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Inverse nodal problems on quantum tree graphs

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We consider inverse nodal problems for the Sturm–Liouville operators on the tree graphs. Can only dense nodes distinguish the tree graphs? In this paper it is shown that the data of dense-nodes uniquely determines the potential (up to a constant) on the tree graphs. This provides interesting results for an open question implied in the paper.

Keywords: Caterpillar graph; Tree graph; Sturm–Liouville operator; Inverse nodal problem

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1. Introduction

In these years there is a lot of interest in the study of Sturm–Liouville operators on graphs. On the one hand, the problem is a natural extension of the classical Sturm–Liouville operators on an interval; on the other hand, it has a number of applications in networks, spider webs and interlocking springs. Quantum graphs was introduced by Kottos and Smilansky [**14**]. Later, Kuchment studied [**16**] the eigenvalue properties of the periodic boundary value problem for the carbon atom in graphite.

In this work, we consider the inverse nodal problems on the tree graphs with Neumann boundary conditions by using dense nodal data, which amounts to nodes (zeros) of eigenfunctions. The inverse nodal problem was posed and solved for Sturm–Liouville problems by McLaughlin [**20**], who showed that the knowledge of a dense subset of nodal points of eigenfunctions on the whole interval alone can determine the potential function of the Sturm–Liouville problem up to a constant. This is the so-called inverse nodal problem.

From the physical point of view this corresponds to finding, e.g., the density of a string or a beam from the zero-amplitude positions of their eigenvibrations. These problems are related to some questions in mechanics and mathematical physics (see, e.g., [**20**]). Inverse nodal problems for Sturm–Liouville operators on an interval have been studied fairly completely in [**3**, **11**, **18**–**20**] and other papers.

Differential operators on graphs (networks, trees) often appear in natural sciences and engineering (see [**15**, **21**, **22**] and the references therein). The single spectrum does not determine the graph (topology) in general, especially if all edges have the same length (see e.g. [12, 17]). In [9] the author considered an inverse spectral problem for a star graph of Krein strings, where the known spectral data including the spectrum associated with the whole graph, the spectra associated with the individual edges are used to show that these spectral quantities uniquely determine the weight within the class of Borel measures on the graph. In [1, 2, 25] the authors solved inverse spectral problems for Sturm–Liouville operators on graphs, where the recovery of the differential operators on the edges of an a priori known graph, was done using the so-called Weyl functions.

On inverse nodal problems for differential operators on graphs there are only a few findings. The works [6, 23, 24] prove that the set of nodal points uniquely determines the boundary conditions and the potential on a star graph. As well as in the work [8] the authors give a construction of the potential on a tree as a limit of a sequence of functions dependent on the eigenvalues and its associated nodal data.

Inspired by the inverse node problems on the star graph we assert that the eigenvalue data in the inverse nodal uniqueness problem on a tree graph can be removed. We are interested in the inverse nodal problem on the tree graphs. In this paper we study more complicated tree graphs and show that the data of dense-nodes can uniquely determine the potential on the tree graphs. *However, for the general trees with possibly different edge lengths the inverse problems only using nodal data are still open.*

The results in this paper are the first in determining the edge potential of the tree with the same edge length from only the nodal data, it is possible due to finding a subsequences of eigenvalues and its relatively precise estimates in [13]. However, for more general graphs, especiallygraphs with different edge length, the inverse nodal problems are open. If one obtains relative precise asymptotic expression of infinitely many sub-eigenvalues then one can recover the potential on the graphs from only the nodal data.

This paper is organized as follows. Section 2 deals with some caterpillar graph, its eigenvalues, the oscillation of eigenfunctions and inverse nodal problems on a caterpillar graph. In §3 we investigate the corresponding results on the connected equilateral tree graphs.

2. Caterpillar graph

A graph like figure 1 is called a caterpillar graph, we consider a compact caterpillar graph $G(V, E)$ with the set of vertices $V = \{v_j\}_{j=1}^{6}$ and the set of edges $E = \{e_j\}_{j=1}^{5}$, where v_3 and v_4 are two internal vertices, and v_1, v_2, v_5, v_6 are boundary vertices.

We suppose that the length of each edge is equal to 1. Each edge $e_j \in E$ is parameterized by the parameter $x \in [0, 1]$; below we identify the value x of the parameter with the corresponding point on the edge. It is convenient for us to choose the following orientations. By choosing an interior vertex v_4 as the root then the graph $G(V, E)$ possesses a fixed orientation (see figure 1). Local coordinates for the edges identify each edge with $[0, 1]$ so that the local coordinate increases as the distance to the root decreases.

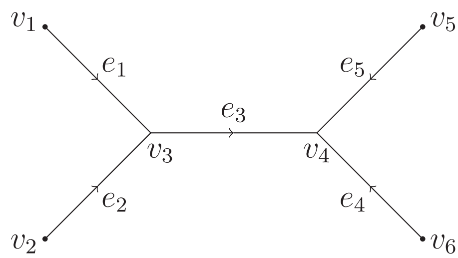

Figure 1. Oriented caterpillar graph.

