline{t}_l\wedge\widehat{d\overline{t}_k}\\ &= (-1)^N\sum\overline{\eta}^k_j\wedge\widehat{dt_j}\wedge u_k\wedge d\overline{t}\end{aligned}$$

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contains a factor $d\bar{t}$. On the other hand, the push forward of an $(n+m,n+m-1)$-form is of bidegree $(m,m-1)$. Therefore, it follows that

$$f_*(\partial\tilde{u}\wedge\overline{\tilde{u}}e^{-\varphi})=0. \qquad \square$$

Thus, the following is derived:

$$(-1)^n\overline{\partial}\partial\widetilde{T}_u^H=(-1)^N c_N f_*(\overline{\partial}^{\varphi}\tilde{u}\wedge\partial^{\varphi}\overline{\tilde{u}}e^{-\varphi})+c_N f_*(\tilde{u}\wedge\overline{\partial}\partial^{\varphi}\overline{\tilde{u}}e^{-\varphi}).$$

By the equation $\overline{\partial}\partial^{\varphi}+\partial^{\varphi}\overline{\partial}=\partial\overline{\partial}\varphi$, we obtain

$$c_N f_*(\tilde{u}\wedge\overline{\partial}\partial^{\varphi}\overline{\tilde{u}}e^{-\varphi})=c_N f_*(\tilde{u}\wedge\overline{\tilde{u}}\wedge\partial\overline{\partial}\varphi e^{-\varphi})-c_N f_*(\tilde{u}\wedge\partial^{\varphi}\overline{\partial\tilde{u}}e^{-\varphi}),$$

and by the vanishing $f_*(\tilde{u}\wedge\overline{\partial\tilde{u}}e^{-\varphi})=0$, we obtain

$$0=\partial f_*(\tilde{u}\wedge\overline{\partial\tilde{u}}e^{-\varphi})=f_*(\partial\tilde{u}\wedge\overline{\partial\tilde{u}}e^{-\varphi})+(-1)^N f_*(\tilde{u}\wedge\partial^{\varphi}\overline{\partial\tilde{u}}e^{-\varphi}).$$

Hence, the following calculation result is obtained:

$$\begin{aligned}(-1)^n\overline{\partial}\partial\widetilde{T}_u^H=(-1)^N c_N f_*(\overline{\partial}^{\varphi}\tilde{u}\wedge\partial^{\varphi}\overline{\tilde{u}}e^{-\varphi})\\ +c_N f_*(\tilde{u}\wedge\overline{\tilde{u}}\wedge\partial\overline{\partial}\varphi e^{-\varphi})+(-1)^N c_N f_*(\partial\tilde{u}\wedge\overline{\partial\tilde{u}}e^{-\varphi}).\end{aligned}$$

Note that with the choice of representatives of our sections u_j furnished by Proposition 3.1, we have that $\overline{\partial}^{\varphi}\tilde{u}\wedge\partial^{\varphi}\overline{\tilde{u}}=0$ at $t=0$. In fact, $\overline{\partial}^{\varphi}\tilde{u}=\sum\overline{\partial^{\varphi}u_j}\wedge\widehat{dt_j}$ and

$$\overline{\partial}^{\varphi}\tilde{u}\wedge\partial^{\varphi}\overline{\tilde{u}}=\sum\overline{\partial^{\varphi}u_j}\wedge\widehat{dt_j}\wedge\partial^{\varphi}u_k\wedge\widehat{d\bar{t}_k}=0$$

at $t=0$, where $\partial^{\varphi}u_j\wedge\widehat{dt_k}=0$ at $t=0$ for all k.

Lemma 3.4. *We obtain that*

$$(-1)^N c_N f_*(\partial\tilde{u}\wedge\overline{\partial\tilde{u}}e^{-\varphi})=ic_n(-1)^n\Big(\int_{X_0}\sum\eta_k^j\wedge\overline{\eta}_j^k e^{-\varphi}\Big)dV_t$$

at $t=0$. In particular, if η_k^j is primitive on X_0, that is, $\eta_k^j\wedge\omega=0$ on X_0, then we can write $\eta_k^j=\sum\eta_{jkl}\widehat{dz_l}\wedge d\bar{z}_l$ and this integral value is

$$-i(-1)^n\Big(\int_{X_0}\sum\eta_{jkl}\overline{\eta}_{kjl}e^{-\varphi}dV_z\Big)dV_t.$$

Proof. Here $\partial\tilde{u}=\sum\overline{\eta}_j^l\wedge dt_l\wedge\widehat{dt_j}$, then

$$\partial\tilde{u}\wedge\overline{\partial\tilde{u}}=-(-1)^{nm}(-1)^{n^2}\sum\eta_k^j\wedge\overline{\eta}_j^k\wedge dt\wedge d\bar{t}.$$

Therefore, we get

$$\begin{aligned}(-1)^N c_N\partial\tilde{u}\wedge\overline{\partial\tilde{u}}&=-(-1)^N c_N(-1)^{nm}(-1)^{n^2}\sum\eta_k^j\wedge\overline{\eta}_j^k\wedge dt\wedge d\bar{t}\\ &=ic_n c_m(-1)^n\sum\eta_k^j\wedge\overline{\eta}_j^k\wedge dt\wedge d\bar{t}\\ &=ic_n(-1)^n\sum\eta_k^j\wedge\overline{\eta}_j^k\wedge dV_t,\end{aligned}$$

where $ic_N=(-1)^N(-1)^{nm}c_n c_m$.

If η_k^j is primitive, we can write $\eta_k^j = \sum \eta_{jkl}\widehat{dz_l}\wedge d\overline{z}_l$. Then we have

$$\begin{aligned}\eta_k^j \wedge \overline{\eta}_j^k &= \sum \eta_{jkl}\widehat{dz_l}\wedge d\overline{z}_l \wedge \overline{\eta}_{kj\mu}\widehat{d\overline{z}_\mu}\wedge dz_\mu \\ &= \sum \eta_{jkl}\overline{\eta}_{kjl}\widehat{dz_l}\wedge d\overline{z}_l\wedge \widehat{d\overline{z}_l}\wedge dz_l \\ &= (-1)^{2n-1}\sum \eta_{jkl}\overline{\eta}_{kjl} dz_l \wedge \widehat{dz_l}\wedge d\overline{z}_l \wedge \widehat{d\overline{z}_l} \\ &= -\sum \eta_{jkl}\overline{\eta}_{kjl} dz\wedge d\overline{z},\end{aligned}$$

where $c_n dz\wedge d\overline{z} = dV_z$. ∎

Hence, we obtain the following calculation:

$$i\partial\overline{\partial}\widetilde{T}_u^H = -(-1)^n c_N f_*(\tilde{u}\wedge\overline{\tilde{u}}\wedge i\partial\overline{\partial}\varphi e^{-\varphi}) + c_n\Big(\int_{X_0}\sum \eta_k^j\wedge\overline{\eta}_j^k e^{-\varphi}\Big)dV_t$$

at $t=0$. Let $u_j = U_j dz$ and $\varphi_{jk} := \partial_{t_j}\overline{\partial}_{t_k}\varphi$. Here, if φ is plurisubharmonic, then $c_N\hat{u}\wedge\overline{\hat{u}}\wedge i\partial\overline{\partial}\varphi = \sum \varphi_{jk}U_j\overline{U}_k dV_z\wedge dV_t \geq 0$. By $\tilde{u} = \sum \overline{u}_j\wedge\widehat{dt_j} = \sum \overline{U}_j d\overline{z}\wedge\widehat{dt_j}$ and $d\overline{z}\wedge dz = (-1)^{n^2} dz\wedge d\overline{z} = (-1)^n dz\wedge d\overline{z}$, we have that

$$c_N\tilde{u}\wedge\overline{\tilde{u}}\wedge i\partial\overline{\partial}\varphi = (-1)^n\sum \varphi_{jk}\overline{U}_j U_k dV_z\wedge dV_t = (-1)^n c_N\hat{v}\wedge\overline{\hat{v}}\wedge i\partial\overline{\partial}\varphi,$$

where $\hat{v} = \sum \overline{U}_j\wedge dz\wedge\widehat{dt_j}$ and that

$$\begin{aligned}(-1)^n c_N f_*(\tilde{u}\wedge\overline{\tilde{u}}\wedge i\partial\overline{\partial}\varphi e^{-\varphi}) &= c_N f_*(\hat{v}\wedge\overline{\hat{v}}\wedge i\partial\overline{\partial}\varphi e^{-\varphi}) \\ &= f_*\big(\sum \varphi_{jk}\overline{U}_j U_k e^{-\varphi} dV_z\wedge dV_t\big) \\ &= \Big(\int_{X_0}\sum \varphi_{jk}\overline{U}_j U_k e^{-\varphi} dV_z\Big)dV_t \\ &\geq 0,\end{aligned}$$

if φ is plurisubharmonic.

3.4 Proof of Theorem 1.1 and projectivized bundles

Let V be a holomorphic vector bundle of finite rank r over a compact complex manifold Y. Let $\pi:\mathbb{P}(V)\to Y$ be a projectivized bundle whose fiber at $t\in Y$ is the projective space of lines in V_t^*, that is, $\mathbb{P}(V_t^*)$. For any point $t\in Y$, we get $\pi^{-1}(t) = \mathbb{P}(V_t^*)\cong\mathbb{P}^{r-1}$, then $\mathbb{P}(V)$ is a holomorphically locally trivial fibration. This projectivized bundle carries the tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ over $\mathbb{P}(V)$ whose restriction to any fiber $\mathbb{P}(V_t^*)$ is identical to $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$.

