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Abstract

We give explicit formulas for the Hilbert series of residual intersections of a scheme in terms of the Hilbert series of its conormal modules. In a previous paper, we proved that such formulas should exist. We give applications to the number of equations defining projective varieties and to the dimension of secant varieties of surfaces and three-folds.

Introduction

Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded module over a Noetherian standard graded algebra R over a field k. The *Hilbert series* (sometimes called the Hilbert–Poincaré series) of M, which we will denote by $[\![M]\!]$, is the Laurent series

$$[\![M]\!] = \sum (\dim_k M_i)\, t^i.$$

If $Z \subset \mathbb{P}^n := \mathbb{P}^n_k$ is a scheme, then the Hilbert series of Z is by definition the Hilbert series of the homogeneous coordinate ring of Z. Of course, this Hilbert series contains the data of the Hilbert polynomial of Z as well.

Sometimes interesting geometric information (such as the dimension of a secant variety) can be described in terms of *residual intersections* in the sense of Artin and Nagata [AN72], and the purpose of this paper is to compute the Hilbert series of such schemes. Here is the definition: let $X \subset Y \subset \mathbb{P}^n$ be closed subschemes of $\mathbb{P}^n$, let R be the homogeneous coordinate ring of Y, and let $I_X \subset R$ be the ideal of X in Y. A scheme $Z \subset Y$ is called an s-residual intersection of X in Y if Z is defined by a (not necessarily saturated) ideal of the form $(f_1, \dots, f_s) :_R I_X$, with $f_1, \dots, f_s$ homogeneous elements in I_X, and Z is of codimension at least s in Y.

We wish to derive formulas for the Hilbert series of Z in terms of information about X and the degrees of the polynomials f_i. In our previous paper [CEU01], we showed that this is sometimes possible in principle: under certain hypotheses the Hilbert series of Z does not vary if we change the polynomials f_i, keeping their degrees fixed. In the present paper we make this more precise by giving formulas, under somewhat stronger hypotheses, for the Hilbert series of Z in terms of the degrees of the f_i and the Hilbert series of finitely many modules of the form $\omega_R/I_X^j\omega_R$, where ω_R denotes the graded canonical module of R.

For example, suppose that $Y = \mathbb{P}^n$ and that X has codimension g and is locally a complete intersection (for instance, smooth). If $f_1, \dots, f_s$ are homogeneous elements of degree d of $I := I_X$ such that $\mathfrak{R} := (f_1, \dots, f_s) : I$ has codimension $\geqslant s$, then the Hilbert series of the homogeneous

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coordinate ring of the scheme Z defined by $\mathfrak{R}$ differs from that of a complete intersection defined by s forms of degree d by

$$\sum_{j=g}^{s}(-1)^{n+j}\binom{s}{j}t^{jd-n-1}[\![R/I^{j-g+1}]\!](t^{-1}) + \text{ a polynomial.}$$

The polynomial remainder term is present because we have made assumptions only on the scheme, and not on the homogeneous coordinate ring. Here the expression $[\![R/I^{j-g+1}]\!](t^{-1})$ denotes the Laurent series obtained by writing $[\![R/I^{j-g+1}]\!]$ as a rational function in t, substituting t^{-1} for t, and rewriting the result as a Laurent series. The formula above is a special case of Theorem 1.4(b), where the assumptions on Y and X are relaxed considerably and the forms $f_1, \dots, f_s$ are allowed to have different degrees.

In applications, one sometimes only needs to know whether the s-residual intersection Z has codimension exactly s in Y. For example, we will use such information to say something about the number of equations defining projective varieties, and to determine when the secant varieties of certain (possibly singular) surfaces and smooth three-folds have dimension less than the expected one. For this purpose it suffices to know just one coefficient of the Hilbert polynomial of Z, that corresponding to the degree of the codimension s component of Z. More generally, we show how to use partial information about X to compute the first few coefficients of the Hilbert polynomial of Z.

Consider the case where Y is equidimensional and locally Gorenstein and X has codimension g in Y. Further, suppose that locally in codimension $i < s$, the subscheme $X \subset Y$ can be defined by i equations and that, for every closed point p of X and every $j \leqslant s-g$,

$$\operatorname{depth} \mathcal{I}^j_{X,p}/\mathcal{I}^{j+1}_{X,p} \geqslant \dim X - j.$$

If Z is any s-residual intersection of X in Y, then the Hilbert polynomial of Z can be written in terms of the degrees of the f_i, the Hilbert polynomial of Y, and the Hilbert polynomials of $\omega_Y/\mathcal{I}^j_X\omega_Y$ for $j \leqslant s-g+1$ (the explicit formula is given in Theorem 1.9(b)). Moreover, if X satisfies our hypotheses only up to some codimension r in Y, then the formula gives the first $r-s+1$ coefficients of the Hilbert polynomial of Z.

Our formulas are derived in § 1, which is the technical heart of the paper. To prove them, we need to adapt the arguments of Ulrich [Ulr94]. The delicate point is the use in that paper of the fact that ω_R is free of rank one if R is Gorenstein. Since our rings are not necessarily Gorenstein, the canonical module ω_R must be brought into play and the modules $\omega_R/I^j_X\omega_R$ need to be considered. For other work along these lines, see Cumming [Cum07]. Further complications arise from the fact that we cannot assume R to be Cohen–Macaulay, in order to allow for applications to secant varieties, for instance.

In case Y is arithmetically Gorenstein, the sheaves $\mathcal{I}^j_X/\mathcal{I}^{j+1}_X$ themselves play the crucial role in our formulas. If X were locally a complete intersection scheme, then $\mathcal{I}_X/\mathcal{I}^2_X$ would be a vector bundle and $\mathcal{I}^j_X/\mathcal{I}^{j+1}_X$ would be its jth symmetric power, so it is reasonable to hope that for 'nice' ideals I the Hilbert series of the first few conormal modules should determine the rest. We prove a general theorem of this kind in § 2, and carry out the reduction in some particular cases. For instance, if $s = \operatorname{codim}_Y X$, then the degree of an s-residual intersection scheme Z in $Y = \mathbb{P}^n$ can be calculated immediately from Bézout's theorem: $\deg Z = (\prod_i \deg f_i) - \deg X$. This was extended to a formula in the case $s = \operatorname{codim}_Y X + 1$ by Stückrad [Stü92] and to the case $s = \operatorname{codim}_Y X + 2$ by Huneke and Martin [HM95]. Our formula gives an answer in general, and

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we work this out explicitly for the case $s = \operatorname{codim}_Y X + 3$. These types of results also lead to criteria for when the subscheme $X \subset \mathbb{P}^n$ can be defined by s equations.

In § 3, we apply our results to the study of secant loci. Our general theorems about smooth three-folds give conditions for the degeneracy of the secant locus in terms of Chern classes and in terms of certain Hilbert coefficients. We also recover the analogous criteria for surfaces with mild singularities, a case treated earlier with different methods by Dale [Dal85] and others.

1. Formulas for the Hilbert series of residual intersections

Without imposing global assumptions, one can only expect partial information about the Hilbert series and the Hilbert polynomial of a residual intersection. In this context, the following question arises: which kind of relationship between two modules guarantees that a certain number of their first Hilbert coefficients coincide? For the leading coefficient, the degree of the modules, it suffices to require that the two modules are isomorphic locally at every prime ideal of maximal dimension in their support, but for other coefficients a more stringent notion of equivalence is needed. To introduce such a notion for rings that are not necessarily equidimensional, we define the *true codimension* of a prime ideal $\mathfrak{p}$ in a ring R to be $\dim R - \dim R/\mathfrak{p}$.

Definition 1.1. Let R be a graded ring of finite Krull dimension, M and N finitely generated graded R-modules, and r an integer. We say that M and N are *equivalent up to true codimension* r, and write $M \underset{r}{\cong} N$, if there exist finitely generated graded R-modules $W_1, \ldots, W_n$ with $W_1 = M$, $W_n = N$ and homogeneous linear maps $W_i \to W_{i+1}$ or $W_{i+1} \to W_i$, for $1 \leqslant i \leqslant n-1$, that are isomorphisms locally at every prime ideal of true codimension $\leqslant r$. A homogeneous linear map that is an isomorphism locally up to true codimension r will be denoted by $\xrightarrow[r]{\sim}$.

Saying that $M \underset{r}{\cong} N$ is of course much stronger than saying that M and N are isomorphic locally at each prime ideal of true codimension $\leqslant r$. For example, any two modules M and N that represent line bundles on a projective variety of dimension r satisfy the latter condition, but $M \underset{r}{\cong} N$ implies that they represent isomorphic line bundles. The need to provide explicit maps that are isomorphisms locally in some true codimension between modules that are not in fact isomorphic is what makes the work in this section delicate.

If R is a Noetherian standard graded algebra over a field and M a finitely generated graded R-module, we will denote the Hilbert series of M by $[\![M]\!]$. Recall that $[\![M]\!]$ is an element of the ring $\mathbb{Z}[t, t^{-1}, (1-t)^{-1}] \subset \mathbb{Z}[\![t]\!][t^{-1}]$. If $M \underset{r}{\cong} N$, then $[\![M]\!] - [\![N]\!]$, considered as a rational function, has a pole of order less than $\dim R - r$ at 1; in this case we write $[\![M]\!] \underset{r}{\equiv} [\![N]\!]$ and say that the two series are *r-equivalent*. Thus, if $r = \dim R - 1 =: d$, then $[\![M]\!] \underset{r}{\equiv} [\![N]\!]$ means that the Hilbert polynomials of M and N agree, and in general if $r < \dim R$, then $[\![M]\!] \underset{r}{\equiv} [\![N]\!]$ means that the Hilbert polynomials of M and N, written in the form

$$a_d \binom{t+d}{d} + a_{d-1} \binom{t+d-1}{d-1} + \cdots,$$

have the same coefficients a_i for $i \geqslant d - r$.

We extend the notation $\underset{r}{\equiv}$ to arbitrary series in $\mathbb{Z}[t, t^{-1}, (1-t)^{-1}]$, by the same requirement, as soon as $\dim R$ is clear from the context.

In the sequel, we will often use the notation $[\![M]\!](t^{-1})$, which makes sense because the substitution $t \mapsto t^{-1}$ is a well-defined automorphism of the ring $\mathbb{Z}[t, t^{-1}, (1-t)^{-1}]$ since $(1 - t^{-1})^{-1} = -t(1-t)^{-1} \in \mathbb{Z}[t, t^{-1}, (1-t)^{-1}]$.

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LEMMA 1.2. *Let R be an equidimensional Noetherian standard graded algebra over a field, ω_R its graded canonical module, and M a finitely generated graded R-module. If for each prime ideal $\mathfrak{p} \in \operatorname{Supp}_R(M)$ with $\dim R_{\mathfrak{p}} \leqslant r$ the ring $R_{\mathfrak{p}}$ is Cohen–Macaulay and the module $M_{\mathfrak{p}}$ is Cohen–Macaulay of codimension i, then*

$$[\![\operatorname{Ext}^i_R(M, \omega_R)]\!](t) \underset{r}{\equiv} (-1)^{\dim M}[\![M]\!](t^{-1}).$$

Proof. We map a standard graded polynomial ring S homogeneously onto R and write $c := \operatorname{codim}_S R$. One has $\operatorname{Ext}^j_S(R, \omega_S)_{\mathfrak{p}} = 0$ for $j \neq c$ whenever $\mathfrak{p} \in \operatorname{Supp}_R(M)$ with $\dim R_{\mathfrak{p}} \leqslant r$, because $R_{\mathfrak{p}}$ is Cohen–Macaulay and R is equidimensional. Therefore,

$$\operatorname{Ext}^i_R(M, \omega_R) \simeq \operatorname{Ext}^i_R(M, \operatorname{Ext}^c_S(R, \omega_S)) \underset{r}{\cong} \operatorname{Ext}^{c+i}_S(M, \omega_S),$$

as can be seen from a spectral sequence argument or, more directly, by considering a homogeneous resolution of ω_S by graded-injective S-modules.

To compute the Hilbert series of the module $\operatorname{Ext}^{c+i}_S(M, \omega_S)$ up to r-equivalence, dualize a minimal homogeneous free S-resolution of M into ω_S and observe that by the Cohen–Macaulay assumption on M, all the cohomology modules other than $\operatorname{Ext}^{c+i}_S(M, \omega_S)$ are supported in codimension $>r$ in R. □

We next adapt some results from [CEU01] and [Ulr94] to our context.