An integrable function $Y(x,\lambda)$ on $G(V,E)$ may be represented as $Y(x,\lambda) = \{y_j(x)\}_{j=\overline{1,5}}$, where the function $y_j(x), x \in [0,1]$, is defined on the edge e_j. Let $q = \{q_j\}_{j=\overline{1,5}}$ be a square integrable real-valued function on $G(V,E)$. Consider the following differential equations on $G(V,E)$:

$$-y_j''(x) + q_j(x)y_j(x) = \lambda y_j(x), \quad j = \overline{1,5}, \tag{2.1}$$

where λ is the spectral parameter, the functions $y_j(x), y_j'(x), j = \overline{1,5}$, are absolutely continuous on $[0,1]$ and satisfy the matching conditions in the internal vertices v_3 and v_4: solutions are required to be continuous at the vertices v_3 and v_4, and in the local coordinate pointing outward, the sum of derivatives is zero at the vertices v_3 and v_4, respectively. That is, in the internal vertex v_3:

$$\left.\begin{array}{ll} y_1(1) = y_2(1) = y_3(0) & \text{(continuity condition)}, \\ y_1'(1) + y_2'(1) = y_3'(0) & \text{(Kirchhoff's condition)} \end{array}\right\} \tag{2.2}$$

and in the internal vertex v_4:

$$\left.\begin{array}{ll} y_3(1) = y_4(1) = y_5(1) & \text{(continuity condition)}, \\ y_3'(1) + y_4'(1) + y_5'(1) = 0 & \text{(Kirchhoff's condition)}, \end{array}\right\} \tag{2.3}$$

as well as Neumann conditions $y_j'(0) = 0$ $(i = 1, 2, 4, 5)$, which are assumed to hold at the pendant vertices.

2.1. Eigenvalues on a caterpillar graph

Let us consider the problem $B := B(q)$ on $G(V,E)$ for equation (2.1) with matching conditions (2.2) and (2.3) in the internal vertices, as well as Neumann conditions at the pendant vertices v_1, v_2, v_5, v_6.

Denote by $S_j(x,\lambda)$ and $C_j(x,\lambda), j = \overline{1,5}$, the solutions of the equation (2.1) on the edge e_j satisfying the initial conditions

$$S_j(0,\lambda) = C_j'(0,\lambda) = 0, \quad S_j'(0,\lambda) = C_j(0,\lambda) = 1.$$

For each fixed $x \in [0,1]$, the functions $S_j^{(v)}(x,\lambda)$ and $C_j^{(v)}(x,\lambda), j = \overline{1,5}, v = 0,1$, are entire in λ of the order $\frac{1}{2}$. Moreover, one gets (see, e.g., [**10**, Chap. 1] for

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details) the following asymptotical formulae as $|\lambda| \to \infty$, uniformly in $x \in [0, 1]$:

$$S_j(x, \lambda) = \frac{\sin \rho x}{\rho} - \frac{\cos \rho x}{\rho^2}[q_j]_x + \frac{\kappa_1(\rho)}{\rho^2}, \tag{2.4}$$

$$S'_j(x, \lambda) = \cos \rho x + \frac{\sin \rho x}{\rho}[q_j]_x + \frac{\kappa_2(\rho)}{\rho}, \tag{2.5}$$

$$C_j(x, \lambda) = \cos \rho x + \frac{\sin \rho x}{\rho}[q_j]_x + \frac{\kappa_3(\rho)}{\rho}, \tag{2.6}$$

$$C'_j(x, \lambda) = -\rho \sin \rho x + [q_j]_x \cos \rho x + \kappa_4(\rho), \tag{2.7}$$

where $\lambda = \rho^2$, $[q_j]_x = \frac{1}{2}\int_0^x q_j(t)dt$ and $\tau = \text{Im}\rho$, $\kappa_i(\rho) = o(1)$ $(i = 1, 2, 3, 4)$ for large real ρ, and $\kappa_i \in \mathcal{L}^x$ ($\mathcal{L}^x$ is the class of entire functions of exponential potential type less than x, belonging to $L^2(\mathbb{R})$ for real ρ).

Put

$$Y(x, \lambda) = \{y_i(x)\}_{i=\overline{1,5}}, \quad y_i(x) = \begin{cases} A_i(\lambda)C_i(x, \lambda), & i = 1, 2, 4, 5, \\ A_3(\lambda)C_3(x, \lambda) + B_3(\lambda)S_3(x, \lambda). \end{cases}$$

Then the function $Y(x, \lambda)$ satisfies equations and the boundary conditions. If λ^* is an eigenvalue of the problem then the function $Y(x, \lambda^*)$ is an eigenfunction. Submitting $Y(x, \lambda)$ into the matching conditions (2.2) and (2.3) we obtain a linear systems about the variables $A_i(\lambda)$ with $i = 1, 2, 4, 5$ and $A_3(\lambda)$ and $B_3(\lambda)$ appearing in $Y(x, \lambda)$. λ is an eigenvalue of the problem if and only if the determinant $\Delta(\lambda)$ of the coefficients of this linear systems about the variables $A_i(\lambda)$ with $i = 1, 2, 4, 5$ and $A_3(\lambda)$ and $B_3(\lambda)$ vanishes.