We shall apply Proposition 3.3 to the line bundles $\mathcal{O}_{\mathbb{P}(V)}(k)\to\mathbb{P}(V)$ where $k\in\mathbb{Z}$. Let $E(k)$ be the vector bundle whose fiber over a point $t\in Y$ is the space of global holomorphic sections of $K_{\mathbb{P}(V_t^*)}\otimes\mathcal{O}_{\mathbb{P}(V)}(k)$, that is,

$$\begin{aligned}E(k) &:= \bigcup_{t\in Y} H^0(\mathbb{P}(V_t^*), K_{\mathbb{P}(V_t^*)}\otimes\mathcal{O}_{\mathbb{P}(V)}(k)|_{\mathbb{P}(V_t^*)}) \\ &= \pi_*(K_{\mathbb{P}(V)/Y}\otimes\mathcal{O}_{\mathbb{P}(V)}(k)),\end{aligned}$$

where $E(k)_t = H^0(\mathbb{P}(V_t^*), K_{\mathbb{P}(V_t^*)}\otimes\mathcal{O}_{\mathbb{P}(V)}(k)|_{\mathbb{P}(V_t^*)})\cong H^0(\mathbb{P}^{r-1},\mathcal{O}_{\mathbb{P}^{r-1}}(k-r))$. If $k<r$, then each fiber $E(k)_t$ is zero. Berndtsson asserted the following fact:

$$E(r+m) = S^m(V)\otimes\det V,$$

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where $S^m(V)$ is the mth symmetric power of V, and showed the following theorem using Theorem 3.2.

THEOREM 3.5 (cf. [1, Th. 1.3]). *Let V be a (finite rank) holomorphic vector bundle over a complex manifold. If $\mathcal{O}_{\mathbb{P}(V)}(1)$ has a smooth (semi)-positive metric, then $V \otimes \det V$ has a smooth canonical Hermitian metric which is Nakano (semi)-positive.*

Here, the vector bundle V is called ample in the sense of Hartshorne (see [15]) if the tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ is ample. Replacing $\mathcal{O}_{\mathbb{P}(V)}(r+1)$ by $\mathcal{O}_{\mathbb{P}(V)}(r+m)$, we also get that $S^m(V) \otimes \det V$ is Nakano (semi)-positivity for any $m \in \mathbb{N}$.

It is a well-known Griffiths conjecture that an ample vector bundle is Griffiths-positive, that is, has a smooth Griffiths-positive Hermitian metric. From Demailly–Skoda's theorem (see [10]) that if V is Griffiths (semi)-positive, then $V \otimes \det V$ is (dual) Nakano (semi)-positive, this theorem may be regarded as indirect evidence of Griffiths conjecture. After that, it was shown that $S^m(V) \otimes \det V$ has Nakano-positive metric and dual Nakano-positive metric (see [18, Cor. 4.12]). Griffiths conjecture is known when Y is a compact curve (cf. [27]), and it was recently shown to hold under a certain condition for the L^2 metric (see [21]). Since the Kodaira–Spencer forms vanish under certain condition in [21], we obtain the following theorem for dual Nakano positivity of the canonical Hermitian metric which is a different metric in [18].

THEOREM 3.6. *Let V be an ample holomorphic vector bundle of rank r over a complex manifold Y. If the canonical isomorphism*

$$K^{-1}_{\mathbb{P}(V)/Y} \cong \mathcal{O}_{\mathbb{P}(V)}(r) \otimes \pi^* \det V^*$$

becomes an isometry for an positive metric on $\mathcal{O}_{\mathbb{P}(V)}(1)$ and some Hermitian metric on $\det V$, then for any $m \in \mathbb{N}$ and for a smooth (semi)-positive Hermitian metric h on $\mathcal{O}_{\mathbb{P}(V)}(r+m)$, the smooth canonical Hermitian metric H induced by h on $S^m(V) \otimes \det V$ is dual Nakano (semi)-positive.

We prove this below. Let $(t_1, \dots, t_m)$ and $(z_1, \dots, z_n)$ be the local coordinates on Y and the fibers, respectively. By the ampleness of V, there is a smooth positive Hermitian metric $h_{O(1)}$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$. We write locally for the curvature of the positively curved metric

$$\begin{aligned} \omega_{\mathbb{P}(V)} &:= -i\partial\overline{\partial} \log h_{O(1)} \\ &= i\Big(g_{\alpha\overline{\beta}} dz_\alpha \wedge d\overline{z}_\beta + h^{O(1)}_{k\overline{\beta}} dt_k \wedge d\overline{z}_\beta + h^{O(1)}_{\alpha\overline{l}} dz_\alpha \wedge d\overline{z}_l + h^{O(1)}_{k\overline{l}} dz_k \wedge d\overline{z}_l\Big). \end{aligned}$$

Thus, the Kähler forms on each fibers are given by $\omega_t := i \sum g_{\alpha\overline{\beta}} dz_\alpha \wedge d\overline{z}_\beta$ and the induced metric on $K^{-1}_{\mathbb{P}(V)/Y}$ can be written as $\det(g_{\alpha\overline{\beta}})$. Here, this positive metric $h_{O(1)}$ induces the above canonical isomorphism.

According to [25], we denote the horizontal lift of a tangent vector $\partial/\partial t_j$ on the base Y by v_j. It is given by

$$v_j = \frac{\partial}{\partial t_j} + \sum a^\alpha_j \frac{\partial}{\partial z_\alpha} \quad \text{and} \quad a^\alpha_j = -\sum g^{\overline{\beta}\alpha} h^{O(1)}_{j\overline{\beta}}.$$

For a fibration $\pi : \mathbb{P}(V) \to Y$, we obtain the Kodaira–Springer forms by

$$\theta_j := \overline{\partial}(v_j)|_{X_t},$$

where $\theta_j \in \rho_t(\partial/\partial t_j)$.

Proposition 3.7 (cf. [21, Prop. 1]). *Under the assumption of Theorem 3.6, the Kodaira–Spencer forms θ_j are harmonic, hence zero.*

Since it is a projectivized bundle, we get $\mathscr{H}^{0,1}(\mathbb{P}(V_t^*), T^{1,0}_{\mathbb{P}(V_t^*)}) \cong H^{0,1}(\mathbb{P}(V_t^*), T^{1,0}_{\mathbb{P}(V_t^*)}) \cong H^{0,1}(\mathbb{P}^{r-1}, T^{1,0}_{\mathbb{P}^{r-1}}) = 0$. Then the value of Kodaira–Spencer map is zero. Here $\{\theta_j\} = \rho_t(\partial/\partial t_j) = 0$. By the forms θ_j is harmonic, θ_j is zero as differential forms.

Proof of Theorem 3.6. From the Kodaira–Spencer forms, θ_j are zero and the definition of the complex structure in $E(r+m)$, for any local holomorphic section $u \in \mathcal{O}(E(r+m))_t$, the restriction of

$$\overline{\partial} u = \sum \eta^j \wedge dt_j$$

to each fiber is zero. In fact, the smooth $(n-1,1)$-forms η^j equal $\theta_j \rfloor u$ in each fiber. In particular, we get $\eta^j = \theta_j \rfloor u = 0$ in each fiber.

By Proposition 3.3, for any local holomorphic section $u_j \in \mathcal{O}(E(r+m))$ such that $D'^H u_j = 0$ at $t = 0$, we have that

$$i\partial\overline{\partial}\widetilde{T}^H_u = -c_N \pi_*(\hat{v} \wedge \overline{\hat{v}} \wedge i\partial\overline{\partial}\varphi e^{-\varphi})$$

at $t = 0$, where $u_j = U_j dz$, $\hat{v} = \sum \overline{U}_j \wedge dz \wedge \widehat{dt_j}$, and $\varphi = -\log h$ on locally. Here $c_N \pi_*(\hat{v} \wedge \overline{\hat{v}} \wedge i\partial\overline{\partial}\varphi e^{-\varphi}) \geq 0$ (resp. > 0) if φ is (strictly) plurisubharmonic.

Hence, this theorem follows from Proposition 2.4. □

Similar to this proof, Theorem 1.1 can be shown from Proposition 3.3, since if Kodaira–Spencer forms θ_j can be taken to be zero, then η^j vanishes where $\theta_j \in \rho_t(\partial/\partial t_j)$. Furthermore, the following corollary is obtained.

Corollary 3.8. *Let X be a compact Kähler manifold, let Y be a complex manifold, and let L be a holomorphic vector bundle over $Z := X \times Y$ equipped with a smooth semi-positive Hermitian metric h. Let $\pi : Z = X \times Y \to Y$ be a natural projection map. Then the smooth canonical Hermitian metric H on $\pi_*(K_{Z/Y} \otimes L)$ has dual Nakano semi-positivity.*

§4. Singular Hermitian metric and positivity

4.1 Singular Hermitian metric on vector bundle and positivity

For any holomorphic vector bundle E, we introduce the definition of singular Hermitian metrics h on E, its various notions of positivity, and the L^2-subsheaf $\mathscr{E}(h)$ of $\mathcal{O}(E)$ analogous to the multiplier ideal sheaf.