LEMMA 1.3. *Let R be a Noetherian positively graded algebra over a factor ring of a local Gorenstein ring, with graded canonical module $\omega := \omega_R$. Assume that R is equidimensional. Let I be a homogeneous ideal of height g, let $f_1, \ldots, f_s$ be forms contained in I of degrees $d_1, \ldots, d_s$, and write $\mathfrak{A}_i := (f_1, \ldots, f_i)$ and $\mathfrak{R}_i := \mathfrak{A}_i : I$. Assume that $\operatorname{ht} \mathfrak{R}_i \geqslant i$ for $1 \leqslant i \leqslant s$ and $\operatorname{ht} I + \mathfrak{R}_i \geqslant i+1$ for $0 \leqslant i \leqslant s-1$. For a fixed integer $r \leqslant \dim R$ and every homogeneous prime ideal $\mathfrak{p}$ with $\dim R_{\mathfrak{p}} = r$, suppose the following.*

- *If $\mathfrak{p} \notin V(I)$, then the elements $f_1, \ldots, f_s$ form a weak regular sequence on $R_{\mathfrak{p}}$ and on $\omega_{\mathfrak{p}}$.*
- *If $\mathfrak{p} \in V(I)$, then $R_{\mathfrak{p}}$ is Gorenstein and* $\operatorname{depth} R_{\mathfrak{p}}/I^j_{\mathfrak{p}} \geqslant \dim R_{\mathfrak{p}}/I_{\mathfrak{p}} - j + 1$ *for $1 \leqslant j \leqslant s-g$.*

The following statements hold.

(a) $(R/\mathfrak{R}_{i-1})(-d_i) \underset{r}{\xrightarrow{\sim}} \mathfrak{A}_i/\mathfrak{A}_{i-1}$ *via multiplication by f_i for $1 \leqslant i \leqslant s$.*

(b) $0 \to (\omega I^j/\omega\mathfrak{A}_{i-1}I^{j-1})(-d_i) \xrightarrow{\cdot f_i} \omega I^{j+1}/\omega\mathfrak{A}_{i-1}I^j \longrightarrow \omega I^{j+1}/\omega\mathfrak{A}_i I^j \to 0$ *is a complex that is exact locally in codimension r in R for $1 \leqslant i \leqslant s$ and $\min\{1, i-g\} \leqslant j \leqslant s-g$.*

(c) $\operatorname{Ext}^i_R(R/\mathfrak{R}_i, \omega) \underset{r}{\cong} (\omega I^{i-g+1}/\omega\mathfrak{A}_i I^{i-g})(d_1+\cdots+d_i)$ *for $0 \leqslant i \leqslant s$, if also* $\operatorname{depth} R_{\mathfrak{p}}/I^{s-g+1}_{\mathfrak{p}} \geqslant \dim R_{\mathfrak{p}}/I_{\mathfrak{p}} - s + g$ *for every homogeneous $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} = r$.*

Before proving Lemma 1.3, we wish to discuss its hypotheses, since they will be used throughout this section. If $s \geqslant g$ and $\mathfrak{R} \neq R$, then the ideal $\mathfrak{R}$ is indeed an s-residual intersection. Quite generally, let I be an ideal of height g in a Noetherian ring and $s \geqslant g$ an integer; a proper ideal $\mathfrak{R}$ is called an *s-residual intersection* of I if $\mathfrak{R} = \mathfrak{A} : I$ for some s-generated ideal $\mathfrak{A} \subset I$ and $\operatorname{ht} \mathfrak{R} \geqslant s$.

If the integers $d_1, \ldots, d_s$ are at least as big as the largest generator degrees of I, then general forms $f_1, \ldots, f_s$ of degrees $d_1, \ldots, d_s$ in I satisfy the assumptions of the lemma on the heights of $\mathfrak{R}_i$ and $I + \mathfrak{R}_i$, provided the residue field of R is infinite and the ideal I satisfies G_s (see [Ulr94, 1.6(a)] or [HM95, 2.8]). The condition G_s means that $I_{\mathfrak{p}}$ can be generated by $\dim R_{\mathfrak{p}}$ elements for

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every $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} < s$; obviously, it suffices to check this condition for homogeneous prime ideals $\mathfrak{p}$. The assumption in the lemma on the heights of $I + \mathfrak{R}_i$ implies that I satisfies G_s.

A *weak regular sequence* on a module is defined like a regular sequence, except that the ideal generated by the elements of the sequence is allowed to act as the unit ideal on the module. A sequence of elements $f_1, \ldots, f_s$ in I that is a weak regular sequence on R and ω locally outside $V(I)$ is also called a *filter regular sequence* with respect to I on R and ω. Again, if $d_1, \ldots, d_s$ are at least as big as the largest generator degrees of I and the residue field of R is infinite, then general forms $f_1, \ldots, f_s$ are filter regular on R and ω with respect to I. Moreover, the assumptions of the lemma on the heights of the ideals $\mathfrak{R}_i$ already imply that $f_1, \ldots, f_s$ are a filter regular sequence on R and ω locally in codimension r in case R is Cohen–Macaulay locally in codimension r outside $V(I)$.

The depth requirements on the ideals $I^j_{\mathfrak{p}}$ in the second itemized assumption and in part (c) of the lemma are more subtle. Assuming that $R_{\mathfrak{p}}$ is Gorenstein, they are clearly satisfied whenever $I_{\mathfrak{p}}$ is a complete intersection. But they also hold if, more generally, $I_{\mathfrak{p}}$ satisfies G_s and is *strongly Cohen–Macaulay*, which means that the Koszul homology modules of a generating set of the ideal are Cohen–Macaulay [HSV83, the proof of 5.1]. The latter condition holds for any Cohen–Macaulay ideal of deviation 2 [AH80, p. 259]. It is also satisfied if the ideal is *licci*, meaning in the linkage class of a complete intersection [Hun82, 1.11]. Standard examples of licci ideals include perfect ideals of height two (see [Apé45] and [Gae52]) and perfect Gorenstein ideals of height three [Wat73].

Finally, we notice that the two itemized assumptions in the lemma and the hypothesis of part (c) pass to not necessarily homogeneous prime ideals $\mathfrak{p}$ with $\dim R_{\mathfrak{p}} \leqslant r$. For the first itemized assumption this is clear and for the other hypotheses it follows because the difference between dimension and depth cannot increase upon localization and remains constant when passing from a prime ideal $\mathfrak{p}$ to a suitable homogeneous prime ideal contained in $\mathfrak{p}$.

Proof of Lemma 1.3. Adjoining a variable to R, I, $\mathfrak{A}$, and localizing, we may suppose that $\operatorname{grade} I > 0$. Notice that the ideal I satisfies G_s. Write $R_i := R/\mathfrak{R}_i$. Whenever $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \leqslant r$, then $R_{i\mathfrak{p}}$ is Cohen–Macaulay for $0 \leqslant i \leqslant s-1$ and, in the setting of part (c), also for $i = s$; see [Ulr94, 2.9(a)]. In addition, for $0 \leqslant i \leqslant s$, the $R_{\mathfrak{p}}$-module $R_{i\mathfrak{p}}$ is zero or has codimension i by [Ulr94, 1.7(a)], whereas, for $1 \leqslant i \leqslant s$, the element f_i is a non-zerodivisor on $R_{i-1_{\mathfrak{p}}}$ by [Ulr94, 1.7(f)], $f_i R_{i-1_{\mathfrak{p}}} :_{R_{i-1_{\mathfrak{p}}}} I = \mathfrak{R}_i R_{i-1_{\mathfrak{p}}}$ by the same reference, and $\mathfrak{A}_{i-1_{\mathfrak{p}}} :_{R_{\mathfrak{p}}} f_i = \mathfrak{R}_{i-1_{\mathfrak{p}}}$ by [Ulr94, 1.7(g)].

Part (a) holds along $V(I)$ by the above equality from [Ulr94, 1.7(g)], and off $V(I)$ because there $f_1, \ldots, f_i$ form a weak regular sequence on R locally in codimension r. Moreover, the sequence of (b) is obviously a complex. That it is exact locally in codimension r holds for primes in $V(I)$ by [Ulr94, 2.7(a)] and for primes outside $V(I)$ because there $f_1, \ldots, f_i$ form a weak regular sequence on ω locally in codimension r.

For the proof of (c), we induct on i. For $i = 0$, our assertion is clear since $\operatorname{grade} I > 0$ and therefore $R/\mathfrak{R}_0 = R$. Assuming that the assertion holds for R_i for some i, $0 \leqslant i \leqslant s-1$, we are going to prove it for R_{i+1}. To this end, we may suppose that $r \geqslant i+1$.

We first wish to prove that

$$\operatorname{Ext}^{i+1}_R(R_i/(f_{i+1}R_i :_{R_i} I), \omega) \underset{r}{\cong} [I \operatorname{Ext}^i_R(R_i, \omega)/f_{i+1}\operatorname{Ext}^i_R(R_i, \omega)](d_{i+1}). \tag{1}$$

Using the exact sequence

$$0 \longrightarrow R_i/(0 :_{R_i} f_{i+1})(-d_{i+1}) \xrightarrow{\cdot f_{i+1}} R_i \longrightarrow R_i/f_{i+1}R_i \longrightarrow 0,$$

we obtain a long exact sequence

$$\cdots \operatorname{Ext}^i_R(R_i,\omega) \xrightarrow{\cdot f_{i+1}} \operatorname{Ext}^i_R(R_i/(0:_{R_i} f_{i+1}),\omega)(d_{i+1}) \to \operatorname{Ext}^{i+1}_R(R_i/f_{i+1}R_i,\omega) \to \operatorname{Ext}^{i+1}_R(R_i,\omega)\cdots.$$

The support in R of $0:_{R_i} f_{i+1}$ has codimension $\geqslant r+1>i$; this holds along $V(I)$ by the explanation at the beginning of this proof [Ulr94, 1.7(f)] and outside $V(I)$ because there $f_1,\ldots,f_{i+1}$ form a weak regular sequence on R locally in codimension r. Thus, $\operatorname{Ext}^i_R(R_i/(0:_{R_i} f_{i+1}),\omega) \underset{r}{\xrightarrow{\sim}} \operatorname{Ext}^i_R(R_i,\omega)$ via the natural map. Furthermore, $\operatorname{Ext}^{i+1}_R(R_i,\omega) \underset{r}{\cong} 0$; indeed, locally up to codimension r in R along $V(I)$, R is Cohen–Macaulay and R_i is zero or Cohen–Macaulay of codimension i, whereas locally up to codimension r off $V(I)$, R_i is defined by the weak regular sequence $f_1,\ldots,f_i$ and hence has projective dimension at most i. We conclude that

$$[\operatorname{Ext}^i_R(R_i,\omega)/f_{i+1}\operatorname{Ext}^i_R(R_i,\omega)](d_{i+1}) \underset{r}{\cong} \operatorname{Ext}^{i+1}_R(R_i/f_{i+1}R_i,\omega).$$

Therefore, to prove (1), it suffices to show that

$$I\ \operatorname{Ext}^{i+1}_R(R_i/f_{i+1}R_i,\omega) \underset{r}{\cong} \operatorname{Ext}^{i+1}_R(R_i/(f_{i+1}R_i:_{R_i} I),\omega).$$

The natural projection $R_i/f_{i+1}R_i \longrightarrow R_i/(f_{i+1}R_i:_{R_i} I)$ induces a map

$$\phi:\ \operatorname{Ext}^{i+1}_R(R_i/(f_{i+1}R_i:_{R_i} I),\omega) \longrightarrow E := \operatorname{Ext}^{i+1}_R(R_i/f_{i+1}R_i,\omega).$$

We prove that locally in codimension r in R, the map ϕ is injective, and its image coincides with IE, which gives

$$\operatorname{im}\phi \underset{r}{\xrightarrow{\sim}} \operatorname{im}\phi + IE \underset{r}{\xleftarrow{\sim}} IE.$$

This is trivial locally off $V(I)$ because on this locus $I=R$ and $f_{i+1}R_i:_{R_i} I = f_{i+1}R_i$. Therefore, we may localize to assume, temporarily, that R is a local ring of dimension at most r and $I\neq R$. Of course, we may suppose that $R_i\neq 0$. In this case R is Cohen–Macaulay, R_i is Cohen–Macaulay of codimension i, and f_{i+1} is regular on R_i, as explained at the beginning of this proof. Write $S := R_i/f_{i+1}R_i$.

