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A THEOREM IN THE PARTITION CALCULUS

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1. **Introduction** If S is an ordered set we write tp S to denote the order type of S and $|S|$ for the cardinal of S. We also write $[S]^k$ for the set $\{X:X \subset S, |X|=k\}$. The partition symbol

$$\alpha \to (\beta_0, \beta_1)^2 \tag{1}$$

connecting the order types α, β_0, β_1 by definition (see [2]) means: *if* tp $S=\alpha$ *and* $[S]^2$ *is partitioned in any way into two sets* K_0, K_1 *then there are* $i<2$ *and* $B \subset S$ *such that* tp $B=\beta_i$ *and* $[B]^2 \subset K_i$. The negation of (1) is written as $\alpha\not\to(\beta_0, \beta_1)^2$.

The purpose of this note is to prove that

$$\omega^{1+\nu h} \to (2^h, \omega^{1+\nu})^2 \tag{2}$$

holds for $h<\omega$ and $\nu<\omega_1$. We have known this result since 1959. It has been quoted in lectures on the partition calculus by Erdös and there is mention of the theorem in the literature ([3], [7], [11]). A proof was given in Milner's thesis [6]. However, we have been asked for details of the proof on several occasions and so it seems desirable to have a reference which is more readily available than [6].

For finite k, (2) gives

$$\omega^{4k+1} \to (4, \omega^{2k+1})^2, \tag{3}$$

and this should be contrasted with the negative relation

$$\alpha \not\to (3, \omega^{2k+1})^2 \qquad (\alpha < \omega^{3k+1}) \tag{4}$$

proved in [7]. We know that (3) is not best possible. For example, it is known that

$$\omega^4 \to (4, \omega^3)^2, \qquad \omega^4 \not\to (5, \omega^3)^2.$$

These results were first proved by A. Hajnal, then by F. Galvin and, more recently by Haddad and Sabbagh [11]. These authors independently discovered a finite algorithm for deciding the truth value of (1) for the case $\alpha<\omega^\omega$. Hajnal and Galvin did not publish their results but a preliminary account of the algorithm is described in the papers by Haddad and Sabbagh ([9], [10], [11]). Quite recently Chang [1] proved that $\omega^\omega\to(3, \omega^\omega)^2$ and Milner (unpublished) generalized this by proving that

$$\omega^\omega \to (m, \omega^\omega)^2 \qquad (m < \omega).$$

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This again shows that (2) is far from being best possible. Even so, it is still the best general positive result of this kind known to us and so it remains of interest.(2).

We should like to express our gratitude to the referee for a number of useful comments. In particular, the proof of (9) follows a suggestion of the referee and is simpler than our original version.

2. The order relation in an ordered set will always be denoted by $<$. If A, B are subsets of the ordered set S, we write $A<B$ if $a<b$ holds for all $a \in A$ and $b \in B$. We also write

$$S = \bigcup_{\nu \in N} A_\nu \quad (<)$$

to indicate that S and N are ordered sets, $S=\bigcup_{\nu\in N} A_\nu$ and $A_\mu < A_\nu$ holds whenever $\mu, \nu \in N$ and $\mu<\nu$. We write tp $A \geq$ tp B if there is a subset $A' \subset A$ which is order isomorphic to B. If α, β are order types we write $\alpha \approx \beta$ if $\alpha \geq \beta$ and $\beta \geq \alpha$.

An order type α is *additively indecomposable* (AI) if $\alpha=\beta+\gamma$ implies that either $\beta \geq \alpha$ or $\gamma \geq \alpha$. α is *right*-AI if $\alpha=\beta+\gamma$, $\gamma \neq 0$ implies $\gamma \geq \alpha$; *left*-AI is similarly defined. The type α is *strongly indecomposable* (SI) if whenever tp $A=\alpha$, $A=B \cup C$, then either tp $B \geq \alpha$ or tp $C \geq \alpha$. Clearly SI implies AI. We say α is right (left)-SI if it is SI and right (left)-AI. It is well known that the AI ordinal numbers are 0 and powers of ω and these are even right-SI (e.g., see [8]).