Moreover, a direct calculation yields the determinant

$$\begin{aligned}\Delta(\lambda) = {} & C_1(1, \lambda)C_2(1, \lambda)[C'_3(1, \lambda)C_4(1, \lambda)C_5(1, \lambda) + C_3(1, \lambda) \\ & C'_4(1, \lambda)C_5(1, \lambda) + C_3(1, \lambda)C_4(1, \lambda)C'_5(1, \lambda)] + [C'_1(1, \lambda) \\ & C_2(1, \lambda) + C_1(1, \lambda)C'_2(1, \lambda)][S'_3(1, \lambda)C_4(1, \lambda)C_5(1, \lambda) \\ & + S_3(1, \lambda)C'_4(1, \lambda)C_5(1, \lambda) + S_3(1, \lambda)C_4(1, \lambda)C'_5(1, \lambda)].\end{aligned} \tag{2.8}$$

Substituting (2.4)–(2.7) into (2.8) we get

$$\Delta(\lambda) = \Delta^0(\lambda) + o\left(\rho^2 \exp(5|\tau|)\right), \ |\lambda| \to \infty, \tag{2.9}$$

where

$$\Delta^0(\lambda) = -\rho^2 \sin \rho \cos^2 \rho (9 \cos^2 \rho - 4). \tag{2.10}$$

Notice that $\Delta^0(\lambda)$ is the characteristic function for the problem $B^0 := B(0)$ with the zero potential. It follows from (2.10) that the problem B^0 has a countable set of eigenvalues $\sigma(B^0) = \{\lambda^0_{ns}\}_{n\geqslant 0, s=\overline{1,5}}$ (counting multiplicities), where $\lambda^0_{ns} = (\rho^0_{ns})^2$,

$$\rho^0_{n1} = n\pi; \quad \rho^0_{n2} = n\pi - \theta; \quad \rho^0_{n3} = n\pi + \theta, \tag{2.11}$$

where $\theta = \arccos \frac{2}{3}$ and

$$\rho^0_{ns} = \left(n + \frac{1}{2}\right)\pi \text{ (multiplicities two)}, \quad s = 4, 5. \tag{2.12}$$

Here we used the fact that for the second-order self-adjoint differential operators on graphs the algebraic and geometric multiplicities of an eigenvalue are equal (see definition 3.3 and theorem 3.5 in [**7**]), so the order of a zero of the function in (2.10) coincides with the multiplicity as an eigenvalue of the problem B^0.

Note that the function $\Delta(\lambda)$ is entire in λ of the order $\frac{1}{2}$, and from the above analysis we know that its zeros coincide with the eigenvalues of the problem B. Applying the standard argument, based on Rouche's theorem (see, e.g., theorem 1.1.3 in [**10**]), we see that the function $\Delta(\lambda)$ has a countable set of eigenvalues $\{\lambda_{ns}\}_{n\geqslant 0, s=\overline{1,5}}$ (counting multiplicities), where $\lambda_{ns} = \rho^2_{ns}$. Combining the arguments in [**4**, **23**] we arrive at the following asymptotic formulas.

Lemma 2.1. *The problem B has a countable set of eigenvalues $\sigma(B) = \{\lambda_{ns}\}_{n\geqslant 0}$, $s = \overline{1,5} := \{1, \cdots, 5\}$. The eigenvalues counting with their multiplicities as $\{\lambda_{ns}\}_{n\geqslant 0, s=\overline{1,5}}$ in the nondecreasing order are numbered: $\lambda_{n_1,s_1} \leqslant \lambda_{n_2,s_2}$, if $(n_1, s_1) < (n_2, s_2)$ (this means that $n_1 < n_2$ or $n_1 = n_2, s_1 < s_2$). All eigenvalues are real and have the asymptotics*

$$\begin{aligned} &\rho_{n1} = n\pi + \frac{\omega}{n\pi} + o\left(\frac{1}{n}\right); \quad \rho_{n2} = n\pi - \theta + \frac{\omega}{n\pi} + o\left(\frac{1}{n}\right); \\ &\rho_{n3} = n\pi + \theta + \frac{\omega}{n\pi} + o\left(\frac{1}{n}\right), \end{aligned} \tag{2.13}$$

where $\omega = \frac{1}{5}\sum_{j=1}^{5}[q_j]_1$, and

$$\rho_{ns} = \left(n + \frac{1}{2}\right)\pi + \frac{\kappa_s}{\left(n + \frac{1}{2}\right)\pi} + o\left(\frac{1}{n}\right), \quad s = 4, 5, \tag{2.14}$$

where κ_s are the roots of the function $f(x)$:

$$f(x) = \sum_{j=4,5}(x - [q_1]_1)(x - [q_j]_1) + \sum_{j=4,5}(x - [q_2]_1)(x - [q_j]_1).$$

2.2. Nodes on a caterpillar graph

At the beginning of this section, we give a lemma.