The natural equivalence of functors $\operatorname{Ext}^{i+1}_R(—,\omega) \simeq \operatorname{Hom}_S(—,\omega_S)$ together with the exact sequence

$$0 \longrightarrow 0:_S I \longrightarrow S \longrightarrow S/(0:_S I) \longrightarrow 0$$

yields a commutative diagram with an exact row

$$\begin{tikzcd}
 & \operatorname{Ext}^{i+1}_R(R_i/(f_{i+1}R_i:_{R_i} I),\omega) \arrow[r, "\phi"] \arrow[d, "\simeq"] & E \arrow[d, "\simeq"] & & \\
0 \arrow[r] & \operatorname{Hom}_S(S/(0:_S I),\omega_S) \arrow[r, "\psi"] & \omega_S \arrow[r] & \operatorname{Hom}_S(0:_S I,\omega_S) \arrow[r] & 0
\end{tikzcd}$$

The last map is surjective because, as explained at the beginning of this proof, $S/(0:_S I) = R_{i+1}$ [Ulr94, 1.7(f)] and R_{i+1} is zero or a maximal Cohen–Macaulay S-module. From the diagram, we see that ϕ is injective and that the desired equality $\operatorname{im}\phi = I\,E$ follows once we have shown that $\operatorname{im}\psi = I\omega_S$. For this, it suffices to prove that

$$\operatorname{coker}\,\psi \simeq \omega_S/I\omega_S; \tag{2}$$

indeed, (2) implies that $I\omega_S \subset \operatorname{im}\psi$ and therefore gives the natural epimorphism of isomorphic modules $\omega_S/I\omega_S \to \operatorname{coker}\,\psi$, which is necessarily an isomorphism.

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We first argue that $\omega_S/I\omega_S$ is a maximal Cohen–Macaulay S-module. From [Ulr94, 2.7(c)], we obtain $\mathfrak{R}_i \cap I^{i-g+2} = \mathfrak{A}_i I^{i-g+1}$, which implies that

$$\mathfrak{A}_i I^{i-g} \cap I^{i-g+2} = \mathfrak{A}_i I^{i-g+1}. \tag{3}$$

Hence, by our induction hypothesis,

$$I\omega_{R_i} \simeq I^{i-g+2}/(\mathfrak{A}_i I^{i-g} \cap I^{i-g+2}) = I^{i-g+2}/\mathfrak{A}_i I^{i-g+1}.$$

But the latter is a maximal Cohen–Macaulay R_i-module according to [Ulr94, 2.7(b)]; hence, $\omega_S/I\omega_S \simeq \omega_{R_i}/I\omega_{R_i}$ is indeed a maximal Cohen–Macaulay S-module.

Therefore,

$$\omega_S/I\omega_S \simeq \operatorname{Hom}_S(\operatorname{Hom}_S(\omega_S/I\omega_S, \omega_S), \omega_S) \simeq \operatorname{Hom}_S(\operatorname{Hom}_S(S/IS, S), \omega_S) \simeq \operatorname{Hom}_S(0 :_S I, \omega_S).$$

Since the last module is isomorphic to $\operatorname{coker}\psi$, this completes the proof of (2) and hence of (1).

Now

$$I \operatorname{Ext}^i_R(R_i, \omega)/f_{i+1}\operatorname{Ext}^i_R(R_i, \omega) \underset{r}{\cong} [\omega I^{i-g+2}/((\omega\mathfrak{A}_i I^{i-g} \cap \omega I^{i-g+2}) + \omega f_{i+1} I^{i-g+1})](d_1 + \cdots + d_i)$$

by our induction hypothesis and, using (3), one sees that

$$I \operatorname{Ext}^i_R(R_i, \omega)/f_{i+1}\operatorname{Ext}^i_R(R_i, \omega) \underset{r}{\cong} (\omega I^{i-g+2}/\omega\mathfrak{A}_{i+1} I^{i-g+1})(d_1 + \cdots + d_i). \tag{4}$$

On the other hand, $R_{i+1} \xrightarrow[r]{\sim} R_i/(f_{i+1}R_i :_{R_i} I)$ as explained at the beginning of this proof [Ulr94, 1.7(f)] and hence

$$\operatorname{Ext}^{i+1}_R(R_{i+1}, \omega) \underset{r}{\cong} \operatorname{Ext}^{i+1}_R(R_i/(f_{i+1}R_i :_{R_i} I), \omega). \tag{5}$$

Now combining (5), (1), and (4) concludes the proof of part (c). □

We are now ready to prove our main result about Hilbert series of residual intersections. By $\sigma_m(t_1, \ldots, t_s)$ we denote the mth elementary symmetric function in $t_1, \ldots, t_s$.

Theorem 1.4. *Let R be an equidimensional Noetherian standard graded algebra over a field. Write $n := \dim R$ and $\omega := \omega_R$, and let I be a homogeneous ideal of height g satisfying G_s for some $s \geqslant g$. Let $f_1, \ldots, f_s$ be forms contained in I of degrees $d_1, \ldots, d_s$, write $\Delta_i := \prod_{k=1}^{i}(1-t^{d_k})$, $\mathfrak{A}_i := (f_1, \ldots, f_i)$, $\mathfrak{R}_i := \mathfrak{A}_i : I$ and $\mathfrak{R} := \mathfrak{R}_s$, and assume that $\operatorname{ht}\mathfrak{R} \geqslant s$. For a fixed integer r with $s \leqslant r \leqslant n$ and every homogeneous $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} = r$, suppose the following.*

- *If $\mathfrak{p} \notin V(I)$, then the elements $f_1, \ldots, f_s$ form a weak regular sequence on $R_{\mathfrak{p}}$ and on $\omega_{\mathfrak{p}}$.*
- *If $\mathfrak{p} \in V(I)$, then $R_{\mathfrak{p}}$ is Gorenstein and $\operatorname{depth} R_{\mathfrak{p}}/I^j_{\mathfrak{p}} \geqslant \dim R_{\mathfrak{p}}/I_{\mathfrak{p}} - j + 1$ for $1 \leqslant j \leqslant s - g$.*

The following statements hold.

(a) *If $R_{\mathfrak{p}}$ is Cohen–Macaulay for every homogeneous $\mathfrak{p} \in V(\mathfrak{R}_g)$ with $\dim R_{\mathfrak{p}} = r$, then*

$$[\![R/\mathfrak{A}]\!](t) \underset{r}{\equiv} \Delta_s[\![R]\!](t) - (-1)^{n-g}\sum_{j=1}^{s-g}(-1)^j\sigma_{g+j}(t^{d_1}, \ldots, t^{d_s})[\![\omega/I^j\omega]\!](t^{-1}).$$

(b) *If $\operatorname{depth} R_{\mathfrak{p}}/I^{s-g+1}_{\mathfrak{p}} \geqslant \dim R_{\mathfrak{p}}/I_{\mathfrak{p}} - s + g$ for every homogeneous $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} = r$ and $R_{\mathfrak{p}}$ is Cohen–Macaulay for every homogeneous $\mathfrak{p} \in V(\mathfrak{R})$ with $\dim R_{\mathfrak{p}} = r$, then*

$$[\![R/\mathfrak{R}]\!](t) \underset{r}{\equiv} \Delta_s[\![R]\!](t) - (-1)^{n-g}\sum_{j=1}^{s-g+1}(-1)^{j-1}\sigma_{g+j-1}(t^{d_1}, \ldots, t^{d_s})[\![\omega/I^j\omega]\!](t^{-1}).$$

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Proof. For every homogeneous $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} = r$, since $R_{\mathfrak{p}}$ is already Cohen–Macaulay if $\mathfrak{p} \in V(I)$, it follows that $R_{\mathfrak{p}}$ is Cohen–Macaulay if $\mathfrak{p} \in V(\mathfrak{A}_g)$ in part (a) and if $\mathfrak{p} \in V(\mathfrak{A})$ in part (b), where $\mathfrak{A} := \mathfrak{A}_s$. Moreover, in part (b), the ring $R_{\mathfrak{p}}$ is Cohen–Macaulay for every homogeneous $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} = r-s$; this is implied by the Cohen–Macaulay assumption along $V(\mathfrak{A})$ because R is equidimensional and $\mathfrak{A}$ is generated by s forms. As before, these properties extend to not necessarily homogeneous prime ideals of possibly smaller height.

We may assume that the ground field is infinite. We reorder the generators of $\mathfrak{A}$ so that $d_1 \geqslant \cdots \geqslant d_s$. By the assumption in part (a), there exist g forms $f_{i_1}, \ldots, f_{i_g}$ in $\mathfrak{A}$ of degrees $d_{i_1}, \ldots, d_{i_g}$ so that $V((f_{i_1}, \ldots, f_{i_g}))$ intersects the non-Cohen–Macaulay locus of R in codimension at least $r+1$. Multiplication with general forms in R produces g elements in $\mathfrak{A}$ of degrees $d_1, \ldots, d_g$ that have the same property, and therefore g general forms in $\mathfrak{A}$ of degrees $d_1, \ldots, d_g$ have this property. Thus, we do not change the assumption on $V(\mathfrak{A}_g)$ in part (a) if we replace $f_1, \ldots, f_s$ by general forms in $\mathfrak{A}$ of degrees $d_1 \geqslant \cdots \geqslant d_s$. Such forms are still a filter regular sequence with respect to I on R and ω locally in codimension r. But they have the additional property that $\operatorname{ht} \mathfrak{R}_i \geqslant i$ and $\operatorname{ht} I + \mathfrak{R}_i \geqslant i+1$ for $0 \leqslant i \leqslant s-1$ (see [Ulr94, 1.6(a)], [HM95, 2.8] or [CEU01, 2.5(a) and (a′)]).

We first prove (b). Using Lemma 1.3(b) and induction on i, we see that

$$[\![\omega \mathfrak{A}_i I^j]\!](t) \underset{r}{\equiv} \sum_{\ell=1}^{i} (-1)^{\ell+1} \sigma_\ell(t^{d_1}, \ldots, t^{d_i}) [\![\omega I^{j-\ell+1}]\!](t)$$

for $0 \leqslant i \leqslant s$ and $i-g \leqslant j \leqslant s-g$. Now Lemma 1.3(c) gives

$$\begin{aligned} [\![\operatorname{Ext}_R^s(R/\mathfrak{R}, \omega)]\!](t) &\underset{r}{\equiv} t^{-(d_1+\cdots+d_s)}([\![\omega I^{s-g+1}]\!] - [\![\omega \mathfrak{A} I^{s-g}]\!])(t) \\ &\underset{r}{\equiv} t^{-(d_1+\cdots+d_s)} \sum_{\ell=0}^{s} (-1)^\ell \sigma_\ell(t^{d_1}, \ldots, t^{d_s}) [\![\omega I^{s-g-\ell+1}]\!](t). \end{aligned} \tag{1}$$

For every $\mathfrak{p} \in V(\mathfrak{R})$ with $\dim R_{\mathfrak{p}} \leqslant r$, the ring $R_{\mathfrak{p}}$ is Cohen–Macaulay and the module $R_{\mathfrak{p}}/\mathfrak{R}_{\mathfrak{p}}$ is Cohen–Macaulay of codimension s according to [Ulr94, 2.9 and 1.7(a)]. Hence, Lemma 1.2 shows that

$$[\![\operatorname{Ext}_R^s(R/\mathfrak{R}, \omega)]\!](t) \underset{r}{\equiv} (-1)^{n-s} [\![R/\mathfrak{R}]\!](t^{-1}).$$

Now Formula (1) yields

$$\begin{aligned} [\![R/\mathfrak{R}]\!](t) &\underset{r}{\equiv} (-1)^{n-s} [\![\operatorname{Ext}_R^s(R/\mathfrak{R}, \omega)]\!](t^{-1}) \\ &\underset{r}{\equiv} (-1)^{n-s} t^{d_1+\cdots+d_s} \sum_{\ell=0}^{s} (-1)^\ell \sigma_\ell(t^{-d_1}, \ldots, t^{-d_s}) [\![\omega I^{s-g-\ell+1}]\!](t^{-1}) \\ &= (-1)^{n-s} \sum_{\ell=0}^{s} (-1)^\ell \sigma_{s-\ell}(t^{d_1}, \ldots, t^{d_s}) [\![\omega I^{s-g-\ell+1}]\!](t^{-1}). \end{aligned}$$

Changing the index of summation, one obtains

$$[\![R/\mathfrak{R}]\!](t) \underset{r}{\equiv} (-1)^{n-g} \sum_{j=-g+1}^{s-g+1} (-1)^{j-1} \sigma_{g+j-1}(t^{d_1}, \ldots, t^{d_s}) [\![\omega I^j]\!](t^{-1}). \tag{2}$$

Since R is Cohen–Macaulay locally in codimension $r-s$, Lemma 1.2 gives $[\![R]\!](t) \underset{r-s}{\equiv} (-1)^n [\![\omega]\!](t^{-1})$. Therefore,

$$\Delta_s [\![R]\!](t) \underset{r}{\equiv} \Delta_s (-1)^n [\![\omega]\!](t^{-1})$$

because Δ_s is divisible by $(1-t)^s$. We obtain

$$\begin{aligned}\Delta_s[\![R]\!](t) \underset{r}{\equiv} \Delta_s(-1)^n[\![\omega]\!](t^{-1}) &= (-1)^{n-s}\sum_{\ell=0}^{s}(-1)^{\ell}\sigma_{s-\ell}(t^{d_1},\ldots,t^{d_s})[\![\omega]\!](t^{-1})\\ &= (-1)^{n-g}\sum_{j=-g+1}^{s-g+1}(-1)^{j-1}\sigma_{g+j-1}(t^{d_1},\ldots,t^{d_s})[\![\omega]\!](t^{-1}). \qquad (3)\end{aligned}$$

Combining (2) and (3) concludes the proof of part (b).