A type α is said to be *scattered* if $\alpha \ngeq \eta$, the order type of the rationals. Laver [5] proved that the scattered types are well-quasiordered and an easy consequence of this (e.g. [4]) is that a scattered set is the union of a finite number of sets whose types are SI. We will say that β is a strong type if, whenever tp $B=\beta$ and $D \subset B$, then there are $n<\omega$ and sets $D_1, \ldots, D_n \subset D$ such that

(5) tp D_i *is SI for* $i = 1, \ldots, n$;

(6) *if* $M \subset D$ *and* tp $(M \cap D_i) \geq$ tp D_i *for* $i = 1, \ldots, n$, *then* tp $M \approx$ tp D.

From Cantor's classical theorem that an ordinal number is expressible as a finite sum of SI ordinal numbers, it follows that an ordinal number α and its reverse α^* are strong types. We mistakenly thought that any scattered type is strong, but the simple example $(\omega^*+\omega)\omega^2$ pointed out to us by R. Laver, shows that this is false. Our theorem stated in the next section, which implies (2), is valid for any strong denumerable type β. We conjecture that the result is true for any denumerable type β.

Added in Proof. F. Galvin has now settled this conjecture. His proof of the stronger result will appear in a later issue of the Bulletin.

(2) **Added in Proof.** Jean Larson has since found a much simpler proof of the relation $\omega^\omega \rightarrow (m, \omega^\omega)^2$ $(m<\omega)$. Eva Nosal has recently obtained several strong results of this kind. In particular, she proved that $\omega^{1+\nu(h+1)-h} \nrightarrow (2^h+1, \omega^{1+\nu})^2$ for $1 \leq h < \omega$ and $2 \leq \nu < \omega$. This shows, rather surprisingly, that in general (2) cannot be substantially improved. For example, (2) gives $\omega^9 \rightarrow (4, \omega^5)^2$ whereas Eva Nosal's negative result gives $\omega^8 \nrightarrow (3, \omega^5)^2$ and $\omega^{12} \nrightarrow (5, \omega^5)^2$.

3. We shall prove the following:

THEOREM. *Let α be right-SI and let β be any strong denumerable type. If* $2 \leq k < \omega$ *and* $\alpha \to (k, \gamma)^2$, *then*

$$\alpha\beta \to (2k, \gamma \vee \omega\beta)^2. \tag{7}$$

REMARKS. (a) In (7) we use the partition symbol with alternatives and the precise meaning of this is the following: If tp $S = \alpha\beta$ and $[S]^2 = K_0 \cup K_1$, then either

(i) there is $X \in [S]^{2k}$ such that $[X]^2 \subset K_0$, or

(ii) there is $C \subset S$ such that tp $C = \gamma$ and $[C]^2 \subset K_1$, or

(iii) there is $Z \subset S$ such that tp $Z = \omega\beta$ and $[Z]^2 \subset K_1$.

(b) If we change the hypothesis on α from right-SI to left-SI, we obtain the analogous result that

$$\alpha\beta \to (2k, \gamma \vee \omega^*\beta)^2.$$

(c) Suppose (2) holds for some integer $h \geq 1$. Applying the above theorem with $k = 2^h$, $\alpha = \omega^{1+\nu h}$, $\beta = \omega^\nu$, $\gamma = \omega^{1+\nu}$, we see that (2) also holds with h replaced by $h+1$. Since (2) holds trivially for $h = 1$, it follows that (2) holds for all $h < \omega$.

Proof of Theorem. Let tp $S = \alpha\beta$, $[S]^2 = K_0 \cup K_1$. If $\alpha = 1$, the hypothesis $\alpha \to (k, \gamma)^2$ implies that $\gamma \leq 1$ and (ii) above holds. Similarly, if $\beta = 0$, then (iii) holds. We may therefore assume that $\alpha > 1$ and $\beta \geq 1$. We shall assume that statements (i) and (ii) in Remark (a) above are both false and deduce (iii).