Definition 4.1 (cf. [2, §3], [23, Def. 2.2.1]). We say that h is a *singular Hermitian metric* on E if h is a measurable map from the base manifold X to the space of nonnegative Hermitian forms on the fibers satisfying $0 < \det h < +\infty$ almost everywhere.

Definition 4.2 (cf. [5, Def. 2.3.1]). Let h be a singular Hermitian metric on E. We define the L^2-subsheaf $\mathscr{E}(h)$ of germs of local holomorphic sections of E by

$$\mathscr{E}(h)_x := \{s_x \in \mathcal{O}(E)_x \mid |s_x|^2_h \text{ is locally integrable around } x\}.$$

If E is a holomorphic line bundle, then we get $\mathscr{E}(h) = \mathcal{O}(E) \otimes \mathscr{I}(h)$. Moreover, we define positivity and negativity such as Griffiths and dual Nakano.

Definition 4.3 (cf. [2, Def. 3.1], [23, Def. 2.2.2]). We say that a singular Hermitian metric h is:

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(1) *Griffiths semi-negative* if $|u|_h$ is plurisubharmonic for any local holomorphic section u of E.
(2) *Griffiths semi-positive* if the dual metric h^* on E^* is Griffiths semi-negative.

For a singular Hermitian metric h on E, the following is already known (see [2], [24]): h being Griffiths semi-negative is equivalent to $T^h_{\xi u}$ being plurisubharmonic, that is, $i\partial\overline{\partial}T^h_{\xi u} \geq 0$ in the sense of currents, for any local section $u \in \mathcal{O}(E)$ and any $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ satisfying $u_j = \xi_j u$ and written $\xi u = (u_1, \dots, u_n)$.

Definition 4.4 (cf. [24, §1]). We say that a singular Hermitian metric h on E is *Nakano semi-negative* if the $(n-1, n-1)$-form T^h_u is plurisubharmonic for any n-tuple of local holomorphic sections $u = (u_1, \dots, u_n)$.

Definition 4.5 (cf. [29, Def. 4.5]). We say that a singular Hermitian metric h on E is *dual Nakano semi-positive* if the dual metric h^* on E^* is Nakano semi-negative.

For singular Hermitian metrics, we cannot always define the curvature currents with measure coefficients (see [24]). However, the above definitions can be defined by not using the curvature currents. In general, the dual of a Nakano negative bundle is not Nakano-positive, then we cannot define Nakano semi-positivity as in the case of Griffiths, but this definition of dual Nakano semi-positivity is natural. The characterization of Nakano semi-positivity using L^2-estimate by the following definition is already known by Deng–Ning–Wang–Zhou's work (see [11, Th. 1.1]).

Definition 4.6 (cf. [11, Def. 1.1]). Let X be a complex manifold of dimension n, and let U be an open subset of X with a Kähler metric ω on U which admits a positive Hermitian holomorphic line bundle. Let (E, h) be a (singular) Hermitian vector bundle over X. We call (E, h) satisfies *the optimal L^2-estimate* on U if for any positive Hermitian holomorphic line bundle (A, h_A) on U, for any $f \in \mathscr{D}^{n,1}(U, E \otimes A)$ satisfying $\overline{\partial} f = 0$ on U and $\int_U \langle B^{-1}_{A,h_A} f, f\rangle_{h\otimes h_A,\omega} dV_\omega < +\infty$, there is $u \in L^2_{n,0}(U, E \otimes A)$ satisfying $\overline{\partial} u = f$ on U and

$$\int_U |u|^2_{h\otimes h_A,\omega} dV_\omega \leq \int_U \langle B^{-1}_{A,h_A} f, f\rangle_{h\otimes h_A,\omega} dV_\omega,$$

where $B_{A,h_A} = [i\Theta_{A,h_A} \otimes \mathrm{id}_E, \Lambda_\omega]$ and $\mathscr{D}$ denotes the space of C^∞ sections with compact support, i.e. $\mathscr{D} = \mathcal{C}^\infty_c$.

In other words, when h is a smooth metric, satisfying the optimal L^2-estimate is equivalent to being Nakano semi-positive. Therefore, since the above optimal L^2-estimate does not depend on curvature $\Theta_{E,h}$, this definition itself can be extended to singular Hermitian metrics, allowing us to define singular Nakano semi-positivity. In the next subsection, we will define singular semi-positivity for torsion-free coherent sheaves.

It is already known that multiplier ideal sheaves are coherent in [20]. After that, Hosono and Inayama proved that the L^2-subsheaf $\mathscr{E}(h)$ is coherent if h is Nakano semi-positive in the singular sense as in Definition 4.6 in [16], [17].

4.2 Singular Hermitian metrics on torsion-free sheaves and positivity

Let X be a complex manifold, and let $\mathscr{F}$ be a torsion-free coherent sheaf on X. Let $X(\mathscr{F}) \subseteq X$ denote the maximal open subset where $\mathscr{F}$ is locally free, then $Z_{\mathscr{F}} := X \setminus X(\mathscr{F})$ is a closed analytic subset of codimension ≥ 2. If $\mathscr{F} \neq 0$, then the restriction of $\mathscr{F}$ to the open subset $X(\mathscr{F})$ is a holomorphic vector bundle F of some rank $r \geq 1$.

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Definition 4.7 (cf. [14, Def. 19.1]). A *singular Hermitian metric* on $\mathscr{F}$ is a singular Hermitian metric h on the holomorphic vector bundle F. We say that a metric is Griffiths semi-positive if h has Griffiths semi-positive on $X(\mathscr{F})$.

Using the natural inclusion $j: X(\mathscr{F}) = X \setminus Z_{\mathscr{F}} \hookrightarrow X$, we define a natural extension of the L^2-subsheaf $\mathscr{E}(h)$ as follows. Here, $j_*\mathcal{O}_{X\setminus Z_{\mathscr{F}}} \cong \mathcal{O}_X$ is already known.

Definition 4.8. Let h be a singular Hermitian metric on $\mathscr{F}$ which is a singular Hermitian metric on F over $X(\mathscr{F})$. We define the extended natural L^2-subsheaf $\mathscr{E}_X(h)$ with respect to h over X by $\mathscr{E}_X(h) := j_*\mathscr{E}(h)$.

The definition of the minimal extension property for singular Hermitian metrics on torsion-free coherent sheaves is already known.

Definition 4.9 (cf. [14, Def. 20.1]). We say that a singular Hermitian metric h on $\mathscr{F}$ has the *minimal extension property* if there exists a nowhere dense closed analytic subset $Z \subseteq X$ with the following two properties:

(1) $\mathscr{F}$ is locally free on $X \setminus Z$, or equivalently, $X \setminus Z \subseteq X(\mathscr{F})$.
(2) For every embedding $\iota : B \hookrightarrow X$ with $x = \iota(0) \in X \setminus Z$, and every $v \in F_x$ with $|v|_h(x) = 1$, there is a holomorphic section $s \in H^0(B, \iota^*\mathscr{F})$ such that

$$s(0) = v \quad \text{and} \quad \frac{1}{\mathrm{Vol}(B)} \int_B |s|_h^2 dV_B \leq 1,$$

where (F,h) denotes the restriction to the open subset $X(\mathscr{F})$.

Based on this definition and Definition 4.6, we define Nakano positivity for singular Hermitian metrics on torsion-free coherent sheaves.

Definition 4.10. We say that a singular Hermitian metric h on $\mathscr{F}$ is *locally L^2-type Nakano semi-positive* if there exists a nowhere dense closed analytic subset $Z \subseteq X$ with the following two properties:

(1) $\mathscr{F}$ is locally free on $X \setminus Z$, or equivalently, $X \setminus Z \subseteq X(\mathscr{F})$.
(2) For any $t \in X \setminus Z$, there exists a Stein open neighborhood $U \subset X \setminus Z$ of t such that a singular Hermitian metric h on E has the optimal L^2-estimate on U.

In other words, U has a Kähler metric ω, and for any smooth strictly pluriharmonic function ψ on U, for any $f \in \mathscr{D}^{n,1}(U, F, he^{-\psi}, \omega)$ satisfying $\overline{\partial} f = 0$ on U and $\int_U \langle B_{\psi,\omega}^{-1} f, f\rangle_{h,\omega} e^{-\psi} dV_\omega < +\infty$, there exists $u \in L^2_{n,0}(U, F, he^{-\psi}, \omega)$ such that $\overline{\partial} u = f$ on U and

$$\int_U |u|_{h,\omega}^2 e^{-\psi} dV_\omega \leq \int_U \langle B_{\psi,\omega}^{-1} f, f\rangle_{h,\omega} e^{-\psi} dV_\omega,$$

where $B_{\psi,\omega} = [i\partial\overline{\partial}\psi \otimes \mathrm{id}_F, \Lambda_\omega]$ and $F := \mathscr{F}|_{X(\mathscr{F})}$ is a holomorphic vector bundle.

In particular, if we can take $Z = X \setminus X(\mathscr{F})$, then we say that h is *full locally L^2-type Nakano semi-positive* on $X(\mathscr{F})$.

Here, the positivity of holomorphic line bundles (A, h_A) in Definition 4.6 can be replaced by $e^{-\psi}$ using smooth strictly plurisubharmonic function ψ in the local case.