To prove part (a), notice that $[\![R/\mathfrak{A}_g]\!](t) \underset{r}{\equiv} \Delta_g[\![R]\!](t)$ because $f_1,\ldots,f_g$ are an R-regular sequence along $V(\mathfrak{A}_g)$ locally in codimension r. Furthermore, by Lemma 1.3(a),

$$[\![R/\mathfrak{A}_i]\!](t) = [\![R/\mathfrak{A}_{i-1}]\!](t) - t^{d_i}[\![R/\mathfrak{R}_{i-1}]\!](t)$$

for $g+1 \leqslant i \leqslant s$. Since $R_{\mathfrak{p}}$ is Cohen–Macaulay for every homogeneous $\mathfrak{p} \in V(\mathfrak{R}_{i-1})$ with $\dim R_{\mathfrak{p}} = r$, we may apply part (b) with $i-1$ in place of s to express $[\![R/\mathfrak{R}_{i-1}]\!](t)$. Now induction on i, with $g \leqslant i \leqslant s$, yields

$$[\![R/\mathfrak{A}_i]\!](t) \underset{r}{\equiv} \Delta_i[\![R]\!](t) - (-1)^{n-g}\sum_{j=1}^{i-g}(-1)^{j}\sigma_{g+j}(t^{d_1},\ldots,t^{d_i})[\![\omega/I^j\omega]\!](t^{-1}). \qquad \square$$

Remark 1.5. If in Theorem 1.4, R is a polynomial ring in n variables, then the formulas of that theorem take the following form:

$$[\![R/\mathfrak{A}]\!](t) \underset{r}{\equiv} \Delta_s[\![R]\!](t) - (-t)^{-n}\sum_{j=1}^{s-g}(-1)^{g+j}\sigma_{g+j}(t^{d_1},\ldots,t^{d_s})[\![R/I^j]\!](t^{-1});$$

$$[\![R/\mathfrak{R}]\!](t) \underset{r}{\equiv} \Delta_s[\![R]\!](t) - (-t)^{-n}\sum_{j=1}^{s-g+1}(-1)^{g+j-1}\sigma_{g+j-1}(t^{d_1},\ldots,t^{d_s})[\![R/I^j]\!](t^{-1}).$$

The next goal is to turn our formulas for Hilbert series of residual intersections into information about Hilbert polynomials or some coefficients thereof. For this, several observations of a numerical nature are needed.

Lemma 1.6. *Write $\Delta_s = (1-t)^s\sum_{k\geqslant 0}c_k(d_1,\ldots,d_s)(1-t)^k$. Then*

$$c_k(d_1,\ldots,d_s) = (-1)^k\sum_{\substack{i_1\geqslant 1,\ldots,i_s\geqslant 1\\ i_1+\cdots+i_s=k+s}}\prod_{j=1}^{s}\binom{d_j}{i_j}.$$

Proof. Write $P_j(t) := \sum_{\ell=0}^{d_j-1}t^{\ell}$ and notice that $\Delta_s = (1-t)^s\prod_{j=1}^{s}P_j(t)$. Taking derivatives, one obtains $P_j^{(m)}(1) = \sum_{\ell=0}^{d_j-1}m!\binom{\ell}{m} = m!\binom{d_j}{m+1}$. Hence,

$$\begin{aligned}\left(\prod_{j=1}^{s}P_j\right)^{(k)}(1) &= \sum_{\substack{m_1\geqslant 0,\ldots,m_s\geqslant 0\\ m_1+\cdots+m_s=k}}\frac{k!}{m_1!\cdots m_s!}\prod_{j=1}^{s}P_j^{(m_j)}(1)\\ &= k!\sum_{\substack{m_1\geqslant 0,\ldots,m_s\geqslant 0\\ m_1+\cdots+m_s=k}}\prod_{j=1}^{s}\binom{d_j}{m_j+1}.\end{aligned}$$

This yields the asserted formula because $c_k(d_1,\ldots,d_s) = ((-1)^k/k!)(\prod_{j=1}^{s}P_j)^{(k)}(1)$. $\square$

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Lemma 1.7. *Let P be a numerical polynomial written in the form $P(t)=\sum_{i=0}^{m}(-1)^i e_i\binom{t+m-i}{m-i}$. For an integer d, define the polynomial $Q(t):=P(-t+d)$ and write $Q(t)=\sum_{i=0}^{m}(-1)^i h_i\binom{t+m-i}{m-i}$. Then*

$$h_i=(-1)^m\sum_{k=0}^{i}(-1)^k\binom{d+m+1-k}{i-k}e_k.$$

Proof. We first notice that for integers r and $n\geqslant 0$, one has the following identities of numerical polynomials:

$$\binom{-t+n}{n}=(-1)^n\binom{t-1}{n}, \tag{1}$$

$$\binom{t+r+n}{n}=\sum_{\ell=0}^{n}\binom{r-1+\ell}{\ell}\binom{t+n-\ell}{n-\ell}, \tag{2}$$

where the first equality is obvious and the second one can be easily proved by induction on n.

Now

$$\begin{aligned}
Q(t)&=\sum_{k=0}^{m}(-1)^k e_k\binom{-t+d+m-k}{m-k}\\
&=(-1)^m\sum_{k=0}^{m}e_k\binom{t-d-1}{m-k}\quad \text{by (1)}\\
&=(-1)^m\sum_{k=0}^{m}e_k\binom{t+(-d-1-m+k)+(m-k)}{m-k}\\
&=(-1)^m\sum_{k=0}^{m}e_k\sum_{\ell=0}^{m-k}\binom{-d-1-m+k-1+\ell}{\ell}\binom{t+m-k-\ell}{m-k-\ell}\quad \text{by (2)}\\
&=(-1)^m\sum_{i=0}^{m}\left(\sum_{k=0}^{i}\binom{-d-m-2+i}{i-k}e_k\right)\binom{t+m-i}{m-i},
\end{aligned}$$

where $\binom{-d-m-2+i}{i-k}=(-1)^{i+k}\binom{d+m+1-k}{i-k}$ by (1). $\square$

The Hilbert series $[\![M]\!]$ of a finitely generated graded module M over a Noetherian standard graded algebra over a field is an element of the ring $\mathbb{Z}[t,t^{-1},(1-t)^{-1}]\subset\mathbb{Z}[\![t]\!][t^{-1}]$. In general, any $S(t)\in\mathbb{Z}[t,t^{-1},(1-t)^{-1}]$ can be written in the form

$$S(t)=\sum_{i=0}^{D-1}(-1)^i e_i\frac{1}{(1-t)^{D-i}}+F,$$

with $D\in\mathbb{Z}$, $e_i\in\mathbb{Z}$, $F\in\mathbb{Z}[t,t^{-1}]$, and this expression is unique once D is fixed. The coefficients e_i can be computed as $e_i(M)=(1/i!)(d^iQ/dt^i)(1)$, where $Q(t):=S(t)(1-t)^D$. We call

$$P(t)=\sum_{i=0}^{D-1}(-1)^i e_i\binom{t+D-1-i}{D-1-i}\in\mathbb{Q}[t]$$

the *polynomial associated to* $S(t)$. Its significance is that if we write $S(t)=\sum_{n\in\mathbb{Z}}c_nt^n$, then $c_n=P(n)$ for $n\gg 0$.

Remark 1.8. Let $S(t) \in \mathbb{Z}[t, t^{-1}, (1-t)^{-1}]$ and let $P(t)$ be the polynomial associated to $S(t)$. If d is any integer, then $-P(-t+d)$ is the polynomial associated to $t^d S(t^{-1})$.

Proof. One uses Lemma 1.7. □

In the case where $S(t) = [\![M]\!](t)$, we can take D to be any integer $\geqslant \dim M$, and we define $e_i^D(M) := e_i$. If $D = \dim M$, we simply write $e_i(M) := e_i^D(M)$. The coefficient $e_0(M)$ gives the multiplicity (or degree) of M provided that $\dim M > 0$, whereas, for a zero-dimensional module, $\operatorname{length}(M) = e_0(M[x])$ with x a new variable of degree one. The polynomial associated to $[\![M]\!](t)$ is the *Hilbert polynomial* of M, which we denote by $[M](t)$. Here is our main result about the coefficients of the Hilbert polynomial of a residual intersection.

THEOREM 1.9. *For any $\ell \in \mathbb{Z}$, write $e_\ell(d_1, \dots, d_s) := \sum_{\substack{i_1 \geqslant 1, \dots, i_s \geqslant 1 \\ i_1 + \cdots + i_s = \ell + s}} \prod_{j=1}^s \binom{d_j}{i_j}$.*

(a) *If the assumptions of Theorem 1.4(a) are satisfied, then, for $0 \leqslant i \leqslant \min\{r-s, n-s-1\}$,*

$$e_i^{n-s}(I/\mathfrak{A}) = \sum_{k=0}^{i} e_{i-k}(d_1, \dots, d_s)\, e_k(R) - (-1)^{s-g} e_{i+s-g}(R/I)$$
$$- (-1)^{s-g} \sum_{j=1}^{s-g} \sum_{k=0}^{i+s-g} (-1)^{j+k} \sum_{1 \leqslant i_1 < \cdots < i_{g+j} \leqslant s} \binom{d_{i_1} + \cdots + d_{i_{g+j}} + n - g - k}{i + s - g - k} e_k(\omega/I^j\omega).$$

(b) *If the assumptions of Theorem 1.4(b) are satisfied, then, for $0 \leqslant i \leqslant \min\{r-s, n-s-1\}$,*

$$e_i^{n-s}(R/\mathfrak{R}) = \sum_{k=0}^{i} e_{i-k}(d_1, \dots, d_s)\, e_k(R)$$
$$+ (-1)^{s-g} \sum_{j=1}^{s-g+1} \sum_{k=0}^{i+s-g} (-1)^{j+k} \sum_{1 \leqslant i_1 < \cdots < i_{g+j-1} \leqslant s} \binom{d_{i_1} + \cdots + d_{i_{g+j-1}} + n - g - k}{i + s - g - k} e_k(\omega/I^j\omega).$$

Proof. We only prove part (a). We write the numerical polynomial

$$(-1)^{n-g} \sum_{j=1}^{s-g} (-1)^j \sum_{1 \leqslant i_1 < \cdots < i_{g+j} \leqslant s} -[\omega/I^j\omega](-t + d_{i_1} + \cdots + d_{i_{g+j}})$$

in the form $\sum_{\ell=0}^{n-g-1} (-1)^\ell h_\ell \binom{t+n-g-1-\ell}{n-g-1-\ell}$. Lemma 1.7 gives

$$h_{i+s-g} = \sum_{j=1}^{s-g} (-1)^j \sum_{1 \leqslant i_1 < \cdots < i_{g+j} \leqslant s} \sum_{k=0}^{i+s-g} \binom{d_{i_1} + \cdots + d_{i_{g+j}} + n - g - k}{i + s - g - k} (-1)^k e_k(\omega/I^j\omega).$$

Now our assertion follows from Theorem 1.4(a) together with Lemma 1.6 and Remark 1.8.

One proves part (b) is a similar way, using Theorem 1.4(b) in place of Theorem 1.4(a). □

Remark 1.10. If in Theorem 1.9, R is a polynomial ring in n variables, then the formulas in that theorem take the following form:

$$e_i^{n-s}(I/\mathfrak{A}) = e_i(d_1, \dots, d_s) - (-1)^{s-g} e_{i+s-g}(R/I)$$
$$- (-1)^{s-g} \sum_{j=1}^{s-g} \sum_{k=0}^{i+s-g} (-1)^{j+k} \sum_{1 \leqslant i_1 < \cdots < i_{g+j} \leqslant s} \binom{d_{i_1} + \cdots + d_{i_{g+j}} - g - k}{i + s - g - k} e_k(R/I^j)$$

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for $0 \leqslant i \leqslant \min\{r-s, n-s-1\}$;

$$e_i^{n-s}(R/\mathfrak{R}) = e_i(d_1, \dots, d_s) + (-1)^{s-g} \sum_{j=1}^{s-g+1} \sum_{k=0}^{i+s-g} (-1)^{j+k} \sum_{1 \leqslant i_1 < \cdots < i_{g+j-1} \leqslant s} \binom{d_{i_1} + \cdots + d_{i_{g+j-1}} - g - k}{i+s-g-k} e_k(R/I^j)$$

for $0 \leqslant i \leqslant \min\{r-s, n-s-1\}$.