Throughout the proof B denotes a fixed set of type β and the letter A (A', A_ν, etc.) always denotes a subset of S of type α. If $x \in S$ and $i < 2$ we define $K_i(x) = \{y \in S : \{x, y\} \in K_i\}$, also if $X \subset S$ we define $K_i(X) = \bigcap_{x \in X} K_i(x)$.

(8) *If $A \subset S$, then there is $X \in [A]^k$ such that $[X]^2 \subset K_0$.*

This follows from the hypothesis $\alpha \to (k, \gamma)^2$ and the assumed falsity of (ii).

(9) *Suppose $D \subset B$, $A_\nu \subset S (\nu \in D)$, $A \subset S$. For $x \in A$ let*

$$M(x) = \{\nu \in D : \text{tp}(K_1(x) \cap A_\nu) \geq \alpha\}.$$

Then

$$\text{tp}\{x \in A : \text{tp } M(x) \geq \text{tp } D\} \geq \alpha.$$

We prove this first with the added assumption that tp D is SI. Suppose the conclusion is false. Then tp $A' \geq \alpha$, where $A' = \{x \in A : \text{tp } M(x) \ngeq \text{tp } D\}$. By (8) there is $X \in [A']^k$ such that $[X]^2 \subset K_0$. From the assumption that tp D is SI it follows that there is $\nu \in D - \bigcup_{x \in X} M(x)$. Then tp $(K_1(x) \cap A_\nu) \ngeq \alpha$ for $x \in X$ and hence tp $(K_0(X) \cap A_\nu) \geq \alpha$. Therefore, by (8) again, there is $Y \in [K_0(X) \cap A_\nu]^k$ such that $[Y]^2 \subset K_0$. This gives the contradiction that $|X \cup Y| = 2k$ and $[X \cup Y]^2 \subset K_0$.

Assume now that D is any subset of B. Since β is strong there are sets $D_1, \ldots, D_n \subset D$ such that (5) and (6) hold. Applying (9) successively to $D_1, \ldots, D_n$, we

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see that there is $A'' \subset A$ such that tp $(M(x) \cap D_i) \geq$ tp D_i for all $x \in A''$ and $i=1, \ldots, n$. It follows from (6) that tp $M(x) \geq$ tp D for $x \in A''$ and this completes the proof of (9).

As a special case of (9) (with tp $D=1$) we have:

(9′) *If* $A, A' \subset S$, *then* tp $\{x \in A' : \text{tp } (K_1(x) \cap A) \geq \alpha\} \geq \alpha$.

(10) *Let F be a finite subset of B, $S' = \bigcup (\nu \in B) A_\nu(<)$, $A \subset S$. Then there are $x_0 \in A$ and a strictly increasing map $g: B \to B$ such that $g(\nu) = \nu$ $(\nu \in F)$ and* tp $(K_1(x_0) \cap A_{g(\nu)}) \geq \alpha (\nu \in B)$.

We may write $B = D_0 \cup \{\nu_1\} \cup D_1 \cup \cdots \cup \{\nu_p\} \cup D_p(<)$, where $F = \{\nu_1, \ldots, \nu_p\}$. For $x \in A$, put $M(x) = \{\nu \in B : \text{tp } (K_1(x) \cap A_\nu) \geq \alpha\}$. By a finite number of applications of (9′) it follows that there is $A' \subset A$ such that $F \subset M(x)$ for all $x \in A'$. If the assertion (10) is false, then for each $x \in A'$ there is $\lambda(x) \leq p$ such that tp $(M(x) \cap D_{\lambda(x)}) \ngeq$ tp $D_{\lambda(x)}$. Since α is strongly indecomposable, there is $A'' \subset A'$ such that $\lambda(x) = \lambda$ for all $x \in A''$. Then tp $(M(x) \cap D_\lambda) \ngeq$ tp $D_\lambda (x \in A'')$, a contradiction against (9).