Definition 4.11. We say that a singular Hermitian metric h on $\mathscr{F}$ is *locally L^2-type Nakano positive* if there exists a nowhere dense closed analytic subset $Z \subseteq X$ with the following two properties:

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(1) $\mathscr{F}$ is locally free on $X \setminus Z$, or equivalently, $X \setminus Z \subseteq X(\mathscr{F})$.
(2) For any $t \in X \setminus Z$, there exist a Stein open neighborhood $U \subset X \setminus Z$ of t and a smooth strictly plurisubharmonic function ψ on U such that a singular Hermitian metric he^{ψ} on $F := \mathscr{F}|_{X(\mathscr{F})}$ has the optimal L^2-estimate on U.

In particular, if we can take $Z = X \setminus X(\mathscr{F})$, then we say that h is *full locally L^2-type Nakano positive* on $X(\mathscr{F})$.

DEFINITION 4.12. We say that a singular Hermitian metric h on $\mathscr{F}$ is *globally Nakano semi-positive* if there exists a nowhere dense closed analytic subset $Z \subseteq X$ with the following two properties:

(1) $\mathscr{F}$ is locally free on $X \setminus Z$, or equivalently, $X \setminus Z \subseteq X(\mathscr{F})$.
(2) h is Nakano semi-positive on $X \setminus Z$ as in [17, Def. 1.1].

Nakano semi-positivity for singular Hermitian metrics is defined by L^2-estimates, so there is a drawback in that it cannot be derived from local positivity to global positivity. The Nakano semi-positivity of [17, Def. 1.1] establishes the vanishing theorem, making it a stronger definition than Definition 4.10, as it is globally defined.

§5. Nakano positivity of canonical singular Hermitian metric

Let $f : X \to Y$ be a projective surjective morphism between two connected complex manifolds, with $\dim X = n+m$ and $\dim Y = m$, but there may be singular fiber. Let $L \to X$ be a holomorphic line bundle equipped with a singular Hermitian metric h which is pseudo-effective.

5.1 Canonical singular Hermitian metric on direct image sheaves

In this subsection, we define the canonical singular Hermitian metric on the direct image sheaf $\mathcal{E} := f_*(K_{X/Y} \otimes L \otimes \mathscr{I}(h))$ in the same way as in [14].

Construct a Hermitian metric of $\mathcal{E}$ over a Zariski-open subset $Y \setminus \mathscr{Z}_{\mathcal{E}}$ where everything is nice, and then to extend it over the bad locus $\mathscr{Z}_{\mathcal{E}}$. First, we choose a nowhere dense closed analytic subset $Z \subseteq \mathscr{Z}_{\mathcal{E}}$ with the following three properties:

(1) The morphism f is submersion over $Y \setminus \mathscr{Z}_{\mathcal{E}}$.
(2) Both $\mathcal{E}$ and the quotient sheaf $f_*(K_{X/Y} \otimes L)/\mathcal{E}$ are locally free on $Y \setminus \mathscr{Z}_{\mathcal{E}}$.
(3) On $Y \setminus \mathscr{Z}_{\mathcal{E}}$, the locally free sheaf $f_*(K_{X/Y} \otimes L)$ has the base change property.

By the base change theorem, the third condition will hold as long as the coherent sheaves $R^i f_*(K_{X/Y} \otimes L)$ are locally free on $Y \setminus \mathscr{Z}_{\mathcal{E}}$. The restriction of $\mathcal{E}$ to the open subset $Y \setminus \mathscr{Z}_{\mathcal{E}}$ is a holomorphic vector bundle E of some rank $r \geq 1$. The second and third conditions together guarantee that

$$E_t := \mathcal{E}|_t \subseteq f_*(K_{X/Y} \otimes L)|_t = H^0(X_t, K_{X_t} \otimes L|_{X_t})$$

whenever $t \in Y \setminus \mathscr{Z}_{\mathcal{E}}$. Here, note that $Z_{\mathcal{E}} = X \setminus X(\mathcal{E}) \subseteq \mathscr{Z}_{\mathcal{E}}$.

LEMMA 5.1 (cf. [14, Lem. 22.1]). *For any $t \in Y \setminus \mathscr{Z}_{\mathcal{E}}$, we have inclusions*

$$H^0(X_t, K_{X_t} \otimes L|_{X_t} \otimes \mathscr{I}(h|_{X_t})) \subseteq E_t \subseteq H^0(X_t, K_{X_t} \otimes L|_{X_t}).$$

Here, we can immediately see that the two subspaces

$$H^0(X_t, K_{X_t} \otimes L|_{X_t} \otimes \mathscr{I}(h|_{X_t})) \subseteq E_t$$

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are equal for almost everywhere $t \in Y \setminus \mathscr{Z}_{\mathcal{E}}$. But unless $\mathcal{E} = 0$, the two subspaces are different, for example, at points where $h|_{X_t}$ is identically equal to $+\infty$.

On each E_t with $t \in Y \setminus \mathscr{Z}_{\mathcal{E}}$, we can define a singular Hermitian metric H as follows. For any element $\alpha \in E_t \subseteq H^0(X_t, K_{X_t} \otimes L|_{X_t})$, we can integrate over the compact complex manifold X_t and define the inner product of α with respect to H by

$$|\alpha|_H^2(t) := \int_{X_t} |\alpha|_h^2 \in [0, +\infty].$$

Clearly, $|\alpha|_H(t) < +\infty$ if and only if $\alpha \in H^0(X_t, K_{X_t} \otimes L|_{X_t} \otimes \mathscr{I}(h|_{X_t}))$. By Ehresmann's fibration theorem and Fubini's theorem, the function $t \mapsto |s|_H(t)$ is measurable for any local holomorphic section s of E.

From the discussion in [14], the singular Hermitian metric H over $Y \setminus \mathscr{Z}_{\mathcal{E}}$ is well defined on the entire open set $Y(\mathcal{E})$. Then we say that this extended metric H on E over $Y(\mathcal{E})$ is a *canonical singular Hermitian metric* of $\mathcal{E}$.

Definition 5.2. We define the set Σ_H on Y related to the unboundedness of H by

$$\Sigma_H := \{t \in Y \mid \mathcal{E}_t \subsetneq H^0(X_t, K_{X_t} \otimes L|_{X_t})\}.$$

Here, for any $t \in Y \setminus \mathscr{Z}_{\mathcal{E}}$, if $\mathscr{I}(h|_{X_t}) = \mathcal{O}_{X_t}$, then $t \notin \Sigma_H$ and $H(t)$ is bounded by $\int_{X_t} e^{-\varphi} < +\infty$, where $h = e^{-\varphi}$ on local. Let $\Sigma_h := \{t \in Y \mid \int_{X_t} e^{-\varphi} = +\infty\}$ be a set related to the unboundedness of h, then we have that $\Sigma_H \setminus \mathscr{Z}_{\mathcal{E}} \subseteq \Sigma_h \setminus \mathscr{Z}_{\mathcal{E}}$.

Example 5.3. If X is a compact Kähler and $L \to X$ is nef and big, then there exists a singular Hermitian metric h on L such that $\Sigma_H \subseteq \mathscr{Z}_{\mathcal{E}}$ for the canonical singular Hermitian metric H induced by h.

In fact, from the analytical characterization of nef and big line bundles (see [8, Chap. 6]), there exists a singular Hermitian metric h on L such that $\mathscr{I}(h) = \mathcal{O}_X$ and $i\Theta_{L,h} \geq \varepsilon\omega$ in the sense of currents for some $\varepsilon > 0$, where ω is a Kähler metric on X. Therefore, from Lemma 5.1 and $\mathcal{E} := f_*(K_{X/Y} \otimes L \otimes \mathscr{I}(h)) = f_*(K_{X/Y} \otimes L)$, we have $\mathcal{E}|_t = E_t = H^0(X_t, K_{X_t} \otimes L|_{X_t})$ for any $t \in Y \setminus \mathscr{Z}_{\mathcal{E}}$.

5.2 Locally L^2-type Nakano (semi)-positivity of H

Recall that $f : X \to Y$ is a projective surjective morphism between two connected complex manifolds. For the canonical singular Hermitian metric H on the direct image sheaf $\mathcal{E} = f_*(K_{X/Y} \otimes L \otimes \mathscr{I}(h))$ induced by the singular Hermitian metric h which is pseudo-effective, the following theorem is known with respect to the positivity property.

Theorem 5.4 (cf. [14, Th. 21.1]). *The direct image sheaf $\mathcal{E} = f_*(K_{X/Y} \otimes L \otimes \mathscr{I}(h))$ has a canonical singular Hermitian metric H. This metric is Griffiths semi-positive and satisfies the minimal extension property.*

In this subsection, we show that this metric H on $\mathcal{E}$ has locally Nakano (semi)-positivity. This proof is inspired by the proof of the smooth case using L^2-estimates in [11, Th. 1.6].

Theorem 5.5. *Let H be a canonical singular Hermitian metric on $\mathcal{E} = f_*(\omega_{X/Y} \otimes L \otimes \mathscr{I}(h))$ which is induced by h. If X is projective and there exists an analytic subset $A \subsetneq Y$ such that $\Sigma_H \subseteq A$, then H is full locally L^2-type Nakano semi-positive on $Y(\mathcal{E})$.*

Proof. First, we prove that H is locally L^2-type Nakano semi-positive, that is, for any $t \in Y \setminus \mathscr{Z}_{\mathcal{E}}$, there exists a Stein open neighborhood $U \subset Y \setminus \mathscr{Z}_{\mathcal{E}}$ of t such that H has the

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optimal L^2-estimate on U. Here, $E := \mathcal{E}|_{Y\setminus Z_{\mathcal{E}}}$ is a holomorphic vector bundle, and f is proper submersion over U from the construction of $\mathscr{Z}_{\mathcal{E}}$.

By projectivity of X, there exists an analytic subset D such that $S := X \setminus D$ is Stein and that $L|_S$ is trivial. Let $\varphi := -\log h|_S$, then φ is plurisubharmonic function on S and $h = e^{-\varphi}$ on S. By [12, Th. 5.5], there exist a sequence of smooth plurisubharmonic functions $(\varphi_\nu)_{\nu\in\mathbb{N}}$ on S decreasing to φ almost everywhere pointwise. Here, there is a smooth exhaustive strictly plurisubharmonic function Ψ on S such that $\sup_S \psi = +\infty$.

Let $X_U = f^{-1}(U)$, then $X_U \setminus D$ is also Stein by f is holomorphic. In fact, there is a strictly plurisubharmonic function Φ on U which is exhaustive and smooth by Stein-ness of U. Thus, the function $\Psi + f^*\Phi$ on $X_U \setminus D$ is strictly plurisubharmonic, smooth, and exhaustive. We take a local coordinate $(t_1,\dots,t_m,z_1,\dots,z_n)$ on X_U near $f^{-1}(t) = X_t$, where $t_1,\dots,t_m$ is the standard coordinate on $U \subset \mathbb{C}^m$. Let $\widetilde{\omega} = i\sum_{j=1}^m dt_j \wedge d\bar{t}_j + i\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ and $\omega = i\sum_{j=1}^m dt_j \wedge d\bar{t}_j$.

For any smooth strictly plurisubharmonic function ψ on U and any section $g \in \mathscr{D}^{m,1}(U,E,he^{-\psi},\omega)$ satisfying $\overline{\partial} g = 0$ on U and $\int_U \langle [i\partial\overline{\partial}\psi \otimes \mathrm{id}_E, \Lambda_\omega]^{-1} g, g\rangle_{H,\omega} e^{-\psi} dV_\omega < +\infty$, we can write $g(t) = \sum_{j=1}^m g_j(t) d\bar{t}_j \wedge dt$, with $g_j(t) \in E_t \subseteq H^0(X_t, K_{X_t} \otimes L|_{X_t})$. We can identify g as a smooth compact supported $(n+m,1)$-form $\widetilde{g}(t,z) := \sum_{j=1}^m g_j(t,z) d\bar{t}_j \wedge dt$ on X, with $g_j(t,z)$ begin holomorphic section $K_{X_t} \otimes L|_{X_t}$. We have the following observations:

- $\overline{\partial}_z g_j(t,z) = 0$ for any fixed $t \in U$, since $g_j(t,z)$ are holomorphic sections of $K_{X_t} \otimes L|_{X_t}$,
- $\overline{\partial}_t g_j = 0$, since g is a $\overline{\partial}$-closed form on U.

Here, we obtain that

$$\begin{aligned}
i\partial\overline{\partial} f^*\psi &= \sum_{j,k=1}^m \frac{\partial^2\psi}{\partial t_j \partial \bar{t}_k} dt_j \wedge d\bar{t}_k,\\
[i\partial\overline{\partial} f^*\psi, \Lambda_{\widetilde{\omega}}]\widetilde{g} &= \sum_{j,k} \frac{\partial^2\psi}{\partial t_j \partial \bar{t}_k} g_j(t,z) dt \wedge d\bar{t}_k,\\
[i\partial\overline{\partial} f^*\psi, \Lambda_{\widetilde{\omega}}]^{-1}\widetilde{g} &= \sum_{j,k} \psi^{jk} g_j(t,z) dt \wedge d\bar{t}_k,\\
\langle [i\partial\overline{\partial} f^*\psi, \Lambda_{\widetilde{\omega}}]^{-1}\widetilde{g}, \widetilde{g}\rangle_{\widetilde{\omega}} dV_{\widetilde{\omega}} &= \sum_{j,k} \psi^{jk} c_n g_j \wedge \overline{g}_k c_m dt \wedge d\bar{t}.
\end{aligned}$$

at any $t \in U$, where $(\psi^{jk}) = (\frac{\partial^2\psi}{\partial t_j \partial \bar{t}_k})^{-1}$. By Fubini's theorem, we have

$$\begin{aligned}
\int_{X_U\setminus D} \langle [i\partial\overline{\partial} f^*\psi, \Lambda_{\widetilde{\omega}}]^{-1}\widetilde{g}, \widetilde{g}\rangle_{\widetilde{\omega}} e^{-\varphi_\nu - f^*\psi} dV_{\widetilde{\omega}} &= \int_{X_U\setminus D} \sum_{j,k} \psi^{jk} c_n g_j \wedge \overline{g}_k e^{-\varphi_\nu - f^*\psi} c_m dt \wedge d\bar{t}\\
&\le \int_{X_U} \sum_{j,k} \psi^{jk} c_n g_j \wedge \overline{g}_k e^{-\varphi - f^*\psi} c_m dt \wedge d\bar{t}\\
&= \int_U (g_j, g_k)_H(t) \psi^{jk} e^{-\psi} c_m dt \wedge d\bar{t}\\
&= \int_U \langle [i\partial\overline{\partial}\psi \otimes \mathrm{id}_E, \Lambda_\omega]^{-1} g, g\rangle_{H,\omega} e^{-\psi} dV_\omega\\
&< +\infty,
\end{aligned}$$

where by $(\bullet,\bullet)_H(t)$, we mean that pointwise inner product with respect to H.

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Note that, acting on $\bigwedge^{n+m,1} T_X^* \otimes L|_{X_U \setminus D} = \bigwedge^{n+m,1} T_X^*|_{X_U \setminus D}$, we have

$$[i\partial\overline{\partial}\varphi_\nu + i\partial\overline{\partial} f^*\psi, \Lambda_{\widetilde{\omega}}] \geq [i\partial\overline{\partial} f^*\psi, \Lambda_{\widetilde{\omega}}] \geq 0, \tag{*}$$

over $X_U \setminus D$ for any $\nu \in \mathbb{N}$.

From Hörmander's L^2-estimate, that is, Theorem 2.5, there exists a solution $\widetilde{v}_\nu \in L^2_{n+m,0}(X_U \setminus D, e^{-\varphi_\nu}, \widetilde{\omega})$ such that $\overline{\partial}\widetilde{v}_\nu = \widetilde{g}$ on $X_U \setminus D$ and satisfies the following estimate:

$$\begin{aligned} \int_{X_U \setminus D} |\widetilde{v}_\nu| e^{-\varphi_\nu - f^*\psi} dV_{\widetilde{\omega}} &= \int_{X_U \setminus D} c_{n+m} \widetilde{v}_\nu \wedge \overline{\widetilde{v}}_\nu e^{-\varphi_\nu - f^*\psi} \\ &\leq \int_{X_U \setminus D} \langle [i\partial\overline{\partial}\varphi_\nu + i\partial\overline{\partial} f^*\psi, \Lambda_\omega]^{-1} \widetilde{g}, \widetilde{g} \rangle_{\widetilde{\omega}} e^{-\varphi_\nu - f^*\psi} dV_{\widetilde{\omega}} \\ &\leq \int_{X_U \setminus D} \langle [i\partial\overline{\partial} f^*\psi, \Lambda_{\widetilde{\omega}}]^{-1} \widetilde{g}, \widetilde{g} \rangle_{\widetilde{\omega}} e^{-\varphi_\nu - f^*\psi} dV_{\widetilde{\omega}} \\ &\leq \int_U \langle [i\partial\overline{\partial}\psi \otimes \mathrm{id}_E, \Lambda_\omega]^{-1} g, g \rangle_{H,\omega} e^{-\psi} dV_\omega < +\infty. \end{aligned}$$

Letting $\widetilde{v}_\nu = 0$ on D, then we have $\overline{\partial}\widetilde{v}_\nu = \widetilde{g}$ on X_U by Lemma 5.6. Since $e^{-\varphi_\nu}$ increases and converges to $h = e^{-\varphi}$ on S as ν tends to $+\infty$, the sequence $(\widetilde{v}_\nu)_{\nu_1 \leq \nu \in \mathbb{N}}$ forms a bounded sequence in $L^2_{n+m,0}(X_U, e^{-\varphi_{\nu_1}}, \widetilde{\omega})$. Therefore, we can obtain a weakly convergence subsequence in $L^2_{n+m,0}(X_U, e^{-\varphi_{\nu_1}}, \widetilde{\omega})$. By using a diagonal argument, we get a subsequence $(\widetilde{v}_{\nu_k})_{k \in \mathbb{N}}$ of $(\widetilde{v}_\nu)_{\nu_1 \leq \nu \in \mathbb{N}}$ converging weakly in $L^2_{n+m,0}(X_U, e^{-\varphi_{\nu_1}}, \widetilde{\omega})$ for any ν_1, where $\widetilde{v}_{\nu_k} \in L^2_{n+m,0}(X_U, e^{-\varphi_{\nu_k}}, \widetilde{\omega}) \subset L^2_{n+m,0}(X_U, e^{-\varphi_{\nu_1}}, \widetilde{\omega})$.

We denote by $\widetilde{v}$ the weakly limit of $(\widetilde{v}_{\nu_k})_{k \in \mathbb{N}}$, then $\widetilde{v}$ satisfies $\overline{\partial}\widetilde{v} = \widetilde{g}$ on X_U and

$$\int_{X_U \setminus D} |\widetilde{v}|^2 e^{-\varphi_{\nu_k} - f^*\psi} dV_\omega \leq \int_U \langle [i\partial\overline{\partial}\psi, \Lambda_{\omega_0}]^{-1} g, g \rangle_{H,\omega_0} e^{-\varphi} dV_{\omega_0} < +\infty$$

for any $k \in \mathbb{N}$. Taking weakly limit $k \to +\infty$ and using the monotone convergence theorem, we have the following L^2-estimate:

$$\begin{aligned} \int_{X_U} |\widetilde{v}|_h^2 e^{-f^*\psi} dV_\omega &= \int_{X_U \setminus D} |\widetilde{v}|_h^2 e^{-f^*\psi} dV_\omega \\ &\leq \int_U \langle [i\partial\overline{\partial}\psi, \Lambda_{\omega_0}]^{-1} g, g \rangle_{H,\omega_0} e^{-\psi} dV_{\omega_0} < +\infty, \end{aligned}$$

that is, $\widetilde{v} \in L^2_{n+m,0}(X_U, L, h, \omega)$.

Here we write $\widetilde{v}(t,z) = \widetilde{V}(t,z) dz \wedge dt$, then $\frac{\partial \widetilde{V}}{\partial \overline{z}_j} = 0$, that is, $\overline{\partial}\widetilde{v}|_{X_t} = 0$ for any fixed $t \in U$, since $\overline{\partial}\widetilde{v} = \widetilde{g}$ on X_U. This means that $\widetilde{V}(t,\cdot) dz \in H^0(X_t, \omega_{X_t} \otimes L|_{X_t})$. We can identify $\widetilde{v}$ as an $(m,0)$-form $v(t) := V(t) dt$ on U, with $V(t) = \widetilde{V}(t,\cdot) dz \in H^0(X_t, \omega_{X_t} \otimes L|_{X_t})$.

Fubini's theorem implies the following:

$$\int_{X_U} |\widetilde{v}|_h^2 e^{-f^*\psi} dV_\omega = \int_{X_U} c_{n+m} \widetilde{v} \wedge \overline{\widetilde{v}} e^{-\varphi - f^*\psi} = \int_U ||v||_H^2 e^{-\psi} dV_{\omega_0}.$$

Therefore, we obtain that

$$\int_U ||v||_{H,\omega_0}^2 e^{-\psi} dV_{\omega_0} \leq \int_U \langle [i\partial\overline{\partial}\psi, \Lambda_{\omega_0}]^{-1} g, g \rangle_{H,\omega_0} e^{-\psi} dV_{\omega_0} < +\infty.$$

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Here, by the boundedness of the integral of $||v||^2_H$, for any almost everywhere $t\in U$, we have that $||v||^2_H(t)<+\infty$, that is, $V(t)\in H^0(X_t,K_{X_t}\otimes L|_{X_t}\otimes\mathscr{I}(h|_{X_t}))\subseteq\mathcal{E}_t$.

Form the assumption $\Sigma_H\subseteq A$, replacing $v=0$, that is, $V=0$, on A then for any $t\in U$ we get $V(t)\in H^0(X_t,K_{X_t}\otimes L|_{X_t})=\mathcal{E}_t$. By the Lebesgue measure of A is zero, this means that $v\in L^2_{m,0}(U,E,H,\omega_0)$ and $\overline{\partial}v=g$ on $U\setminus A$. From Lemma 5.6, we get $\overline{\partial}v=g$ on U. Hence, we showed that H satisfies the optimal L^2-estimate on U.

Finally, we prove that H is full locally L^2-type Nakano semi-positive on $Y(\mathcal{E})$. By $Z_{\mathcal{E}}:=Y\setminus Y(\mathcal{E})$ and $Z_{\mathcal{E}}\subseteq\mathscr{Z}_{\mathcal{E}}$, there exists an analytic subset B such that $\mathscr{Z}_{\mathcal{E}}=Z_{\mathcal{E}}\bigcup B$. Therefore, it is sufficient to show that for any $t\in B\setminus Z_{\mathcal{E}}$, there exists a open neighborhood $U\subset Y(\mathcal{E})$ of t such that H has the optimal L^2-estimate on U. This can be shown in the same way as above by using Lemma 5.6. □

Lemma 5.6 (cf. [6, Lem. 6.9]). *Let Ω be an open subset of $\mathbb{C}^n$, and let Z be a complex analytic subset of Ω. Assume that u is a $(p,q-1)$-form with L^2_{loc} coefficients and g is a (p,q)-form with L^1_{loc} coefficients such that $\overline{\partial}u=g$ on $\Omega\setminus Z$ (in the sense of currents). Then $\overline{\partial}u=g$ on Ω.*

Lemma 5.7 (cf. [6, Th. 1.5]). *Let X be a Kähler manifold, and let Z be an analytic subset of X. Assume that Ω is a relatively open subset of X possessing a complete Kähler metric. Then $\Omega\setminus Z$ carries a complete Kähler metric.*

By using Lemma 5.7 and Demailly's approximation theorem (see [7]), the following can be shown similarly as above. Here, we do not use Demailly's approximation theorem in the proof of Theorem 5.5 because the left term of $(*)$ is not necessarily semi-positive and Hörmander's L^2-estimate cannot be used.

Theorem 5.8. *Let H be a canonical singular Hermitian metric on $\mathcal{E}:=f_*(K_{X/Y}\otimes L\otimes\mathscr{I}(h))$ which is induced by h. We assume that X is compact Kähler and h is big. If there exists an analytic subset $A\subsetneq Y$ such that $\Sigma_H\subseteq A$, then the H is full locally L^2-type Nakano positive on $Y(\mathcal{E})$.*

Here, the L^2-subsheaf $\mathscr{E}(H)$ of H is a subsheaf of $E=\mathcal{E}|_{Y(\mathcal{E})}$ over $Y(\mathcal{E})$. For the natural inclusion $j:Y\setminus Z_{\mathcal{E}}=Y(\mathcal{E})\hookrightarrow Y$, the natural extended L^2-subsheaf with respect to H over Y is defined by $\mathscr{E}_Y(H):=j_*\mathscr{E}(H)$ as in Definition 4.8.

Theorem 5.9. *Let $f:X\to Y$ be a projective surjective morphism between two connected complex manifolds, and let L be a holomorphic line bundle on X equipped with a pseudo-effective metric h. Let H be a canonical singular Hermitian metric on $f_*(K_{X/Y}\otimes L\otimes\mathscr{I}(h))$. If X is projective and there exists an analytic subset $A\subsetneq Y$ such that $\Sigma_H\subseteq A$, then the natural extended L^2-subsheaf $\mathscr{E}_Y(H)$ over Y is coherent.*

Proof. From Theorem 5.5 and [17, Prop. 4.4], the L^2-subsheaf $\mathscr{E}(H)$ over $Y(\mathcal{E})$ is coherent. For the natural inclusion $j:Y\setminus Z_{\mathcal{E}}=Y(\mathcal{E})\hookrightarrow Y$, we are already known $j_*\mathcal{O}_{Y\setminus Z_{\mathcal{E}}}\cong\mathcal{O}_Y$ since the analytic set $Z_{\mathcal{E}}:=Y\setminus Y(\mathcal{E})$ is codimension ≥ 2. By Riemann's extension theorem, the sheaf $j_*\mathscr{E}(H)=\mathscr{E}_Y(H)$ is also coherent. □

Corollary 5.10. *Let H be a canonical singular Hermitian metric on $\mathcal{E}=f_*(K_{X/Y}\otimes L\otimes\mathscr{I}(h))$ which is induced by a pseudo-effective metric h on L. Let $B_H\subseteq Y(\mathcal{E})\setminus\Sigma_H$ be an open subset. Here, $\mathcal{E}|_{Y(\mathcal{E})}=E$ is holomorphic vector bundle. If X is projective, then for any local Stein open subset $U\subset B_H$, the metric H satisfies the optimal L^2-estimate on U. Moreover, the L^2-subsheaf $\mathscr{E}_Y(H)$ is coherent on B_H.*

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REMARK 5.11. This theorem and corollary hold even if the situation is that X is compact Kähler and h is big by Theorem 5.8.

COROLLARY 5.12. *Let $\mathscr{F}$ be a torsion-free coherent sheaf on complex manifold X equipped with a singular Hermitian metric h. If h is full locally L^2-type Nakano semi-positive on $X(\mathscr{F})$, then the natural extended L^2-subsheaf $\mathscr{E}_X(h)$ is coherent.*

Here, when L is nef and big, the following is known from [30].

REMARK 5.13 (cf. [30, Th. 1.3 and Cor. 1.4]). Let $f : X \to Y$ be a smooth fibration of smooth projective varieties with connected fibers. If a holomorphic line bundle L on X is nef and big, then the holomorphic vector bundle $f_*(K_{X/Y} \otimes L)$ is also nef and Viehweg-big and has a canonical singular Hermitian metric H induced by a nef and big singular metric h as in Example 5.3 and there exists a proper analytic subset Z such that H is smooth and Nakano positive on $Y \setminus Z$.

Moreover, we have the following cohomology vanishing

$$H^q(Y, f_*(K_X \otimes L)) = 0$$

for any integers $q \leq 1$. Here, we have $\Sigma_H = \emptyset$ by Example 5.3.

§6. The minimal extension property and Nakano semi-positivity

In this section, we study the relation between the minimal extension property and Nakano semi-positivity and prove the following theorem. For holomorphic line bundles, the two properties are equivalent from the optimal Ohsawa-Takegoshi L^2-extension theorem (see [4], [13]) and the proof of [14, Th. 21.1]. In the case of holomorphic vector bundles, the Ohsawa–Takegoshi L^2-extension theorem follows from Nakano semi-positivity, so it is likely to have the minimal extension property if it is Nakano semi-positive. However, it turns out that in general the converse does not hold true. This phenomenon is first mentioned in [16] for the positivity called *weak Ohsawa–Takegoshi* in a close concept instead of the minimal extension property. The previous result pertains to smooth metrics, and we have shown that the following analogous result does not hold even when extended to singular Hermitian metrics.

THEOREM 6.1. *Let $\mathscr{F}$ be a torsion-free coherent sheaf on a complex manifold X. Even if $\mathscr{F}$ has a singular Hermitian metric satisfying the minimal extension property, it does not necessarily have a singular Hermitian metric h which is globally Nakano semi-positive and satisfying $\nu(-\log\det h, x) < 2$ for any point $x \in X(\mathscr{F})$.*

Here, this symbol ν denotes the Lelong number and is defined by

$$\nu(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{\log|z - x|}$$

for a plurisubharmonic function φ and some coordinate $(z_1, \ldots, z_n)$ around x. It is already known that if $\nu(-\log\det h, x) < 2$ then $\mathscr{E}(h)_x = \mathcal{O}(E)_x$.

6.1 Exact sequences of torsion-free coherent sheaves and positivity

Consider the inheritance of positivity in exact sequences. The following is already known for the minimal extension property.

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Proposition 6.2 (cf. [19, Props. 6 and 7]). *Let*

$$0 \longrightarrow \mathscr{S} \overset{j}{\hookrightarrow} \mathscr{F} \overset{g}{\twoheadrightarrow} \mathscr{Q} \longrightarrow 0$$

be an exact sequence of torsion-free coherent sheaves. Then we have the following.

(a) *Let h be a singular Hermitian metric on $\mathscr{S}$ which satisfies the minimal extension property. If j is generically an isomorphism, then h extends to a singular Hermitian metric $h_{\mathscr{F}}$ on $\mathscr{F}$ satisfying the minimal extension property,*

(b) *If $\mathscr{F}$ has a singular Hermitian metric satisfying the minimal extension property, then the induced metric $h_{\mathscr{Q}}$ has also the minimal extension property.*

For Griffiths and Nakano positivity of smooth metrics, the following is known.

Proposition 6.3 (cf. [9, Chap. VII, Prop. 6.10]). *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of hermitian vector bundles. Then we have the following*

(a) $E \geq_{Grif} 0 \Longrightarrow Q \geq_{Grif} 0$, (b) $E \leq_{Grif} 0 \Longrightarrow S \leq_{Grif} 0$, (c) $E \leq_{Nak} 0 \Longrightarrow S \leq_{Nak} 0$, and analogous implications hold true for strictly positivity.

In particular, a Nakano semi-positive metric of E does not necessarily induce a Nakano semi-positive metric of Q.

Here, for the inheritance of semi-positivity from E to Q, Nakano semi-positivity has a counterexample (see Proposition 6.7), but by rephrasing condition (c), we find the following with respect to dual Nakano positivity.

Corollary 6.4. *Let $g : E \twoheadrightarrow Q$ be a quotient onto a holomorphic vector bundle. Then if E is dual Nakano (semi)-positive then Q is also dual Nakano (semi)-positive.*

Proof. There exists a holomorphic vector bundle S such that $0 \to S \to E \to Q \to 0$ is an exact sequence of holomorphic vector bundles. Then the sequence $0 \to Q^* \to E^* \to S^* \to 0$ is also exact. Here, E^* is Nakano (semi)-negative by the assumption. By (c) of Proposition 6.3, Q^* is Nakano (semi)-negative. □

We consider the positivity of singular Hermitian metrics. For Griffiths positivity, [14, Prop. 19.3] is already known, and we obtain the following proposition for (dual) Nakano positivity.

Proposition 6.5 (cf. [14, Prop. 19.3]). *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism between two torsion-free coherent sheaves that is generically an isomorphism. If $\mathscr{F}$ has a singular Griffiths semi-positive Hermitian metric, then so does $\mathscr{G}$.*

Proposition 6.6. *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of holomorphic vector bundles. Let h be a singular Hermitian metric on E. Then we have:*

(a) *If h is Nakano semi-negative, then S has a natural induced singular Hermitian metric which is Nakano semi-negative.*

(b) *If h is dual Nakano semi-positive, then Q has a natural induced singular Hermitian metric which is dual Nakano semi-positive.*

In particular, Proposition 6.3 and Corollary 6.4 follow from this proposition.

Proof. (a) We define the natural singular Hermitian metric h_S of S induced from h by $|u|_{h_S} := |ju|_h$ for any section u of S. By the assumption, for any local holomorphic section $s_j \in \mathcal{O}(E)$, the $(n-1,n-1)$-form $T^h_u = \sum (s_j, s_k)_h \widehat{dz_j \wedge d\overline{z}_k}$ is plurisubharmonic,

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that is, $i\partial\overline{\partial}T^h_u \geq 0$. For any local holomorphic section $u_k \in \mathcal{O}(S)$, image ju_k is also a local holomorphic section of E, that is, $ju_k \in \mathcal{O}(E)$. Then, from the equality

$$T^{h_S}_u = \sum (u_j, u_k)_{h_S} \widehat{dz_j \wedge d\overline{z}_k} = \sum (ju_j, ju_k)_h \widehat{dz_j \wedge d\overline{z}_k} = T^h_{ju},$$

we have that $T^{h_S}_u$ is also plurisubharmonic, that is, h_S is Nakano semi-negative.

(b) Here, the sequence $0 \to Q^* \to E^* \to S^* \to 0$ is also exact. Similarly to the proof of (a), Q^* has a Nakano semi-negative singular Hermitian metric. ∎

6.2 A concrete example

We consider the following exact sequence of holomorphic vector bundles:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \overset{j}{\hookrightarrow} \underline{V} := \mathbb{P}^n \times \mathbb{C}^{n+1} \overset{g}{\twoheadrightarrow} Q := \underline{V}/\mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow 0.$$

From this sequence, we get $\det \underline{V} = \det Q \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ and get isomorphisms

$$\det Q \cong \mathcal{O}_{\mathbb{P}^n}(1), \quad T_{\mathbb{P}^n} = Q \otimes \mathcal{O}_{\mathbb{P}^n}(1) \cong Q \otimes \det Q,$$

where $\det \underline{V}$ is also trivial. By Griffiths semi-positivity of $\underline{V}$ and Corollary 6.4, the bundle Q is dual Nakano semi-positive and then Griffiths semi-positive. Therefore, $T_{\mathbb{P}^n}$ is Nakano semi-positive from Demailly–Skoda's theorem (see [10]), and is Griffiths positive from $Q \geq_{Grif} 0$ and $\det Q \cong \mathcal{O}_{\mathbb{P}^n}(1) > 0$. But the tangent bundle $T_{\mathbb{P}^n}$ has no smooth Nakano positive metric. In fact, if $T_{\mathbb{P}^n} >_{Nak} 0$, then $H^q(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) = 0$ for any $q \geq 1$ by the Nakano vanishing theorem. However, this contradicts the following:

$$H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) \cong H^1(\mathbb{P}^n, T^*_{\mathbb{P}^n}) = H^{1,1}(\mathbb{P}^n, \mathbb{C}) = \mathbb{C}.$$

PROPOSITION 6.7. *We have that Q has no smooth Griffiths-positive Hermitian metric and no singular Hermitian metric which is globally Nakano semi-positive and satisfying $\nu(-\log \det h, x) < 2$ for any point $x \in \mathbb{P}^n$.*

Proof. First, if Q has a smooth Griffiths-positive Hermitian metric, then $T_{\mathbb{P}^n} \cong Q \otimes \det Q$ has a smooth Nakano-positive Hermitian metric by Demailly–Skoda's theorem. Second, if Q has a smooth Nakano semi-positive Hermitian metric, then $T_{\mathbb{P}^n} \cong Q \otimes \det Q$ has a smooth Nakano-positive Hermitian metric by $\det Q \cong \mathcal{O}_{\mathbb{P}^n}(1)$ is positive line bundle. But these contradict that $T_{\mathbb{P}^n}$ is not Nakano-positive.

Finally, if Q has a singular Hermitian metric h which is globally Nakano semi-positive and satisfying $\nu(-\log \det h, x) < 2$ for any point $x \in \mathbb{P}^n$, then from the vanishing theorem (see [28, Th. 6.1]) for singular Nakano semi-positivity, we have

$$H^q(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathscr{E}(h)) = 0$$

for $q \geq 1$. By the fact that if $\nu(-\log \det h, x) < 2$, then $\mathscr{E}(h) = \mathcal{O}_{\mathbb{P}^n}(Q)$ (see the proof of [28, Th. 6.2]), we get

$$0 = H^q(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathscr{E}(h)) \cong H^q(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \det Q \otimes Q) \cong H^q(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}).$$

But this vanishing contradicts that $H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) \cong \mathbb{C}$. ∎

Proof of Theorem 6.1. Let I_V be a trivial Hermitian metric on $\underline{V}$, then I_V has the minimal extension property by the optimal Ohsawa–Takegoshi L^2-extension theorem (see [4], [13]). From Proposition 6.2, the induced Hermitian metric h' on Q has the minimal extension property. Then this theorem is shown by Proposition 6.7. ∎

Finally, we ascertain by concrete calculations that the naturally induced smooth metric h_Q of Q has indeed the minimal extension property. Here, this metric h_Q induced from $I_{\underline{V}}$ and g defined by $|u|_{h_Q} := |g^*u|_{I_V}$ for any section u of Q.

Let $a \in \mathbb{P}^n$ be fixed. Choose an orthonormal basis $(e_0, e_1, \ldots, e_n)$ of $\mathbb{C}^{n+1}$ such that $a = [e_0]$. Consider the natural embedding $\mathbb{C}^n \hookrightarrow \mathbb{P}^n : 0 \mapsto a$ which sends $z = (z_1, \ldots, z_n) \mapsto [e_0 + z_1e_1 + \cdots + z_ne_n]$. Then $\varepsilon(z) = e_0 + z_1e_1 + \cdots + z_ne_n$ defines a nonzero hol section of $\mathcal{O}_{\mathbb{P}^n}(-1)|_{\mathbb{C}^n}$. The adjoint homomorphism $g^* : Q \to \underline{V}$ is C^∞ and can be described as the orthogonal splitting of the above exact sequence. The images $(\tilde{e}_1, \ldots, \tilde{e}_n)$ of $(e_1, \ldots, e_n)$ in Q define a local holomorphic frame of $Q|_{\mathbb{C}^n}$, and we already know that $gg^* = \mathrm{id}_{\underline{V}}$ and

$$g^* \cdot \tilde{e}_j = e_j - \frac{\langle e_j, \varepsilon\rangle}{|\varepsilon|^2}\varepsilon = e_j - \frac{\overline{z}_j}{1+|z|^2}\varepsilon = e_j - \zeta_j\varepsilon,$$

where put $\zeta_j = \frac{\overline{z}_j}{1+|z|^2}$ (see [9, Chap. V]). By $gg^* = \mathrm{id}_{\underline{V}}$ and $\varepsilon \in \ker g$, we get $\tilde{e}_j = gg^*\tilde{e}_j = g(e_j - \zeta_j\varepsilon) = ge_j$. From these, the matrix representations of g and g^* with respect to frames $(\tilde{e}_1, \ldots, \tilde{e}_n)$ and $(e_1, \ldots, e_n)$ are as follows.

$$g = \begin{pmatrix} -z_1 & \\ \vdots & I_n \\ -z_n & \end{pmatrix}, \quad g^* = \begin{pmatrix} 0 \\ I_n \end{pmatrix} + G^*, \quad G^* = \begin{pmatrix} -\zeta_1 & \cdots & \cdots & -\zeta_n \\ -\zeta_1 z_1 & & & -\zeta_n z_1 \\ \vdots & & & \vdots \\ -\zeta_1 z_n & \cdots & \cdots & -\zeta_n z_n \end{pmatrix},$$

where we can write $G^* = (-\zeta_1\varepsilon, \cdots, -\zeta_n\varepsilon)$. In this setting, we prove the following.

PROPOSITION 6.8. *There exists a smooth Hermitian metric h_Q on $Q = \underline{V}/\mathcal{O}_{\mathbb{P}^n}(-1)$ such that h_Q has the minimal extension property.*

Proof. Let I_V be a trivial Hermitian metric on $\underline{V}$, then I_V has the minimal extension property by the optimal Ohsawa–Takegoshi theorem. We define the natural smooth Hermitian metric h_Q of Q induced from I_V by $|u|_{h_Q} := |g^*u|_{I_V}$ for any section u of Q. We show that h_Q has the minimal extension property. By the minimal extension property of I_V, for any $a \in \mathbb{P}^n$ and any $v \in Q_a$ with $|v|_{h_Q} = |g^*v|_{I_V} = 1$, there is a holomorphic section $s \in H^0(B, \underline{V})$ such that

$$s(0) = g^*v \quad \text{and} \quad \frac{1}{\mathrm{Vol}(B)}\int_B |s|^2_{I_V} dV_B \le 1,$$

where $g^*v \in \underline{V}_a$. From $gg^* = \mathrm{id}_{\underline{V}}$, then the composition gs is a holomorphic section, that is, $gs \in H^0(B, Q)$, and $gs(0) = gg^*v = v$. Hence, if $|gs|^2_{h_Q} = |g^*gs|^2_{I_V} \le |s|^2_{I_V}$ on B, then h_Q has the minimal extension property.

We can write $s = \sum_{j=0}^n s_je_j = \sigma_0\varepsilon + \sum_{j=0}^n \sigma_je_j \in H^0(B, \underline{V})$, where $\sigma_0 = s_0$, $\sigma_j = s_j - s_0z_j$, and $s_j \in \mathcal{O}(B)$. Then we have that $gs = \sum_{j=1}^n \sigma_jge_j = \sum_{j=1}^n \sigma_j\widetilde{e}_j$ and

$$g^*gs = \sum_{j=1}^n \sigma_jg^*\widetilde{e}_j = \sum_{j=1}^n \sigma_j(e_j - \zeta_j\varepsilon) = \Big(\sum_{j=1}^n \zeta_j\sigma_j\Big)e_0 + \sum_{j=1}^n \Big(\sigma_j - z_j\sum_{k=1}^n \zeta_k\sigma_k\Big)e_j,$$

$$|g^*gs|^2_{I_V} = \Big|\sum_{j=1}^n \zeta_j\sigma_j\Big|^2 + \sum_{j=1}^n \Big|\sigma_j - z_j\sum_{k=1}^n \zeta_k\sigma_k\Big|^2$$

$$\le (1+|z|^2)\Big|\sum_{j=1}^n \zeta_j\sigma_j\Big|^2 + \sum_{j=1}^n |\sigma_j|^2 = \frac{1}{1+|z|^2}\Big|\sum_{j=1}^n \overline{z}_j\sigma_j\Big|^2 + \sum_{j=1}^n |\sigma_j|^2$$

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$$\leq \frac{1}{1+|z|^2}\Big(\Big|\sum_{j=1}^n s_j\overline{z}_j\Big|^2+|s_0|^2|z|^4\Big)+\sum_{j=1}^n\Big(|s_j|^2+|s_0|^2|z_j|^2\Big)$$
$$=\frac{|z|^2}{1+|z|^2}\Big(\sum_{j=1}^n|s_j|^2+|s_0|^2|z|^2\Big)+\sum_{j=1}^n|s_j|^2+|s_0|^2|z|^2$$
$$=\frac{1+2|z|^2}{1+|z|^2}\Big(\sum_{j=1}^n|s_j|^2+|s_0|^2|z|^2\Big),$$

where $\sum_{j=1}^n\overline{z}_j\sigma_j=\sum_{j=1}^n(s_j\overline{z}_j-s_0|z_j|^2)=\sum_{j=1}^n s_j\overline{z}_j-s_0|z|^2$. Therefore, if

$$\frac{1+2|z|^2}{1+|z|^2}\Big(\sum_{j=1}^n|s_j|^2+|s_0|^2|z|^2\Big)\leq|s|^2_{I_V}=\sum_{j=0}^n|s_j|^2,$$

that is, $|z|^2\sum_{j=1}^n|s_j|^2\leq|s_0|^2(1-2|z|^4)$, then we obtain $|g^*gs|^2_{I_V}\leq|s|^2_{I_V}$.

Here, s_j is expressed as a scalar multiple of s_0 for any j. In fact, by the optimal Ohsawa–Takegoshi extension theorem (see [4], [13]) for trivial line bundle, there is a holomorphic function $f\in\mathcal{O}(B)$ such that

$$f(0)=1\quad\text{and}\quad\frac{1}{\mathrm{Vol}(B)}\int_B|f|^2dV_B\leq 1.$$

We write $g^*v=\sum_{j=0}^n w_je_j\in\underline{V_a}$ where $1=|g^*v|^2_{I_V}=\sum_{j=0}^n|w_j|^2$ and $w_j\in\mathbb{C}$. By changing the subscript of the local trivial frame (e_j), $w_0\neq 0$ can be assumed. Therefore, we can take $s_j:=w_jf=\frac{w_j}{w_0}f\in\mathcal{O}(B)$. Indeed, it is $s(0)=\sum_{j=0}^n w_jf(0)e_j=\sum_{j=0}^n w_je_j=g^*v$ and $|s|^2_{I_V}=(\sum_{j=0}^n|w_j|^2)|f|^2=|f|^2$.

Thus, the condition $|z|^2\sum_{j=1}^n|s_j|^2\leq|s_0|^2(1-2|z|^4)$ is sufficient for $2|z|^2+|z|^2(1-1/|w_0|^2)-1\leq 0$. Since w_0 is taken as one of the nonzeros in $\{w_0,\dots,w_n\}$ that satisfy $\sum_{j=0}^n|w_j|^2=1$, we get $|w_0|^2\geq\frac{1}{1+n}$. Hence, if the radius of B is taken to be smaller than $(-n+\sqrt{n^2+8})/4>0$, which is a solution of $2r^2+nr-1=0$, then we have that $|gs|^2_{h_Q}=|g^*gs|^2_{I_V}\leq|s|^2_{I_V}$ on B for any solution s of the optimal Ohsawa–Takegoshi extension theorem for any $g^*v\in\underline{V}_a$. ∎

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