Proof. Notice that $\omega/I^j\omega \simeq (R/I^j)(-n)$. Now one proceeds as in the proof of Theorem 1.9. Alternatively, one can rewrite the formulas of Theorem 1.9 by successively using the two identities $e_k((R/I^j)[-\ell-1]) = e_k((R/I^j)[-\ell]) + e_{k-1}((R/I^j)[-\ell])$ and

$$\sum_{k=0}^{m} (-1)^k \binom{N+1}{m-k} (e_k + e_{k-1}) = \sum_{k=0}^{m} (-1)^k \binom{N}{m-k} e_k,$$

where e_k are any integers with $e_{-1} = 0$. □

Our first application of Theorem 1.9 deals with the height of residual intersections.

Corollary 1.11. *Let R be an equidimensional Noetherian standard graded algebra over a field, with graded canonical module $\omega := \omega_R$, and assume that R is Gorenstein locally in codimension r for some $r \leqslant n =: \dim R$. Let I be a homogeneous ideal of height g satisfying G_s for some s with $g \leqslant s \leqslant \min\{r, n-1\}$. Suppose that $\operatorname{Ext}_R^{g+j}(R/I^j, R)$ has codimension $\geqslant r+1$ in R for $1 \leqslant j \leqslant s-g$, and that $\operatorname{depth} R_\mathfrak{p}/I_\mathfrak{p}^j \geqslant \dim R_\mathfrak{p}/I_\mathfrak{p} - j$ for $1 \leqslant j \leqslant s-g-1$ and for every homogeneous $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} = s$. Let $f_1, \dots, f_s$ be forms contained in I of degrees $d_1, \dots, d_s$ and write $\mathfrak{R} = (f_1, \dots, f_s) : I$.*

Then $\operatorname{ht} \mathfrak{R} \geqslant r+1$ if and only if $\operatorname{ht} \mathfrak{R} \geqslant s$ and

$$(-1)^{s-g} e_0(R) \prod_{j=1}^{s} d_j = e_{s-g}(R/I) + \sum_{j=1}^{s-g} \sum_{k=0}^{s-g} (-1)^{j+k} \sum_{1 \leqslant i_1 < \cdots < i_{g+j} \leqslant s} \binom{d_{i_1} + \cdots + d_{i_{g+j}} + n - g - k}{s-g-k} e_k(\omega/I^j\omega).$$

Proof. Assume that $\operatorname{ht} \mathfrak{R} \geqslant s$. Our hypothesis on the Ext-modules implies that the minimal primes of I have height g or $\geqslant s+1$. Hence, for every homogeneous $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} = s$, one has $\operatorname{ht} I_\mathfrak{p} = g$. Therefore, local duality, our assumption on the Ext-modules, and the hypothesis about local depths imply the stronger inequalities $\operatorname{depth} R_\mathfrak{p}/I_\mathfrak{p}^j \geqslant \dim R_\mathfrak{p}/I_\mathfrak{p} - j + 1$ in the range $1 \leqslant j \leqslant s-g$. It follows that the hypotheses of Theorem 1.9(a) are satisfied locally in codimension s. The theorem shows that the above equality involving Hilbert coefficients is equivalent to $\operatorname{ht} \mathfrak{R} \geqslant s+1$. On the other hand, our assumptions, most notably the condition on the Ext-modules, guarantee that locally in codimension r the ideal I is '$(s-1)$-residually S_2', which in turn implies that it cannot have a proper s-residual intersection of height $\geqslant s+1$; see [CEU01, 4.2 and 3.4(a)]. Therefore, $\operatorname{ht} \mathfrak{R} \geqslant r+1$. □

An immediate consequence of the corollary above is a criterion for when a projective variety of codimension g can be defined by s equations of given degrees. Earlier applications along these lines can be found in [Stü92, Theorem 1] and [HM95, 4.20], where the cases $s = g+1$ and $s = g+2$ were treated. (The authors of the second paper agreed with us that an assumption on R/I^2 should be added in their result.)

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COROLLARY 1.12. *Let $X \subset \mathbb{P}^N_k$ be a subscheme of codimension g. Write R for the polynomial ring in $N+1$ variables over k and $I \subset R$ for the saturated ideal defining X. Let s be an integer with $g \leqslant s \leqslant N$. Assume that I satisfies G_s, that the modules $\operatorname{Ext}_R^{g+j}(R/I^j, R)$ have finite length for $1 \leqslant j \leqslant s-g$, and that $\operatorname{depth} R_{\mathfrak{p}}/I^j_{\mathfrak{p}} \geqslant \dim R_{\mathfrak{p}}/I_{\mathfrak{p}} - j$ for $1 \leqslant j \leqslant s-g-1$ and for every homogeneous $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} = s$.*

Then X can be defined scheme-theoretically by s forms of degrees $d_1, \dots, d_s$ if and only if there are forms $f_1, \dots, f_s$ of degrees $d_1, \dots, d_s$ in I with $\operatorname{ht}((f_1, \dots, f_s) : I) \geqslant s$, and furthermore

$$(-1)^{s-g} \prod_{j=1}^{s} d_j = e_{s-g}(R/I) + \sum_{j=1}^{s-g} \sum_{k=0}^{s-g} (-1)^{j+k} \sum_{1 \leqslant i_1 < \cdots < i_{g+j} \leqslant s} \binom{d_{i_1} + \cdots + d_{i_{g+j}} - g - k}{s-g-k} e_k(R/I^j).$$

Proof. One applies Corollary 1.11 with $r := n-1 = N$ and uses the fact that $\omega/I^j\omega \simeq (R/I^j)(-n)$. □

2. Hilbert series of powers of ideals and degrees of residual intersections

2.1 Computing Hilbert series of powers

The main results of the previous section all require information about the Hilbert series of the powers of an ideal. This leads to the following question, which we are going to address now: to what extent do the Hilbert series of the first powers of an ideal determine the Hilbert series of *all* its powers?

LEMMA 2.1. *Let R be a standard graded Cohen–Macaulay algebra over an infinite field k, let I be a homogeneous ideal, generated by forms of degrees at most d, and let r be an integer with $0 \leqslant r \leqslant \dim R$. Assume that I satisfies G_{r+1} and I is strongly Cohen–Macaulay locally in codimension r in R.*

Given $d_i \geqslant d$ for $1 \leqslant i \leqslant r+1$, there exists a Zariski dense open subset U of the affine k-space $I_{d_1} \times \cdots \times I_{d_{r+1}}$ such that for every $(f_1, \dots, f_{r+1}) \in U$, the following conditions hold.

(a) *The ideal $(f_1, \dots, f_{r+1})$ coincides with I locally in codimension r in R.*

(b) *$f_1, \dots, f_{r+1}$ is a d-sequence locally in codimension r in R.*

Proof. For part (a), we refer to [Ulr94, 1.6(a)] or [HM95, 2.8], whereas (b) follows from [Hun83, 3.1] and [CEU01, 3.6(b)]. □

We choose degrees $d_1, \dots, d_{r+1}$ and forms $f_1, \dots, f_{r+1}$ as in Lemma 2.1. We consider one of the approximation complexes, the $\mathcal{M}$-complex, associated to these forms and its graded strands

$$\mathcal{M}_p : \quad 0 \longrightarrow H_p \otimes S_0 \to H_{p-1} \otimes S_1 \longrightarrow \ \cdots \ \longrightarrow H_0 \otimes S_p \longrightarrow 0.$$

Here $p \geqslant 0$ is an integer, H_q stands for the qth homology of the Koszul complex $\mathbf{K}(f_1, \dots, f_{r+1}; R)$, and S_q denotes the qth symmetric power of the free module $R(-d_1) \oplus \cdots \oplus R(-d_{r+1})$. These graded strands are complexes of graded R-modules with homogeneous maps of degree zero, they are acyclic locally in codimension r in R, and their zeroth homology satisfies $H_0(\mathcal{M}_p) \xrightarrow[r]{\sim} I^p/I^{p+1}$; see [HSV83] for all these facts.

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We conclude that

$$[\![I^p/I^{p+1}]\!](t) \underset{r}{\equiv} \sum_{i=0}^{p}(-1)^i s_{p-i}(t^{d_1},\dots,t^{d_{r+1}})[\![H_i]\!](t), \qquad (1)_p$$

where s_m stands for the sum of all monomials of degree m in $r+1$ variables (the complete symmetric function). These equalities express, up to r-equivalence, the Hilbert series of the modules $R/I,\dots,I^p/I^{p+1}$ in terms of the ones of $H_0,\dots,H_p$ and vice versa.

If I has height g, then $H_q=0$ for $q>r+1-g$. Hence, we see that the Hilbert series of all the modules I^p/I^{p+1} are determined, up to r-equivalence, by knowing the Hilbert series of I^p/I^{p+1}, up to r-equivalence, for $0\leqslant p\leqslant r+1-g$.

We now assume that, in addition to the hypotheses of Lemma 2.1, R is Gorenstein with a-invariant $a:=a(R)$ and I has pure codimension g. In this case we can use the self-duality of the homology of the Koszul complex to see that only half of the information about the Koszul homology is needed. Indeed, the structure of graded alternating algebra on the homology of the Koszul complex gives a homogeneous map

$$H_p \longrightarrow \operatorname{Hom}_{R/I}(H_{r+1-g-p},H_{r+1-g}),$$

and this map is an isomorphism locally in codimension r in R because of the strong Cohen–Macaulayness assumption; see [Her74, 2.4.1].

On the other hand, one has

$$\begin{aligned}\operatorname{Hom}_{R/I}(H_{r+1-g-p},H_{r+1-g}) &\simeq \operatorname{Hom}_{R/I}(H_{r+1-g-p},\operatorname{Ext}^g_R(R/I,R)[-d_1-\cdots-d_{r+1}])\\ &\simeq \operatorname{Hom}_{R/I}(H_{r+1-g-p},\omega_{R/I})[-a-d_1-\cdots-d_{r+1}].\end{aligned}$$

Notice that R/I is equidimensional and that locally in codimension r in R, either $H_{r+1-g-p}$ is zero or else R/I and $H_{r+1-g-p}$ are both Cohen–Macaulay of the same dimension. Hence, Lemma 1.2 shows that

$$[\![H_p]\!](t) \underset{r}{\equiv} t^{a+d_1+\cdots+d_{r+1}}(-1)^{\dim R/I}[\![H_{r+1-g-p}]\!](t^{-1}). \qquad (2)_p$$

Moreover, the Euler characteristic of the homology of the Koszul complex depends only upon the degrees $d_1,\dots,d_{r+1}$, namely

$$\sum_{p=0}^{r+1-g}(-1)^p[\![H_p]\!](t) = \Delta_{r+1}[\![R]\!](t) \underset{r}{\equiv} 0, \qquad (3)$$

where the asserted r-equivalence holds because $\Delta_{r+1}:=\prod_{k=1}^{r+1}(1-t^{d_k})$ is divisible by $(1-t)^{r+1}$.

We now have all the formulas needed to effectively compute the Hilbert series of the higher conormal modules I^p/I^{p+1} up to r-equivalence.

Notice that the result of the computation does not depend on the choice of $d_1,\dots,d_{r+1}$. Thus, we may choose $d_i=0$ for all i. The intermediate steps have no meaning (for instance, $[\![H_p]\!]$ may have negative coefficients), but the information we extract from the computation is the same. Thus, the above formulas take the following simpler form:

$$[\![I^p/I^{p+1}]\!](t) \underset{r}{\equiv} \sum_{i=0}^{p}(-1)^i\binom{r+p-i}{r}[\![H_i]\!](t), \qquad (1)_p$$

$$[\![H_p]\!](t) \underset{r}{\equiv} t^a(-1)^{\dim R/I}[\![H_{r+1-g-p}]\!](t^{-1}), \qquad (2)_p$$

$$\sum_{p=0}^{r+1-g}(-1)^p[\![H_p]\!](t) \underset{r}{\equiv} 0. \qquad (3)$$

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THEOREM 2.2. *Let R be a standard graded Gorenstein algebra over a field, with a-invariant a, let I be a homogeneous ideal of pure codimension g, and let r be an integer with $g \leqslant r \leqslant \dim R$. Assume that I satisfies G_{r+1} and that I is strongly Cohen–Macaulay locally in codimension r in R. Given the Hilbert series of I^p/I^{p+1} for $0 \leqslant p \leqslant \lfloor (r-g)/2 \rfloor$ up to r-equivalence, the Hilbert series of I^p/I^{p+1} can be computed for all p, up to r-equivalence, by the formulas $(1)_p$, $(2)_p$, and (3) above.*

Proof. Write $q = r + 1 - g$. First, using $(1)_p$ for $0 \leqslant p \leqslant \lfloor (q-1)/2 \rfloor$, we obtain the Hilbert series of H_p, for p in the same range, up to r-equivalence. Then, from $(2)_p$, we get the Hilbert series of $H_q, \ldots, H_{q-\lfloor (q-1)/2 \rfloor}$ up to r-equivalence. Therefore, we know the Hilbert series of all H_p up to r-equivalence if q is odd; and of all but one, namely $H_{q/2}$, if q is even. In case q is even, we obtain the Hilbert series of $H_{q/2}$ up to r-equivalence by using (3).