We now conclude the proof of the theorem.

Since β is denumerable and nonzero, there is a sequence $(\gamma_n : n < \omega)$ which repeats each element of B infinitely often, i.e. such that

(11) $$|\{n : \gamma_n = \nu\}| = \aleph_0 \qquad (\nu \in B).$$

Since tp $S = \alpha\beta$, we may write $S = S^{(0)} = \bigcup (\nu \in B) A_\nu^{(o)}(<)$.

Let $n < \omega$ and suppose we have already chosen elements $x_i \in S (i < n)$ and a subset

(12) $$S^{(n)} = \bigcup (\nu \in B) A_\nu^{(n)}(<)$$

of S of order type $\alpha\beta$. Since α is right-SI, $A_{\gamma_n}^{(n)}$ contains a final section A' such that $A_{\gamma_n}^{(n)} \cap \{x_0, \ldots, x_{n-1}\} < A'$. By (10), there are $x_n \in A'$, a strictly increasing map $g_n : B \to B$ and sets $A_\nu^{(n+1)} (\nu \in B)$ such that

(13) $$g_n(\gamma_i) = \gamma_i \qquad (i \leq n),$$

(14) $$A_\nu^{(n+1)} \subset K_1(x_n) \cap A_{g_n(\nu)}^{(n)} \qquad (\nu \in B).$$

From the definition of A', it follows that

(15) $$x_n \in A_{\gamma_n}^{(n)} \subset S^{(n)}$$

and

(16) $$x_i < x_n \quad \text{if} \quad i < n \quad \text{and} \quad x_i \in A_{\gamma_n}^{(n)}.$$

$S^{(n+1)}$ is defined by equation (12) with n replaced by $n+1$. It follows by induction that there are x_n, $A_\nu^{(n)} (\nu \in B)$, $S^{(n)}$ and g_n such that (12)–(16) hold for $n < \omega$.

Let $Z = \{x_n : n < \omega\}$. If $m < n < \omega$, then by (15), (14), and (12) we have that

$$x_n \in S^{(n)} \subset S^{(m+1)} \subset K_1(x_m).$$

Therefore, $[Z]^2 \subset K_1$. To complete the proof of (iii) we have only to show that tp $Z = \omega\beta$.

By (13), we see that

$$g_{j-1}(g_{j-2}(\ldots(g_i(\gamma_i))\ldots)) = \gamma_i \qquad (i < j < \omega). \tag{17}$$

Also, if $i<j$, then by (12), $A^{(j)}_{\gamma_j} \subset A^{(i)}_{\rho}$, where

$$\rho = g_{-1}(g_{j-2}(\ldots(g_i(\gamma_j))\ldots)).$$

Since the g_n are increasing functions, it follows that $\rho \lesseqqgtr \gamma_i$ according as $\gamma_j \lesseqqgtr \gamma_i$. Therefore, by (12) we have for $m, n<\omega$

$$A^{(m)}_{\gamma_m} < A^{(n)}_{\gamma_n} \quad \text{iff} \quad \gamma_m < \gamma_n. \tag{18}$$

By (14) and (17) we have

$$A^{(n)}_{\gamma_n} \subset A^{(m)}_{\gamma_m} \quad \text{if} \quad m \leq n \quad \text{and} \quad \gamma_m = \gamma_n.$$

By (11), the set $\{n:m\leq n<\omega,\ \gamma_m=\gamma_n\}$ is infinite and therefore, by (15) and (16),

$$\mathrm{tp}(Z \cap A^{(m)}_{\gamma_m}) = \omega \qquad (m < \omega). \tag{19}$$

Since $\{\gamma_m:m<\omega\}=B$, it follows from (18) and (19) that the order type of Z is $\omega\beta$. This completes the proof of (iii) and the theorem follows.

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