Lemma 2.2. *The components of the eigenfunction $Y(x, \lambda_{n1})$ with*

$$||Y(\cdot, \lambda_{n1})||_{\bigoplus_{i=1}^{5} L^2(0,1)} = 1,$$

corresponding to the eigenvalues λ_{n1}, are not identically zero on different edges.

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Proof. In fact, from the matching conditions (2.2) and (2.3), we have

$$\begin{aligned}(-1)^n A_1(\lambda_{n1}) + o(1) &= (-1)^n A_2(\lambda_{n1}) + o(1) = A_3(\lambda_{n1}),\\ (-1)^n A_4(\lambda_{n1}) + o(1) &= (-1)^n A_5(\lambda_{n1}) + o(1)\\ &= (-1)^n A_3(\lambda_{n1}) + o(B_3(\lambda_{n1}))\end{aligned} \tag{2.15}$$

and

$$A_1(\lambda_{n1})([q_1]_1 - \omega) + A_2(\lambda_{n1})([q_2]_1 - \omega) + o(1) = B_3(\lambda_{n1}). \tag{2.16}$$

If $A_1(\lambda_{n1}) = 0$ then from (2.15) we get $A_i(\lambda_{n1}) = o(1)$ for $i = 2, 3$. Again, from (2.16), we obtain $B_3(\lambda_{n1}) = o(1)$, which implies from (2.15), that $A_i(\lambda_{n1}) = o(1)$ for $i = 4, 5$. This leads to a contraction to that $Y(x, \lambda_{n1})$ is an eigenfunction corresponding to the eigenvalue λ_{n1}, which is notrival. Similarly, if $A_3(\lambda_{n1}) = 0$ then from (2.15) we get $A_i(\lambda_{n1}) = o(1)$ for $i = 1, 2$. Again, from (2.16), we obtain $B_3(\lambda_{n1}) = o(1)$, which implies from (2.15), that $A_i(\lambda_{n1}) = o(1)$ for $i = 4, 5$. This also leads to a contraction. Therefore, for $i = \overline{1, 5}$ the quantities $A_i(\lambda_{n1})$ can't be zero. Lemma 2.2 is complete. □

Moreover, combining lemma 2.2, (2.15) with (2.16), it yields

$$\frac{B_3(\lambda_{n1})}{A_3(\lambda_{n1})} = [q_1 + q_2]_1 - 2\omega + o(1). \tag{2.17}$$

Using the asymptotic expressions (2.13), (2.4) and (2.6), when $n \to \infty$, we obtain the asymptotics for the components (modulus $A_i(\lambda_{n1})$) of the eigenfunction $Y(x, \lambda_{n1})$, uniformly in $x \in [0, 1]$:

$$C_i(x, \lambda_{n1}) = \cos \rho_{n1}^0 x + ([q_i]_x - \omega x)\frac{\sin \rho_{n1}^0 x}{\rho_{n1}^0} + o\left(\frac{1}{n}\right), \quad i = 1, 2, 4, 5$$

and

$$\begin{aligned}&C_3(x, \lambda_{n1}) + \frac{B_3(\lambda_{n1})}{A_3(\lambda_{n1})} S_3(x, \lambda_{n1})\\ &\quad = \cos \rho_{n1}^0 x + ([q_1 + q_2]_1 - 2\omega + [q_i]_x - \omega x)\frac{\sin \rho_{n1}^0 x}{\rho_{n1}^0} + o\left(\frac{1}{n}\right).\end{aligned}$$

Fix $i = 1, 2, 4, 5$. There exists N_0 such that for $n \geqslant N_0$ the function $C_i(x, \lambda_{n1})$ (or $C_3(x, \lambda_{n1}) + \frac{B_3(\lambda_{n1})}{A_3(\lambda_{n1})} S_3(x, \lambda_{n1})$) has exactly n simple zeros inside the interval $(0, 1)$, that is, $0 < x_{ni}^1 < \cdots < x_{ni}^n < 1$. The sets $X_i := \{x_{ni}^j\}_{n \geqslant N_0}$ $(i = \overline{1, 5}, j = \overline{1, n})$ are called the nodes on the edge e_i with respect to the eigenvalues λ_{n1}.

Taking asymptotic formulae (2.4) and (2.6) into account, we obtain the asymptotic expressions of nodes as follows.

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LEMMA 2.3. *For large n, the following asymptotic formulae for the nodes hold uniformly in j :*

$$x_{ni}^j = \begin{cases} \dfrac{j-\dfrac{1}{2}}{n} + \dfrac{[q_i]_x - \omega x}{n^2\pi^2} + o\left(\dfrac{1}{n^2}\right), & i = 1,2,4,5, \\ \dfrac{j-\dfrac{1}{2}}{n} + \dfrac{[q_1+q_2]_1 + [q_i]_x - 2\omega - \omega x}{n^2\pi^2} + o\left(\dfrac{1}{n^2}\right), & i = 3. \end{cases} \tag{2.18}$$

Proof. Fixing $i = 1,2,4,5$, there exists N_0 such that for $n \geqslant N_0$ the function $C_i(x,\lambda_{n1})$ has exactly n simple zeros inside the interval $(0,1)$. Other cases are similar and omitted.