Thus, we know the Hilbert series of all the modules H_p up to r-equivalence, and we can use $(1)_p$ to obtain the ones of I^p/I^{p+1} for any p. □

The above theorem also shows that for any integer s with $g \leqslant s \leqslant r$, the Hilbert series of I^p/I^{p+1} for $0 \leqslant p \leqslant \lfloor (s-g)/2 \rfloor$ up to s-equivalence determine the Hilbert series of I^p/I^{p+1} for all p up to s-equivalence; in others words, the $s - g + 1$ highest coefficients of the Hilbert polynomials of I^p/I^{p+1} for $0 \leqslant p \leqslant \lfloor (s-g)/2 \rfloor$ yield the $s - g + 1$ highest coefficients of the Hilbert polynomials of all higher conormal modules. Thus, one has the following schematic depiction for the determination of the $r - g + 1$ highest Hilbert coefficients e_i, $0 \leqslant i \leqslant r - g$, of the conormal modules of an ideal satisfying the hypotheses of Theorem 2.2.

Hilb. coeff.	0	·	·	·	·	$r-g$	·	·	
R/I	■	■	■	■	■	■	?	?	⋯
I/I^2	□	□	■	■	■	■	?	?	⋯
I^2/I^3	□	□	□	□	■	■	?	?	⋯
I^3/I^4	□	□	□	□	□	□	?	?	⋯
I^4/I^5	□	□	□	□	□	□	?	?	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

■ : needed as input
□ : may be computed from the others
? : not concerned

COROLLARY 2.3. *If $X \subset \mathbb{P}^N_k$ is an equidimensional local complete intersection subscheme and $\mathcal{I}_X$ denotes the corresponding ideal sheaf, then the Hilbert polynomials of the sheaves $\mathcal{I}_X^p/\mathcal{I}_X^{p+1}$ for all p are determined by the ones for $0 \leqslant p \leqslant \lfloor \dim X/2 \rfloor$.*

Proof. One applies Theorem 2.2 with $r := N$ and $g := N - \dim X$. □

Example 2.4. If $X \subset \mathbb{P}^N_k$ is an equidimensional local complete intersection three-fold and $\mathcal{I}_X$ the corresponding ideal sheaf, then the Hilbert polynomials of the sheaves $\mathcal{I}_X^p/\mathcal{I}_X^{p+1}$ are determined by the Hilbert polynomials of X and of the conormal bundle $\mathcal{I}_X/\mathcal{I}_X^2$. Moreover, the two highest coefficients of the Hilbert polynomial of the conormal bundle are determined by the two highest coefficients of the Hilbert polynomial of X.

2.2 Hilbert polynomials of powers of an ideal

We have seen in Example 2.4 that the Hilbert coefficients of $\mathcal{I}_X^p/\mathcal{I}_X^{p+1}$ are determined by only six of these coefficients, in the case of a local complete intersection three-fold. We are now going to elaborate on this fact by giving explicit formulas for the remaining coefficients. We consider, more generally, an equidimensional subscheme $X \subset \mathbb{P}_k^N$ of codimension g. Write R for the ambient polynomial ring and $I := I_X$. Using Theorem 2.2 and a computer algebra system, one derives the following identities, where the formula for each ith Hilbert coefficient requires that X is a complete intersection locally in codimension $g+i$ in $\mathbb{P}_k^N$.

Formulas 2.5. Set $e_i := e_i(R/I)$ and $f_i := e_i(I/I^2)$. Applying the formulas $(1)_1$, 3, $(2)_2$ above with $r := g+1$ and using Lemma 1.7 and Remark 1.8, one sees that

$$f_0 = e_0(I/I^2) = ge_0, \quad f_1 = e_1(I/I^2) = ge_0 + (g+2)e_1$$

and, similarly,

$$e_0(I^p/I^{p+1}) = \binom{g+p-1}{g-1}e_0, \quad e_1(I^p/I^{p+1}) = \binom{g+p-1}{g}ge_0 + \frac{g+2p}{g+p}\binom{g+p}{g}e_1.$$

One uses the last two equalities, formulas $(1)_2$, $(1)_1$, $(2)_2$ for $r := g+1$, and formulas $(1)_2$, $(1)_1$, (3), $(2)_3$, $(2)_4$ for $r := g+3$, together with Lemma 1.7 and Remark 1.8 to see that

$$\begin{aligned}
e_0(I^2/I^3) &= \frac{g(g+1)}{2}e_0,\\
e_1(I^2/I^3) &= g(g+1)e_0 + \frac{(g+1)(g+4)}{2}e_1,\\
e_2(I^2/I^3) &= \frac{g(g+1)}{2}e_0 + (g+1)e_1 - \frac{g(g+3)}{2}e_2 + (g+2)f_2,\\
e_3(I^2/I^3) &= -\frac{(g+1)(g+2)}{2}e_1 - (g+2)(g+3)e_2 - \frac{(g+3)(g+4)}{2}e_3 + (g+2)f_2 + (g+4)f_3.
\end{aligned}$$

As a first guess, one may hope that, at least with some strong hypotheses on X, the Hilbert polynomials of the conormal sheaves are determined by the Hilbert polynomial of X. This is not even true for complete intersections, as the following computation shows.

Suppose that $X \subset \mathbb{P}_k^N$ is a global complete intersection of codimension g and dimension at least three, let e_i be the ith Hilbert coefficient of X, and denote by $\sigma_1, \dots, \sigma_g$ the elementary symmetric functions in the degrees of the defining equations of X. Setting

$$\alpha_1 = \sigma_1 - g, \quad \alpha_2 = \sigma_1^2 - 2\sigma_2 - g, \quad \alpha_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 - g,$$

one obtains

$$\begin{aligned}
e_0 &= \sigma_g,\\
e_1 &= \frac{\sigma_g}{2}\alpha_1,\\
e_2 &= \frac{\sigma_g}{24}(3\alpha_1^2 - 6\alpha_1 + \alpha_2),\\
e_3 &= \frac{\sigma_g}{48}(\alpha_1^3 - 6\alpha_1^2 + 8\alpha_1 + \alpha_1\alpha_2 - 2\alpha_2).
\end{aligned}$$

Notice that these formulas imply that e_3 is a rational function of the other three coefficients.

Remark 2.6. If e_i denotes the ith Hilbert coefficient of a global complete intersection $X \subset \mathbb{P}_k^N$ of dimension at least three, then

$$e_3 = -e_2 + \frac{e_1 e_2}{e_0} - \frac{e_1}{6} + \frac{e_1^2}{2e_0} - \frac{e_1^3}{3e_0^2}.$$

Now, using the expansion

$$t^{d_1} + \cdots + t^{d_g} = g + (g + \alpha_1)(t-1) + (\alpha_2 - \alpha_1)\frac{(t-1)^2}{2} + (\alpha_3 - 3\alpha_2 + 2\alpha_1)\frac{(t-1)^3}{6} + \cdots,$$

one can compute the coefficients $e_i(I/I^2)$ for $0 \leqslant i \leqslant 3$. The only place where α_3 appears is in $e_3(I/I^2) = (\sigma_g/6)\alpha_3 + \cdots$.

If one chooses two collections of degrees such that the first, second, and fourth elementary symmetric functions are equal but the third one differs, one gets an example of two complete intersections of dimension three in $\mathbb{P}_k^7$ having the same Hilbert polynomial (but distinct Hilbert functions!) such that the constant terms of the Hilbert polynomials of their conormal bundles are distinct. Such examples were given to us by Benjamin de Weger; the two 'smallest' ones are (1,6,7,22)–(2,2,11,21) and (2,6,7,15)–(3,3,10,14). He also gave an infinite collection of them, and Noam Elkies gave a rational parametrization of all the solutions (after a linear change of coordinates, the solutions are parametrized by a quadric in $\mathbb{P}_{\mathbb{C}}^5$).

2.3 Degrees of residual intersections

Let $X \subset \mathbb{P}_k^N$ be an equidimensional subscheme of dimension $D-1$ and codimension g that is a complete intersection locally in codimension s in $\mathbb{P}_k^N$. We are now going to give formulas for the degree of the codimension s part of an s-residual intersection of X that require less input data than the formulas of § 1. We restrict ourselves to the case where $\delta := s - g$ is at most three. As before, R denotes the ambient polynomial ring, $\mathfrak{R} := (f_1, \ldots, f_s) : I$ is an s-residual intersection of $I := I_X$ given by s homogeneous polynomials $f_1, \ldots, f_s$ of degrees $d_1, \ldots, d_s$, and σ_m stands for the mth elementary symmetric function in $d_1, \ldots, d_s$. By combining Theorem 1.9(b), in the version of Remark 1.10, with Formulas 2.5, we obtain the following.

- If $\delta = 0$, $e_0^D(R/\mathfrak{R}) = \sigma_s - e_0(R/I)$, by Bézout's theorem.
- If $\delta = 1$, $e_0^{D-1}(R/\mathfrak{R}) = \sigma_s - (\sigma_1 - g)e_0(R/I) + 2e_1(R/I)$; see also [Stü92, Theorem 5].
- If $\delta = 2$,

$$e_0^{D-2}(R/\mathfrak{R}) = \sigma_s - \left[\sigma_2 - g\sigma_1 + \binom{g+1}{2}\right]e_0(R/I) + (2\sigma_1 - (g+1))e_1(R/I) + ge_2(R/I) - e_2(I/I^2);$$

  see also [HM95, 4.11].
- If $\delta = 3$,

$$\begin{aligned} e_0^{D-3}(R/\mathfrak{R}) = \sigma_s - &\left[\sigma_3 - g\sigma_2 + \binom{g+1}{2}\sigma_1 - \binom{g+2}{3}\right]e_0(R/I) \\ &+ (2\sigma_2 - (g+1)\sigma_1)e_1(R/I) + (g\sigma_1 - (g+2)^2)e_2(R/I) \\ &- 2(g+2)e_3(R/I) - (\sigma_1 - g - 2)e_2(I/I^2) + 2e_3(I/I^2). \end{aligned}$$

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3. Applications to secant varieties

We now apply the results of § 1 to say something about the dimension of secant varieties and to derive relations among certain Hilbert coefficients in the case of surfaces and three-folds. For the next result, we recall that a local algebra essentially of finite type over a field k is said to be *licci* if it is isomorphic to $S/\mathfrak{B}$, where S is a regular local ring essentially of finite type over k and $\mathfrak{B}$ is an S-ideal in the linkage class of a complete intersection.

Theorem 3.1. *Let k be a perfect field, $X \subset \mathbb{P}^N_k$ an equidimensional subscheme of dimension two with at most isolated licci Gorenstein singularities, A its homogeneous coordinate ring, $\omega := \omega_A$ the graded canonical module, and $\Omega := \Omega_{A/k}$ the module of differentials.*

(a) *One has*

$$e_0(A)^2 + 14e_0(A) - 16e_1(A) + 4e_2(A) \geqslant e_2(\omega \otimes_A \omega) + e_2(\Omega).$$

(b) *In case the singularities of X have embedding codimension at most two, then equality holds in* (a) *if and only if the secant variety of X is deficient, i.e.*

$$\dim \mathrm{Sec}(X) < 5.$$

Proof. We may assume that k is infinite. We define a ring R and an R-ideal I via the exact sequence

$$0 \longrightarrow I \longrightarrow R := A \otimes_k A \xrightarrow{\text{mult}} A \longrightarrow 0.$$

Recall that $\Omega \simeq I/I^2$. The ring R is an equidimensional standard graded k-algebra of dimension six with $\omega_R = \omega \otimes_k \omega$. The ideal I has height three and is generated by linear forms. Moreover, I satisfies G_5 and, in the setting of (b), even G_6. Indeed, for any $\mathfrak{p} \in V(I)$, one has $\mu(I_\mathfrak{p}) = \mu(\Omega_\mathfrak{p}) \leqslant \mathrm{ecodim}(A_\mathfrak{p}) + \dim A \leqslant \dim R_\mathfrak{p}$ if $\dim R_\mathfrak{p} \leqslant 4$ or, in the setting of (b), $\dim R_\mathfrak{p} \leqslant 5$; here μ denotes minimal number of generators and $\mathrm{ecodim}(A_\mathfrak{p}) := \mathrm{edim}(A_\mathfrak{p}) - \dim A_\mathfrak{p}$ stands for the embedding codimension of $A_\mathfrak{p}$. Now let $\mathfrak{A}$ be an R-ideal generated by five general linear forms in I. By [Ulr94, 1.6(a)] or [HM95, 2.8], one has $\mathrm{ht}(\mathfrak{A} : I) \geqslant 5$ as I satisfies G_5, and $\mathrm{ht}(I + (\mathfrak{A} : I)) \geqslant 6$ in (b) as I is G_6. Thus, in the setting of (b), the ideal $\mathfrak{A} : I$ has height at least six if and only if the analytic spread $\ell(I)$ of I is at most five; indeed, $\ell(I) \leqslant 5$ if and only if I is integral over $\mathfrak{A}$, an ideal generated by five general linear forms in I; the latter implies that $\sqrt{\mathfrak{A}} = \sqrt{I}$, and hence $\sqrt{\mathfrak{A} : I}$ equals $\sqrt{I + (\mathfrak{A} : I)}$, which has height at least six; the reverse implication follows from [Ulr92, Proposition 3]. On the other hand, $\dim \mathrm{Sec}(X) = \ell(I) - 1$ because $k[\mathfrak{A}_1]$ is the homogeneous coordinate ring of $\mathrm{Sec}(X)$; see [SU00, § 1]. Thus, we have shown that $\dim R/(\mathfrak{A} : I) < 1$ if and only if $\dim \mathrm{Sec}(X) < 5$. In other words, $e_0^1(I/\mathfrak{A}) = 0$ if and only if $\dim \mathrm{Sec}(X) < 5$.

To compute $e_0^1(I/\mathfrak{A})$, we apply Theorem 1.9(a) with $r = s = 5$ and $g = 3$. We first argue that the hypotheses of Theorem 1.9(a) are satisfied. Notice that the five general linear forms in I that generate $\mathfrak{A}$ are a filter regular sequence with respect to I on R and on ω_R. Moreover, for every homogeneous $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} \leqslant 5$, the ring $A_\mathfrak{p}$ is Gorenstein and $\mathrm{depth}\,(I/I^2)_\mathfrak{p} = \mathrm{depth}\,\Omega_\mathfrak{p} \geqslant \dim A_\mathfrak{p} - 1$. To see the latter, we write $A_\mathfrak{p} \simeq S/\mathfrak{B}$ with S a regular local k-algebra essentially of finite type and $\mathfrak{B}$ an ideal in the linkage class of a complete intersection. Since $A_\mathfrak{p}$ is moreover Gorenstein, [Buc81, 6.2.11 and 6.2.12] implies that $\mathfrak{B}/\mathfrak{B}^2$ is Cohen–Macaulay. Thus, the natural complex

$$0 \longrightarrow \mathfrak{B}/\mathfrak{B}^2 \longrightarrow \Omega_{S/k} \otimes_S A_\mathfrak{p} \simeq \oplus A_\mathfrak{p} \longrightarrow \Omega_{A_\mathfrak{p}/k} \simeq \Omega_\mathfrak{p} \longrightarrow 0$$

is exact and shows that indeed $\mathrm{depth}\ \Omega_\mathfrak{p} \geqslant \dim\ A_\mathfrak{p} - 1$. Next, the non-Gorenstein locus of $R = A \otimes_k A$ is contained in $V(\mathfrak{m} \otimes_k \mathfrak{m})$, where $\mathfrak{m}$ stands for the maximal homogeneous ideal of A.

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Since the standard graded k-algebra $R/(\mathfrak{m}\otimes_k\mathfrak{m})$ has dimension three and $R/(\mathfrak{m}\otimes_k\mathfrak{m}+I)\cong A/\mathfrak{m}^2$ has dimension zero, it follows that three general linear forms of I generate a zero-dimensional ideal in the ring $R/(\mathfrak{m}\otimes_k\mathfrak{m})$. Therefore, $R_\mathfrak{p}$ is Gorenstein whenever $\mathfrak{p}\in\operatorname{Spec}(R)$ is homogeneous with $\dim R_\mathfrak{p}\leqslant 5$ and $\mathfrak{p}$ contains $\mathfrak{A}_3$, an R-ideal generated by three general linear forms in I. In particular, $R_\mathfrak{p}$ is Gorenstein for every homogeneous $\mathfrak{p}\in V(I)$ with $\dim R_\mathfrak{p}\leqslant 5$. Thus, we have shown that Theorem 1.9(a) applies.

The theorem gives

$$e_0^1(I/\mathfrak{A})=e_0(R)-e_2(A)-\sum_{j=1}^{2}\sum_{k=0}^{2}(-1)^{j+k}\binom{5}{j+3}\binom{6+j-k}{2-k}e_k(\omega_R/I^j\omega_R). \tag{1}$$

Since $e_0^1(I/\mathfrak{A})\geqslant 0$ and equality holds if and only if $\dim\operatorname{Sec}(X)<5$, the present theorem will follow once we have shown that the right-hand side of (1) equals

$$e_0(A)^2+14e_0(A)-16e_1(A)+4e_2(A)-e_2(\omega\otimes_A\omega)-e_2(\Omega). \tag{2}$$

There are isomorphisms of R-modules

$$\omega_R/I\omega_R\cong\omega_R\otimes_R R/I\cong(\omega\otimes_k\omega)\otimes_{A\otimes_k A}A\cong\omega\otimes_A(\omega\otimes_A A)\cong\omega\otimes_A\omega,$$

where the next to last isomorphism holds according to [CE56, IX 2.1]. Likewise, since locally in codimension five the ring R is Gorenstein along $V(I)$,

$$I\omega_R/I^2\omega_R\underset{5}{\cong}\omega_R\otimes_R I/I^2\cong(\omega\otimes_k\omega)\otimes_{A\otimes_k A}\Omega\cong\omega\otimes_A(\omega\otimes_A\Omega)\cong(\omega\otimes_A\omega)\otimes_A\Omega.$$

Therefore, the right-hand side of (1) becomes

$$e_0(A)^2-7e_0(A)-e_2(A)-23e_1(\omega^{\otimes 2})+4e_2(\omega^{\otimes 2})+7e_1(\omega^{\otimes 2}\otimes\Omega)-e_2(\omega^{\otimes 2}\otimes\Omega). \tag{3}$$

Here and in what follows tensor products are taken over the ring A.

We are now going to express the Hilbert coefficients $e_1(\omega^{\otimes 2})$, $e_1(\omega^{\otimes 2}\otimes\Omega)$, and $e_2(\omega^{\otimes 2}\otimes\Omega)$ in terms of $e_i(A)$, $e_2(\omega^{\otimes 2})$, and $e_2(\Omega)$. First, notice that for any finitely generated graded A-module M,

$$e_i(M(-1))=e_i(M)+e_{i-1}(M). \tag{4}$$

Moreover, by Lemma 1.2, Remark 1.8, and Lemma 1.7,

$$e_1(\omega)=3e_0(A)-e_1(A)\quad\text{and}\quad e_2(\omega)=3e_0(A)-2e_1(A)+e_2(A). \tag{5}$$

Since ω is free of rank one locally in codimension one in A, there is a complex of graded A-modules

$$0\longrightarrow Z\longrightarrow A(-a)^2\longrightarrow\omega\longrightarrow 0 \tag{6}$$

for some $a\gg 0$ that is exact locally in codimension one. It induces complexes

$$0\longrightarrow Z\otimes A(-ja+a)^j\longrightarrow A(-ja)^{j+1}\longrightarrow\operatorname{Sym}_j(\omega)\underset{1}{\cong}\omega^{\otimes j}\longrightarrow 0 \tag{7}$$

that are likewise exact locally in codimension one. Now (6) yields $e_1(Z)=2e_1(A(-a))-e_1(\omega)$ and then (4), (5), and (7) show that for every $j\geqslant 0$,

$$e_1(\omega^{\otimes j})=3je_0(A)-(2j-1)e_1(A). \tag{8}$$

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Next, we treat the first Hilbert coefficient of $\omega^{\otimes 2} \otimes \Omega$. As Ω is free of rank three locally in codimension one and is generated in degree one, there is an exact sequence of graded A-modules

$$0 \longrightarrow A(-1)^2 \longrightarrow \Omega \longrightarrow C \longrightarrow 0,$$

where C is free of rank one locally in codimension one; see [EE73, p. 282, Theorem A and Remark]. Also recall the fundamental class, a natural map $\wedge^3\Omega \to \omega$ that is an isomorphism off the singular locus of A; see for instance [KW88, § 5]. We obtain

$$C \otimes \bigwedge^2 (A(-1)^2) \xrightarrow[1]{\sim} \bigwedge^3 \Omega \xrightarrow[1]{\sim} \omega,$$

which gives a complex

$$0 \longrightarrow A(-1)^2 \longrightarrow \Omega \longrightarrow \omega(2) \longrightarrow 0$$

that is exact in codimension one. Tensoring with $\omega^{\otimes 2}$ yields

$$0 \longrightarrow \omega^{\otimes 2}(-1)^2 \longrightarrow \omega^{\otimes 2} \otimes \Omega \longrightarrow \omega^{\otimes 3}(2) \longrightarrow 0.$$

Since this complex is exact in codimension one, (4) and (8) imply that

$$e_1(\omega^{\otimes 2} \otimes \Omega) = 21e_0(A) - 11e_1(A). \tag{9}$$

We now turn to the second Hilbert coefficient of $\omega^{\otimes 2} \otimes \Omega$. Write $-^* := \operatorname{Hom}_A(-, A)$. Recall that A is Gorenstein locally in codimension two. Since ω is free of rank one locally in codimension two, we have $\omega^* \xrightarrow[2]{\sim} \operatorname{Hom}_A(\omega^{\otimes 2}, \omega)$, which, by Lemma 1.2, Remark 1.8, and Lemma 1.7, gives

$$e_1(\omega^*) = 3e_0(\omega^{\otimes 2}) - e_1(\omega^{\otimes 2}) \quad \text{and} \quad e_2(\omega^*) = 3e_0(\omega^{\otimes 2}) - 2e_1(\omega^{\otimes 2}) + e_2(\omega^{\otimes 2}). \tag{10}$$

Furthermore,

$$\omega^{\otimes 2} \otimes \omega^* \xrightarrow[2]{\sim} \operatorname{Hom}_A(\omega, \omega^{\otimes 2}) \xleftarrow[2]{\sim} \omega. \tag{11}$$

Let $e := N - 2$. Increasing N if needed, we may assume that $e \geqslant 2$. We define a graded A-module E via the exact sequence

$$0 \longrightarrow E \longrightarrow A(-1)^{e+3} \longrightarrow \Omega \longrightarrow 0, \tag{12}$$

which is split-exact locally in codimension one. Notice that E has rank e, is free locally in codimension one, and is Cohen–Macaulay locally in codimension two by the discussion at the beginning of this proof. Furthermore, (12) and the fundamental class give

$$\left(\bigwedge^e E\right)^{**} \xrightarrow[2]{\sim} \left(\bigwedge^3 \Omega\right)^{*}(-e-3) \xleftarrow[2]{\sim} \omega^*(-e-3); \tag{13}$$

these natural maps of reflexive modules are isomorphisms locally in codimension two, because they are isomorphisms locally in codimension one and A is Cohen–Macaulay locally in codimension two.

As E^* is free locally in codimension one and $\operatorname{rk} E^* - 1 \geqslant 1$, there exists a homogeneous element $f \in E^*$ of degree $c \gg 0$ whose order ideal $(E^*)^*(f)$ has height at least two; see [EE73, p. 282, Theorem A and Remark]. However, the ideals $E^{**}(f)$ and $J := f(E)$ coincide locally in

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codimension one, since E is reflexive locally in codimension one. Hence, $\operatorname{ht} J \geqslant 2$. The map f induces an exact sequence of graded A-modules

$$0 \longrightarrow E_{e-1} \longrightarrow E_e := E \longrightarrow J_e(c_e) := J(c) \longrightarrow 0.$$

Repeating this procedure, if needed, we obtain a filtration

$$E_1 \subset E_2 \subset \cdots \subset E_e \quad \text{with } E_i/E_{i-1} \cong J_i(c_i), \tag{14}$$

where J_i are homogeneous A-ideals of height at least two. Thus, E_i has rank i, is free in codimension one and Cohen–Macaulay in codimension two, and

$$\left(\bigwedge^{i-1} E_{i-1}\right)^{**}(c_i) \xleftarrow[2]{\sim} \left(\left(\bigwedge^{i-1} E_{i-1}\right) \otimes J_i(c_i)\right)^{**} \xrightarrow[2]{\sim} \left(\bigwedge^{i} E_i\right)^{**}.$$

Again, these natural maps of reflexive modules are isomorphisms locally in codimension two, because they are isomorphisms locally in codimension one and A is Cohen–Macaulay locally in codimension two.