Combining the asymptotics $\rho_{n1} = \sqrt{\lambda_{n1}} = n\pi + o\left(\frac{1}{n}\right)$, $n \to \infty$, with (2.6) and (2.7), we obtain that

$$C_i(x,\lambda_{n1}) = \cos\rho_{n1}x + f_{n,i,1}(x),\ C_i'(x,\lambda_{n1}) = -\rho_{n1}\sin\rho_{n1}x + f_{n,i,2}(x), \tag{2.19}$$

where

$$f_{n,i,1}(x) = O\left(\frac{1}{n}\right), \quad f_{n,i,2}(x) = O(1) \quad \text{for } n \to \infty,$$

uniformly on $[0,1]$. Therefore we conclude that

$$f_{n,i,1}'(x) = O(1), \quad n \to \infty,$$

uniformly for $x \in \mathbb{R}$.

Consider the equation $C_i(x,\lambda_{n1}) = 0$ on $(0,1)$, which is equivalent to the equations

$$x = x_{ni}^j(x), \quad x_{ni}^j(x) := \frac{\left(j-\frac{1}{2}\right)\pi}{\rho_{n1}} + f_{n,i,1}^j(x), \quad j \in \mathbb{N}, \tag{2.20}$$

where $f_{n,i,1}^j(x) = (-1)^j \frac{\arcsin f_{n,i,1}(x)}{\rho_{n1}}$, and

$$\left(f_{n,i,1}^j\right)'(x) = O\left(\frac{1}{n}\right), \quad n \to \infty, \tag{2.21}$$

uniformly for $j \in \mathbb{N}$ and $x \in [0,1]$. One can continue $f_{n,i,1}^j(x)$ on $(-\infty,0) \cup (1,\infty)$ by differentiability in any way to satisfy (2.21) uniformly for $j \in \mathbb{N}$ and $x \in [0,1]$.

Consider the equation (2.20) in $\mathbb{R}$. According to (2.21) and the formula

$$x_{ni}^j(x_1) - x_{ni}^j(x_2) = (f_{n,i,1}^j)'(\theta)(x_1 - x_2), \quad \theta \in (x_1,x_2),$$

there exists N_0 such that for $n \geqslant N_0$ the function $x_{ni}^j(x)$ is a contracting mapping in $\mathbb{R}$ for all $j \in \mathbb{N}$. Thus, for each $j \in \mathbb{N}$ the equation (2.20) has a unique solution in $\mathbb{R}$, which is denoted by x_{ni}^j.

Again, taking into account (2.20), we arrive at the formula

$$x_{ni}^{j}=\frac{\left(j-\frac{1}{2}\right)\pi}{\rho_{n1}}+O\left(\frac{1}{n^{2}}\right),\quad n\to\infty,\quad j\in\mathbb{N}. \tag{2.22}$$

Note that, for sufficiently large n, we have $x_{ni}^{1}\in\left(0,\frac{\pi}{\rho_{n1}}\right)$. Introduce the nodal length $l_{ni}^{j}:=x_{ni}^{j+1}-x_{ni}^{j}$. Then from (2.22), we have

$$l_{ni}^{j}=\frac{\pi}{\rho_{n1}}+O\left(\frac{1}{n^{2}}\right),$$

uniformly with respect to j. Hence, we obtain for sufficiently large n, that $x_{ni}^{j}\in\left(\frac{(j-1)\pi}{\rho_{n1}},\frac{j\pi}{\rho_{n1}}\right)$ for $j=\overline{1,n}$. Thus, we have proved that the function $C_i(x,\rho_{n1})$ has exactly n nodes for large positive values of n.

In order to obtain more precise asymptotic expression, we substitute (2.22) into the equation $C_i(x,\lambda_{n1})=0$. In view of (2.22), we have

$$|\sin\rho_{n1}x|\geqslant c_0>0$$

for sufficiently large n and $x=x_{ni}^{j}$, where the constant c_0 does not depend on n and j. Therefore we derive the relation

$$\cot\rho_{n1}x=-\frac{[q_i]_x-\omega x}{\rho_{n1}}+o\left(\frac{1}{\rho_{n1}}\right),$$

which is equivalent to

$$\tan\left(\rho_{n1}x+\frac{\pi}{2}\right)=\frac{[q_i]_x-\omega x}{\rho_{n1}}+o\left(\frac{1}{\rho_{n1}}\right).$$

Using Taylor's expansion for the arctangent, we obtain the following asymptotic formulae for the nodal points as $n\to\infty$:

$$x_{ni}^{j}=\frac{\left(j-\frac{1}{2}\right)\pi}{\rho_{n1}}+\frac{[q_i]_x-\omega x}{\rho_{n1}^{2}}+o\left(\frac{1}{\rho_{n1}^{2}}\right). \tag{2.23}$$

From the asymptotics of ρ_{n1}, we have $\frac{1}{\rho_{n1}}=\frac{1}{n\pi}+o\left(\frac{1}{n^3}\right)$ and $\frac{1}{\rho_{n1}^2}=\frac{1}{n^2\pi^2}+o\left(\frac{1}{n^4}\right)$. Combining the latter formulae with (2.23), we arrive at the expected results (2.18). □

2.3. Inverse nodal problems on a caterpillar graph

Note that for the fixed $i=\overline{1,5}$ the nodal sets X_i is dense in $(0,1)$. Analyzing the asymptotic expressions x_{ni}^{j} we have the following statements.

Fixed $i=\overline{1,5}$ and $x\in[0,1]$. Suppose that $X_i^0\subset X_i$ is dense on $(0,1)$ and choose $\{x_{ni}^{j_{ni}}\}\subset X_i^0$ such that $\lim_{n\to\infty}x_{ni}^{j_{ni}}=x$. Then the following finite limits hold:

$$\lim_{n\to\infty}n^2\pi^2\left(x_{ni}^{j_{ni}}-\frac{j-\frac{1}{2}}{n}\right)\overset{\text{exists}}{=}f_i(x),\quad i=\overline{1,5}, \tag{2.24}$$

where $f_i(x)=[q_i]_x-\omega x$ for $i=1,2,4,5$, and $f_3(x)=[q_1+q_2]_1+[q_3]_x-2\omega-\omega x$.

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Now we can provide a uniqueness theorem and constructive algorithm for the inverse nodal problem on the whole interval $(0, 1)$. For stating the theorem, together with $B := B(q)$ we consider a boundary value problem $\tilde{B} := B(\tilde{q})$ of the same form but with a different potential q. We agree that if a certain symbol α denotes an object related to B, then $\tilde{\alpha}$ will denote an analogous object related to $\tilde{B}$.

Note that the zero sets X_i $(i = \overline{1, 5})$ are defined as shown before lemma 2.3.

THEOREM 2.4. *Fix $i = \overline{1, 5}$. Suppose that $X_i^0 \subset X_i$ is dense on $(0, 1)$ and $X_i^0 = \tilde{X}_i^0$, then $q_i(x) = \tilde{q}_i(x)$ a.e. on $(0, 1)$. Therefore, the data X_i^0 uniquely determines the $q_i(x) - 2\omega$ on the edge e_i.*

The constructive algorithm is as follows. For $i = \overline{1, 5}$, given X_i^0.

(1) *X_i^0 determining $f_i(x)$ from (2.24);*

(2)

$$q_i(x) - 2\omega \overset{a.e.}{=} 2f_i'(x). \tag{2.25}$$

In fact, (2.25) follows from (2.24). If $X_i^0 = \tilde{X}_i^0$ then (2.24) implies that $f_i(x) = \tilde{f}_i(x)$ for $x \in [0, 1]$, and consequently $q_i(x) - \omega = \tilde{q}_i(x) - \tilde{\omega}$ a.e. on $(0, 1)$. Moreover, theorem 2.1 demonstrates that the nodal data on one edge can determine the potential on the edge up to a constant.

3. Finite tree

In this section we consider a connected tree $G(V, E)$ with edges of the equal length. We parametrize each edge with $x \in (0, 1)$. This gives an orientation on $G(V, E)$. We consider a Schrödinger operator with potential $q_j \in L^2(0, 1)$ on the edge e_j and with Neumann (or Kirchhoff) boundary conditions (some times called standard matching conditions), i.e., solutions are required to be continuous at the vertices and, in the local coordinate pointing outward, the sum of derivatives is zero. More formally, one considers the eigenvalue problem (figure 2)

$$-y''(x) + q_j(x)y(x) = \lambda y(x) \tag{3.1}$$

on e_j for all j with the conditions

$$y_j(\kappa_j) = y_k(\kappa_k) \tag{3.2}$$

if e_j and e_k are incident edges attached to a vertex v where $\kappa = 0$ for outgoing edges, $\kappa = 1$ for incoming edges; and in every vertex v

$$y_j'(0) = \sum_{e_j \text{ enters } v} y_j'(1). \tag{3.3}$$

3.1. Eigenvalue and eigenfunction

Section 2 is a special tree graph (called caterpillar graph), however, §3 deals with a general tree graph. All eigenvalues in §2 can be estimated, while in §3 this is not possible, we only obtain a subsequences $\{\lambda_n\}$ of eigenvalues.

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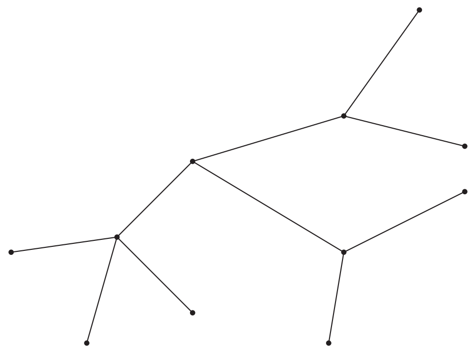

Figure 2. Quantum tree graph.

A connected graph $G(V, E)$ with edges of the equal length is considered in [**13**]. The spectral determinant of Schrödinger operators on $G(V, E)$ with standard matching conditions has a sequence of roots which asymptotically differ by the mean value of the potential from the corresponding sequence of roots of the spectral determinant of the free Schrödinger operator. Precisely, the problem (3.1)–(3.3) has a sequence of eigenvalues

$$\lambda_n = (2n\pi)^2 + \langle q \rangle + o(1) \tag{3.4}$$

for large integer n, and $\langle q \rangle = \frac{1}{|E|}\sum_j \int_0^1 q_j(t)\,\mathrm{d}t$, $|E|$ denotes the number of edges on the tree graphs. Moreover, if $G(V, E)$ is a bipartite graph, the problem (3.1)–(3.3) has a sequence of eigenvalues

$$\lambda_n = (n\pi)^2 + \langle q \rangle + o(1). \tag{3.5}$$

Denote by $S_j(x, \lambda)$ and $C_j(x, \lambda)$ the solutions of the equation (3.1) on the edge e_j satisfying the initial conditions

$$S_j(0, \lambda) = C_j'(0, \lambda) = 0, \quad S_j'(0, \lambda) = C_j(0, \lambda) = 1.$$

For each fixed $x \in [0, 1]$, the functions $S_j^{(v)}(x, \lambda)$ and $C_j^{(v)}(x, \lambda), v = 0, 1$, are entire in λ of the order $\frac{1}{2}$, and these solutions possess the asymptotic expressions (2.4)–(2.7).

Before proving the main result we recall some preliminaries. From (2.4)–(2.7) we know that for $\lambda > 0$ the following estimates hold

$$C_j(x, \lambda) = \cos(\sqrt{\lambda}x) + O\left(\frac{1}{\sqrt{\lambda}}\right), \quad S_j(x, \lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right). \tag{3.6}$$

Suppose that $Y(x, \lambda)$ is a vector function whose components $y_j(x, \lambda)$ satisfy (3.1), and which is given the graph L^2 norm:

$$\|Y(x, \lambda)\|^2 = \sum_{j:e_j\in E} \int_0^1 |y_j(x, \lambda)|^2\,\mathrm{d}x.$$

Each $y_j(x,\lambda)$ may be written as a linear combination

$$y_j(x,\lambda) = A_j(\lambda)C_j(x,\lambda) + B_j(\lambda)S_j(x,\lambda).$$

Suppose $\|Y(x,\lambda)\| = 1$. Then there is a $\lambda_0 > 0$ and a constant C such that (see [**5**, lemma 2.1])

$$|A_j(\lambda)| \leqslant C, \quad |B_j(\lambda)/\sqrt{\lambda}| \leqslant C, \quad \lambda \geqslant \lambda_0.$$

Suppose that λ has the form $\lambda = (2\pi n)^2 + \langle q\rangle + o(1)$ as $n \to \infty$, then there hold (see [**5**, equation (2.3)])

$$\begin{aligned} &C_j(1,\lambda) = 1 + o(1), \quad C_j'(1,\lambda) = [q_j]_1 + o(1),\\ &S_j(1,\lambda) = o(n^{-1}), \quad S_j'(1,\lambda) = 1 + o(1), \end{aligned} \tag{3.7}$$

where $[q_j]_1 = \frac{1}{2}\int_0^1 q_j(x)\,\mathrm{d}x$.

Suppose that $\{Y(x,\lambda_n)\}$ is a sequence of eigenfunctions for (3.1) with norm 1, corresponding to the eigenvalue $\lambda_n = (2\pi n)^2 + \langle q\rangle + o(1)$ as $n \to \infty$. Write the components $y_j(x,\lambda_n)$ as a linear combination

$$y_j(x,\lambda_n) = A_j(\lambda_n)C_j(x,\lambda_n) + B_j(\lambda_n)S_j(x,\lambda_n).$$

Recall that the coefficients $A_j(\lambda_n)$ and $B_j(\lambda_n)/\sqrt{\lambda_n}$ are bounded sequences (see [**5**, lemma 2.1]).

Firstly, we consider the values of $y_j(x,\lambda_n)$ for edges incident on a vertex v. The continuity of Y at the vertex v thus implies [**5**]

$$A_j(\lambda_n) = A_k(\lambda_n) + o(1), \quad n \to \infty, \tag{3.8}$$

for all edges j, k incident on v. Since the graph $G(V,E)$ is connected, equation (3.8) can be extended to all edges j, k.

Secondly, the root vertex is regarded as the top of the graph $G(V,E)$, that is to say, an edge e_j is below an edge $e_k \neq e_j$ if a path from e_j to the root passes through e_k. We label each vertex v of the graph $G(V,E)$ with the combinatorial distance from the root, and label edges with the larger of the vertex labels on the edge. Let M be the maximum label. If the vertex v has label $M-1$, then all its incoming edges e_k join v to a pendant vertex v. If e_j is the outgoing edge for v, then the derivative condition at v gives

$$B_j(\lambda_n) = A_1(\lambda_n)\sum_h [q_h] + o(1), \tag{3.9}$$

where the last sum is taken over all edges e_h which are below e_j on $G(V,E)$.

For vertices v with label $M-2$, and outgoing edge e_j, the derivative condition at v gives

$$B_j(\lambda_n) = A_1(\lambda_n)\sum_l [q_l] + o(1), \tag{3.10}$$

where the last sum is taken over all edges e_l which are below e_j on $G(V,E)$.

Thirdly, $A_j(\lambda_n)$ are bounded away from zero, otherwise by $||Y(x,\lambda_n)|| = 1$ we get a contradiction.

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3.2. Nodes on a tree

Put

$$Y(x,\lambda_n)=\{y_j(x,\lambda_n)\}$$

where

$$y_j(x,\lambda_n)=\begin{cases}A_j(\lambda_n)C_j(x,\lambda_n) \text{ for boundary edges,}\\ A_j(\lambda_n)C_j(x,\lambda_n)+B_j(\lambda_n)S_j(x,\lambda_n) \text{ for other edges.}\end{cases}$$

Using the asymptotic expressions (3.4), (2.4) and (2.6), when $n\to\infty$, we obtain the asymptotics for the components (modulus nonvanishing $A_j(\lambda_n)$) of the eigenfunction $Y(x,\lambda_n)$, uniformly in $x\in[0,1]$:

$$C_j(x,\lambda_n)=\cos\sqrt{\lambda_n}x+[q_j]_x\frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}+o\left(\frac{1}{n}\right) \text{ on boundary edges}$$

and on other edges

$$\begin{aligned}&C_j(x,\lambda_n)+\frac{B_j(\lambda_n)}{A_j(\lambda_n)}S_j(x,\lambda_n)\\&\quad=\cos\sqrt{\lambda_n}x+\left(\sum_l[q_l]_1-l\langle q\rangle+[q_j]_x\right)\frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}+o\left(\frac{1}{n}\right).\end{aligned}$$

Here the sum is taken over all edges e_l which are below e_j on $G(V,E)$.

There exists N_0 such that for $n\geqslant N_0$ the function $C_j(x,\lambda_n)$ (or $C_j(x,\lambda_n)+\frac{B_j(\lambda_n)}{A_j(\lambda_n)}S_j(x,\lambda_n)$) has exactly n simple zeros inside the interval $(0,1)$, that is, $0<x_{nj}^1<\cdots<x_{nj}^n<1$. The sets $X_j:=\{x_{nj}^k\}_{n\geqslant N_0}$ $(k=\overline{1,n})$ are called the nodes on the edge e_j with respect to the eigenvalues λ_n.

Taking asymptotic formulae (2.4) and (2.6) into account, we obtain the asymptotic expressions of nodes as follows.

Lemma 3.1. *For large n, the following asymptotic formulae for the nodes hold uniformly in j:*

$$x_{nj}^k=\begin{cases}\dfrac{k-\frac{1}{2}}{n}+\dfrac{[q_j]_x-\langle q\rangle x}{n^2\pi^2}+o\left(\dfrac{1}{n^2}\right) \text{ on boundary edges,}\\ \dfrac{k-\frac{1}{2}}{n}+\dfrac{\sum_l[q_l]_1+[q_j]_x-l\langle q\rangle-\langle q\rangle x}{n^2\pi^2}+o\left(\dfrac{1}{n^2}\right) \text{ on other edges.}\end{cases} \tag{3.11}$$

3.3. Inverse nodal problems on a tree

Note that for the fixed e_j the nodal sets X_j is dense in $(0,1)$, respectively. Analyzing the asymptotic expression x_{nj}^k we have the following statements.

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Fixed e_j and $x \in [0,1]$. Suppose that $X_j^0 \subset X_j$ is dense on $(0,1)$ and choose $\{x_{nj}^{k_{nj}}\} \subset X_j^0$ such that $\lim_{n\to\infty} x_{nj}^{k_{nj}} = x$. Then the following finite limit holds:

$$\lim_{n\to\infty} n^2\pi^2 \left(x_{nj}^{k_{nj}} - \frac{k-\frac{1}{2}}{n} \right) \overset{\text{exists}}{=} \theta_j(x), \tag{3.12}$$

where $\theta_j(x) = [q_j]_x - \langle q \rangle$ for boundary edges, and $\theta_j(x) = \sum_l [q_l]_1 + [q_j]_x - l\langle q \rangle - \langle q \rangle x$ for other edges.

Now we can provide a uniqueness theorem and constructive algorithm for the inverse nodal problem on the whole interval $(0,1)$.

THEOREM 3.2. *Fix the edge e_j. Suppose that the following conditions is true:*

$$X_j^0 \subset X_j \text{ is dense on } (0,1) \text{ and } X_j^0 = \tilde{X}_j^0,$$

then $q_j(x) = \tilde{q}_j(x)$ a.e. on $(0,1)$. Therefore, the data X_j^0 uniquely determines the potential $q_j(x) - \langle q \rangle$ on the edge e_j.

The constructive algorithm is as follows:

(1) *X_j^0 determining $\theta_j(x)$ from* (3.12);

(2)

$$q_j(x) - \langle q \rangle \overset{a.e.}{=} 2\theta_j'(x). \tag{3.13}$$

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