Since E_1 is reflexive locally in codimension two, it follows that

$$E_1 \xrightarrow[2]{\sim} E_1^{**} \underset{2}{\cong} \left(\bigwedge^{e} E\right)^{**}\left(-\sum_{i=2}^{e} c_i\right),$$

which, together with (13), implies that

$$E_1 \underset{2}{\cong} \omega^*\left(-e-3-\sum_{i=2}^{e} c_i\right). \tag{15}$$

On the other hand, the exact sequence

$$0 \longrightarrow J_i(c_i) \longrightarrow A(c_i) \longrightarrow (A/J_i)(c_i) \longrightarrow 0$$

yields a complex

$$0 \longrightarrow \omega^{\otimes 2} \otimes J_i(c_i) \longrightarrow \omega^{\otimes 2}(c_i) \longrightarrow (\omega^{\otimes 2}/\omega^{\otimes 2} J_i)(c_i) \longrightarrow 0$$

that is exact in codimension two. Since $\operatorname{ht} J_i \geqslant 2$ and $\omega^{\otimes 2}$ is free of rank one locally in codimension two, it follows that $e_0^1((A/J_i)(c_i)) = e_0^1((\omega^{\otimes 2}/\omega^{\otimes 2} J_i)(c_i))$. We conclude that

$$e_2(J_i(c_i)) - e_2(\omega^{\otimes 2} \otimes J_i(c_i)) = e_2(A(c_i)) - e_2(\omega^{\otimes 2}(c_i)). \tag{16}$$

Tensoring (12) and (14) with $\omega^{\otimes 2}$ and using (16) and (15), we obtain

$$\begin{aligned} e_2(\omega^{\otimes 2} \otimes \Omega) - e_2(\Omega) = e_2(\omega^{\otimes 2}(-1)^{e+3}) - e_2(A(-1)^{e+3}) + \sum_{i=2}^{e}(e_2(A(c_i)) - e_2(\omega^{\otimes 2}(c_i))) \\ + e_2\left(\omega^*\left(-e-3-\sum_{i=2}^{e} c_i\right)\right) - e_2\left(\omega^{\otimes 2} \otimes \omega^*\left(-e-3-\sum_{i=2}^{e} c_i\right)\right). \end{aligned} \tag{17}$$

We now combine (17) with (4), (8), (10), (11), and (5) to deduce that

$$e_2(\omega^{\otimes 2} \otimes \Omega) = -12e_0(A) + 8e_1(A) - 5e_2(A) + 5e_2(\omega^{\otimes 2}) + e_2(\Omega). \tag{18}$$

Substituting (8), (9), and (18) into (3), we conclude that (3) and (2) coincide. □

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Remark 3.2. The inequality in Theorem 3.1 can be replaced by

$$e_0(A)^2 + 5e_0(A) - 10e_1(A) + 4e_2(A) \geqslant e_2(\omega^*) + e_2(\Omega).$$

Proof. One uses the equalities (10) and (8) in the proof of Theorem 3.1. □

Corollary 3.3. *Let k be a perfect field, $X \subset \mathbb{P}^4_k$ an equidimensional subscheme of dimension two with at most isolated Gorenstein singularities, A its homogeneous coordinate ring, $\omega := \omega_A$ the canonical module, and $\Omega := \Omega_{A/k}$ the module of differentials. One has*

$$e_0(A)^2 + 14e_0(A) - 16e_1(A) + 4e_2(A) = e_2(\omega \otimes_A \omega) + e_2(\Omega).$$

Corollary 3.4. *Let k be a field, $X \subset \mathbb{P}^N_k$ an equidimensional smooth subscheme of dimension two, H the class of the hyperplane section, K the canonical divisor, and c_2 the second Chern class of the cotangent bundle of X. One has*

$$(H^2)^2 \geqslant 10H^2 + 5HK + K^2 - c_2,$$

and equality holds if and only if $\dim \operatorname{Sec}(X) < 5$.

Proof. We may assume that k is algebraically closed. The Riemannn–Roch theorem in dimension two gives

$$\chi(X, E) = \tfrac{1}{2}(c_1(E)^2 - 2c_2(E) - c_1(E)K_X) + (\operatorname{rk} E)\,\chi(X, \mathcal{O}_X).$$

If D is a divisor, this equality specializes to

$$\begin{aligned}\chi(D + nH) &= \frac{1}{2}H^2n^2 + \left(DH - \frac{1}{2}KH\right)n + \frac{1}{2}(D^2 - KD) + \chi(X, \mathcal{O}_X)\\ &= H^2\binom{n+2}{2} - \frac{1}{2}(3H^2 + KH - 2DH)\binom{n+1}{1}\\ &\quad + \frac{1}{2}(H^2 + KH - 2DH - KD + D^2) + \chi(X, \mathcal{O}_X).\end{aligned}$$

For a rank two vector bundle E, the formula reads

$$\begin{aligned}\chi(E + nH) &= H^2n^2 + (c_1(E)H - KH)n + \frac{1}{2}(c_1(E)^2 - Kc_1(E)) - c_2(E) + 2\,\chi(X, \mathcal{O}_X)\\ &= 2H^2\binom{n+2}{2} - (3H^2 + KH - c_1(E)H)\binom{n+1}{1}\\ &\quad + H^2 + KH - c_1(E)H - \frac{1}{2}Kc_1(E) + \frac{1}{2}c_1(E)^2 - c_2(E) + 2\,\chi(X, \mathcal{O}_X).\end{aligned}$$

Taking $D = 0$, we obtain

$$\begin{aligned}e_0(\mathcal{O}_X) &= H^2,\\ e_1(\mathcal{O}_X) &= \tfrac{3}{2}H^2 + \tfrac{1}{2}KH,\\ e_2(\mathcal{O}_X) &= \tfrac{1}{2}H^2 + \tfrac{1}{2}KH + \chi(X, \mathcal{O}_X)\end{aligned}$$

and, for $D = 2K$,

$$e_2(\omega_X^{\otimes 2}) = \tfrac{1}{2}H^2 - \tfrac{3}{2}KH + K^2 + \chi(X, \mathcal{O}_X).$$

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Finally, taking $E = \Omega_X$, the cotangent sheaf of X, and using the fact that $c_1(\Omega_X) = K$, we deduce that

$$e_2(\Omega_X) = H^2 - c_2(\Omega_X) + 2\,\chi(X, \mathcal{O}_X).$$

Now the assertion of the corollary follows from Theorem 3.1 since $e_i(A) = e_i(\mathcal{O}_X)$, $e_i(\omega^{\otimes 2}) = e_i(\omega_X^{\otimes 2})$, and $e_i(\Omega) = e_i(\Omega_X) + e_i(\mathcal{O}_X)$. $\square$

We now turn to smooth three-folds.

Theorem 3.5. *Let k be a field, $X \subset \mathbb{P}_k^N$ an equidimensional smooth subscheme of dimension three, H the class of the hyperplane section, K the canonical divisor, and c_2 and c_3 the second and third Chern classes of the cotangent bundle of X. One has*

$$(H^3)^2 \geqslant 35H^3 - 11KH^2 - 9K^2H + c_2H - K^3 - \tfrac{1}{12}Kc_2 + \tfrac{1}{2}c_3,$$

and equality holds if and only if $\dim \operatorname{Sec}(X) < 7$.

Proof. We use the notation introduced in the proof of Theorem 3.1, with $\mathfrak{A}$ an ideal generated by seven general linear forms in the ideal I of the diagonal. Again, we see that $\dim R/(\mathfrak{A} : I) < 1$ if and only if $\dim \operatorname{Sec}(X) < 7$.

We will apply Theorem 1.9(a) with $r = s = 7$ and $g = 4$. Since X is smooth, one sees as in the proof of Theorem 3.1 that

$$\begin{aligned} \omega_R/I\,\omega_R &\cong \omega^{\otimes 2}, \\ I\omega_R/I^2\omega_R &\underset{7}{\cong} \omega^{\otimes 2} \otimes \Omega, \\ I^2\omega_R/I^3\omega_R &\underset{7}{\cong} \omega^{\otimes 2} \otimes S_2\Omega, \end{aligned}$$

where $S_2\Omega := \operatorname{Sym}_2(\Omega)$.

Again, since X is smooth, we can apply the Riemann–Roch theorem as in the proof of Corollary 3.4 to derive the following formulas, which express the relevant Hilbert coefficients in terms of the invariants that appear in the statement of the present theorem:

$$\begin{aligned} &e_0(A) = H^3, \\ &e_1(A) = 2H^3 - \tfrac{3}{2}KH^2, \\ &e_2(A) = \tfrac{1}{12}(14H^3 + 9KH^2 + K^2H + c_2H), \\ &e_3(A) = \tfrac{1}{24}(4H^3 + 6KH^2 + 2K^2H + 2c_2H + Kc_2), \end{aligned}$$

$$\begin{aligned} &e_0(\Omega) = 4H^3, \\ &e_1(\Omega) = 8H^3 + KH^2, \\ &e_2(\Omega) = \tfrac{1}{6}(28H^3 + 9KH^2 + 2K^2H - 4c_2H), \\ &e_3(\Omega) = \tfrac{1}{12}(8H^3 + 6KH^2 + 4K^2H - 8c_2H + Kc_2 - 6c_3), \end{aligned}$$

$$\begin{aligned} &e_0(\omega^{\otimes 2}) = H^3, \\ &e_1(\omega^{\otimes 2}) = 2H^3 - \tfrac{3}{2}KH^2, \\ &e_2(\omega^{\otimes 2}) = \tfrac{1}{12}(14H^3 - 27KH^2 + 13K^2H + c_2H), \\ &e_3(\omega^{\otimes 2}) = \tfrac{1}{24}(4H^3 - 18KH^2 + 26K^2H + 2c_2H - 12K^3 - 3Kc_2), \end{aligned}$$

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$e_0(\omega^{\otimes 2} \otimes \Omega) = 4H^3,$
$e_1(\omega^{\otimes 2} \otimes \Omega) = 8H^3 - 7KH^2,$
$e_2(\omega^{\otimes 2} \otimes \Omega) = \frac{1}{6}(28H^3 - 63KH^2 + 38K^2H - 4c_2H),$
$e_3(\omega^{\otimes 2} \otimes \Omega) = \frac{1}{12}(8H^3 - 42KH^2 + 76K^2H - 8c_2H - 48K^3 + 17Kc_2 - 6c_3),$

$e_0(\omega^{\otimes 2} \otimes S_2\Omega) = 10H^3,$
$e_1(\omega^{\otimes 2} \otimes S_2\Omega) = 20H^3 - 19KH^2,$
$e_2(\omega^{\otimes 2} \otimes S_2\Omega) = \frac{1}{6}(70H^3 - 171KH^2 + 119K^2H - 25c_2H),$
$e_3(\omega^{\otimes 2} \otimes S_2\Omega) = \frac{1}{12}(20H^3 - 114KH^2 + 238K^2H - 50c_2H - 186K^3 + 125Kc_2 - 42c_3).$ □

Remark 3.6. Assume that k is infinite and let $D := H^3$ denote the degree of X. The inequality in Theorem 3.5 is equivalent to

$$D^2 \geqslant 7(5D + 3KH^2 + K^2H - c_2H) - 2Kc_2 + K^3 + c_3.$$

In other words, if S and C are a surface and a curve obtained from X by general hyperplane sections, the formula reads

$$D^2 \geqslant 7(5D + 3\chi_C + 12\chi(\mathcal{O}_S) - 2\chi_S) - 48\chi(\mathcal{O}_X) + K^3 + \chi_X.$$

Theorem 3.5 can be converted into a statement about Hilbert coefficients that is analogous to Theorem 3.1.

Corollary 3.7. *Let k be a field, $X \subset \mathbb{P}_k^N$ an equidimensional smooth subscheme of dimension three, A its homogeneous coordinate ring, $\omega := \omega_A$ the canonical module, and $\Omega := \Omega_{A/k}$ the module of differentials. One has*

$$e_0(A)^2 + 391e_0(A) - 246e_1(A) + 66e_2(A) + 50e_3(A) \geqslant 18e_2(\omega \otimes_A \omega) - 2e_3(\omega \otimes_A \omega) - 2e_3(\Omega),$$

and equality holds if and only if $\dim \mathrm{Sec}(X) < 7$.

Proof. One uses the formulas in the proof of Theorem 3.5 to express the relevant Hilbert coefficients in terms of the invariants appearing in the statement of the theorem. □

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Marc Chardin chardin@math.jussieu.fr
Institut de Mathématiques de Jussieu, CNRS and UPMC, 4 place Jussieu,
75005 Paris, France

David Eisenbud de@msri.org
Department of Mathematics, University of California, Berkeley, CA 94720, USA

Bernd Ulrich ulrich@math.purdue.edu
Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA