THE DISTRIBUTION OF CERTAIN SPECIAL VALUES OF THE CUBIC LEGENDRE SYMBOL

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Dedicated to Professor Robert Rankin on the occasion of his 70th birthday

1. Introduction. Let ω be a primitive cube root of unity. We define the cubic residue symbol (Legendre symbol) on $\mathbb{Z}[\omega]$ as follows. Let $\pi \in \mathbb{Z}[\omega]$ be a prime, $(3, \pi) = 1$. For $\alpha \in \mathbb{Z}[\omega]$ such that $(\alpha, \pi) = 1$ we let $\left(\frac{\alpha}{\pi}\right)_3$ be that third root of unity so that

$$\left(\frac{\alpha}{\pi}\right)_3 \equiv \alpha^{(N(\pi)-1)/3} \qquad (\text{mod } \pi).$$

One then extends $(-)_3$ to a function defined on all pairs α, β with $(3\alpha, \beta) = 1$ by requiring that

$$\left(\frac{\alpha}{\beta_1\beta_2}\right)_3 = \left(\frac{\alpha}{\beta_1}\right)_3 \cdot \left(\frac{\alpha}{\beta_2}\right)_3.$$

Since $\mathbb{Z}[\omega]$ is a principal ideal domain this defines $(-)_3$ completely. Each $\alpha \in \mathbb{Z}[\omega]$ such that $(\alpha, 3) = 1$ can be written in the form $\zeta\alpha_1$ where $\zeta^6 = 1$ and $\alpha_1 \equiv 1$ (mod 3). Note that $3 = -(\sqrt{-3})^2$. This decomposition is unique; if $\alpha \equiv 1$ (mod 3) we shall say that α is *primary*.

The law of cubic reciprocity states that

$$\left(\frac{\alpha}{\beta}\right)_3 = \left(\frac{\beta}{\alpha}\right)_3$$

if α, β are primary.

Denote complex conjugation (the non-trivial element of the Galois group of $\mathbf{Q}(\omega)/\mathbf{Q}$) by $\alpha \mapsto \bar{\alpha}$.

If π is a primary prime of $\mathbb{Z}[\omega]$ one sees that

$$\left(\frac{\bar{\pi}}{\pi}\right)_3 = \overline{\left(\frac{\pi}{\bar{\pi}}\right)_3}$$

(by transport of structure) and

$$\left(\frac{\bar{\pi}}{\pi}\right)_3 = \left(\frac{\pi}{\bar{\pi}}\right)_3$$

(by cubic reciprocity). It follows that

$$\left(\frac{\bar{\pi}}{\pi}\right)_3 = 1.$$

Actually this argument does not make use of the fact that π is a prime; one has for all

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primary α that

$$\left(\frac{\bar{\alpha}}{\alpha}\right)_3 = 1$$

if $(\alpha, \bar{\alpha}) = 1$. It is an easy exercise to recover the law of cubic reciprocity from this statement.

One can rephrase this by saying that the value of $\left(\frac{\bar{\alpha}}{\alpha}\right)_3$ for $(\alpha, 3) = 1$, $(\alpha, \bar{\alpha}) = 1$ depends only on the residue class of α (mod 3), i.e. that it is a "congruence function".

If now $i = \sqrt{-1}$ one can likewise define the biquadratic residue symbol $(-)_4$ on $\mathbb{Z}[i]$. In this case the argument is rather more involved but it can be shown that for $\alpha \equiv 1 \pmod{(1+i)^3}$ one has

$$\left(\frac{\bar{\alpha}}{\alpha}\right)_4 = \left(\frac{-2}{\alpha}\right)_4 \left(\frac{2}{a}\right)$$

where $a = \frac{1}{2}(\alpha + \bar{\alpha})$ and $(-)$ is the usual Legendre symbol. This is proved in [**7**, Proposition 2.2].

With these examples one is prompted to ask if this represents a general phenomenon. Indeed one can show that if K/k is any quadratic extension with K totally imaginary, if $\alpha \mapsto \bar{\alpha}$ represents the non-trivial element of the Galois group of K/k, if $\zeta \in K$ is a primitive nth root of unity and if ${}_K(-)_n$ denotes the nth order Legendre symbol on the integers of K (for which see, for example, [**1**, pp. 81–93]) then the function

$$\alpha \mapsto {}_K\left(\frac{\bar{\alpha}}{\alpha}\right)_n$$

defined on a suitable domain is a congruence function if and only if $\zeta\bar{\zeta} = 1$. We shall not give the rather involved precise formulation nor the proof here since this would distract us from the main purpose of the introduction. However the reader can refer to Proposition 4.1 below which gives in a special case a formula from which the assertion follows. Alternatively one can use the method of proof of [**7**, Proposition 2.2] and [**1**, A.23], [**13**] to analyze this question.

Before we describe the nature of the results which can be obtained we do note that although one can formulate many analogous questions when K/k is no longer quadratic no progress has been made on any of these.

The objective of this paper is to use the theory of automorphic forms, in particular the ideas of T. Kubota, to investigate this phenomenon more closely when $n = 3$, $k = \mathbf{Q}(\omega)$ and so K is of the form $k(\sqrt{D})$. This is the simplest case in which

$$\alpha \mapsto {}_K\left(\frac{\bar{\alpha}}{\alpha}\right)_n$$

is not a congruence function. We have chosen it for simplicity of exposition in that much of the necessary background material is already available in [**10**]. We do emphasize that the method is by no means restricted to this case.

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To describe what can be proved consider the case when $K = k(\sqrt{D})$, D primary (i.e. $D \equiv 1(3)$) and $D \equiv 0 \pmod 2$. We form the order $R \subset K$ of elements $a + b\sqrt{D}$ where a, $b \in \mathbb{Z}[\omega]$, a, $b \equiv 0$, 1, 2 (mod 3). Let $U \subset R$ be the subgroup of $R^\times$ of elements $a + b\sqrt{D} \in R^\times$, $b \equiv 0 \pmod 3$. Embed k in $\mathbf{C}$ by taking $\omega = e^{2\pi i/3}$.

Let $q : R \to \mathbf{C}$ be a function satisfying

(i) $q(0) = 0$, $q(-\alpha) = q(\alpha)$,

(ii) $q(a + b\sqrt{D})$ depends only on a, b (mod 3), and

(iii) $\sum_{a(3)} \sum_{b(3)} q(a + b\sqrt{D}) = 0$.

We form the Dirichlet series

$$\psi_q(s) = \sum_\alpha q(\alpha) \,.\, {}_K\!\left(\frac{\bar{\alpha}}{\alpha}\right)_3 \chi(\alpha) N(\alpha)^{-s}$$

where

$$\chi(\alpha) = {}_K\!\left(\frac{2\sqrt{D}}{\alpha}\right)_3^{-1}.$$

The sum is taken over all $\alpha \in R$ such that $(\alpha, \bar{\alpha}) = 1$ (this can be understood either in R or in the ring of integers of K), and taken modulo U multiplicatively. Likewise we define

$$\psi_1(s) = \sum_\alpha {}_K\!\left(\frac{\bar{\alpha}}{\alpha}\right)_3 . \chi(\alpha) \,.\, N(\alpha)^{-s}$$

where the sum is over the same set.

Let ζ_k be the Dedekind zeta-function of k. Then form with $* = 1$ or q

$$\Psi_*(s) = (2\pi)^{-6s}\Gamma(s - 1/3)\Gamma(s - 1/6)\Gamma(s)^2\Gamma(s + 1/6)\Gamma(s + 1/3)\psi_*(s)\zeta_k(6s - 2).$$

Then characteristic for what one can prove is the following:

THEOREM 1.1. *The series giving ψ_1, ψ_q are well-defined and converge in the half-plane* $\mathrm{Re}(s) > 1$. *They have analytic continuations as meromorphic functions of finite order to the entire complex plane with at most a simple pole at $s = 2/3$ in the half-plane* $\mathrm{Re}(s) \geq 1/2$.

The functions $\Psi_*(s)$ satisfy the following functional equations

$$\Psi_q(s)(1 - 3^{2-6s}) = (2^6 3^{12} |D|)^{1-2s} \Psi_q(1 - s)(1 - 3^{6s-4})$$

and

$$\Psi_1(s) = (2^6 3^9 |D|)^{1-2s} \Psi_1(1 - s).$$

At this point it is useful to note that

$$ {}_K\!\left(\frac{\overline{\alpha_1\alpha_2}}{\alpha_1\alpha_2}\right)_3 = {}_K\!\left(\frac{\bar{\alpha}_1}{\alpha_1}\right)_3 . {}_K\!\left(\frac{\bar{\alpha}_2}{\alpha_2}\right)_3 . {}_K\!\left(\frac{\bar{\alpha}_1}{\alpha_2}\right)_3 . {}_K\!\left(\frac{\bar{\alpha}_2}{\alpha_1}\right)_3 .$$

This behaviour is very analogous to that of Gauss sums (cf. [**10**]). Then the $\psi_1(s)$, $\psi_q(s)$ are analogues of the $\psi_p(s, \mu, 1)$ of [**10**]; like the latter they have no Euler product.

We shall not prove Theorem 1.1 here; it is another application of the technique used here to prove Theorem 4.3 and we merely hint at the proof. For our purposes we need a

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rather more general family of Dirichlet series, namely those of Theorem 5.1, and it is then very tedious to find the explicit functional equation.

Since it will be necessary to have reasonable bounds on the Dirichlet series generalizing ψ_1, ψ_q we have to replace the usual Phragmén–Lindelöf-functional equation argument by another. This is described in Section 3 and is the main technical question which we have to answer here. It involves point-wise estimates for Eisenstein series and these have to be as uniform as possible.

In Section 4 we show how the Dirichlet series can be extracted from Eisenstein series following a method of Hecke. From this type of result and standard Tauberian arguments one can estimate asymptotically sums of the form $\sum_{N(\alpha)\leq X} {}_K\left(\frac{\bar{\alpha}}{\alpha}\right)_3 \chi(\alpha)$ (N = absolute norm, the sum is taken modulo U).

One of the objectives of this paper was to show that $\pi \mapsto {}_K\left(\frac{\bar{\pi}}{\pi}\right)_3$ for π *prime* is not given by a congruence function. Although we do not quite achieve this goal the final result, Theorem 5.2, is essentially sufficient for it. The idea is that in order to prove the assertion it suffices to show that

$$\sum_{N(\alpha)\leq X} {}_K\left(\frac{\bar{\alpha}}{\alpha}\right)_3 \cdot \chi(\alpha)\theta(\alpha)\Lambda(\alpha) \ll X^{1-\varepsilon}$$

for some $\varepsilon > 0$, where Λ is the von Mangoldt function and θ a Dirichlet character. This type of estimate can be derived from those for

$$\sum_{\substack{N(\alpha)\leq X \\ \alpha\equiv 0(\delta)}} {}_K\left(\frac{\bar{\alpha}}{\alpha}\right)_3 \chi(\alpha)\theta(\alpha)$$

by sieve methods such as that of Vaughan used in [**4**], [**11**]. Unfortunately to apply this we would need that R had class-number 1, which does not happen.

In order to circumvent this difficulty one has to "invert" a finite number of primes so that the computations are carried out in S-integers for a suitable set S which will form a principal ideal domain. This involves no difficulty of principle but since it necessitates a more thorough analysis of the non-archimedean local case it would increase the length of this paper considerably. For this reason we are forced not to broach this question here but this deficiency shall be rectified in a future publication.

2. Preliminaries. In this section we shall establish the basic concept and notations needed later. Let $\mathbf{Q}(\omega)$, $\mathbb{Z}[\omega]$ be as in the introduction and let

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}[\omega]).$$

Let $\mathfrak{a}$ be an ideal of $\mathbb{Z}[\omega]$ and

$$\Gamma' = \{\gamma \in \Gamma : \gamma \equiv I (\mathrm{mod}\ 3)\},$$

$$\Gamma'_0(\mathfrak{a}) = \left\{\gamma \in \Gamma' : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\mathrm{mod}\ \mathfrak{a})\right\}.$$

Let $(-)$ be the cubic Legendre symbol in $\mathbb{Z}[\omega]$. On Γ' we define κ by

$$\kappa\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \left(\dfrac{c}{d}\right) & (\text{if } c \neq 0), \\ 1 & (\text{if } c = 0). \end{cases}$$

Kubota has shown that this is a homomorphism whose kernel is a non-congruence subgroup. In particular the restriction of κ to $\Gamma'_0(\mathfrak{a})$ is non-trivial.

Let χ be a Dirichlet character of $\mathbb{Z}[\omega]$ to the modulus $\mathfrak{a}$; then the attribution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi(d)$$

defines a character on $\Gamma'_0(\mathfrak{a})$ which we also denote by χ.

The group $SL_2(\mathbf{C})$ acts on the upper half-space $\mathbf{H}^3$. The method by which we shall represent this action is to regard $\mathbf{H}^3$ as the following subset of the Hamiltonian quaternions HQ (with standard generators 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$)

$$\mathbf{H}^3 = \{x + y\mathbf{i} + v\mathbf{k} \in \mathrm{HQ} \mid v > 0\}.$$

We embed $\mathbf{C}$ in HQ by $\sqrt{-1} \mapsto \mathbf{i}$. Then the action of $SL_2(\mathbf{C})$ on $\mathbf{H}^3$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . \, w = (aw + b)(cw + d)^{-1}.$$

Thus Γ also acts on $\mathbf{H}^3$. The boundary of $\mathbf{H}^3$ we take to be $\mathbf{C} \cup \{\infty\}$; on this $SL_2(\mathbf{C})$ acts by the standard Möbius action.

Let Γ_∞ (resp. Γ'_∞) be the stabilizer of ∞ in Γ (resp. Γ').

The cusps of $\Gamma'_0(\mathfrak{a})$ are the $\Gamma'_0(\mathfrak{a})$ classes of $\{\gamma(\infty) \mid \gamma \in \Gamma\} = \mathbf{Q}(\omega) \cup \{\infty\}$. These are finite in number and are in one-to-one correspondence with $\Gamma'_0(\mathfrak{a})\backslash\Gamma/\Gamma_\infty$. If p is a cusp we will say that $\gamma \in \Gamma$ *represents* p if p is the class of γ in $\Gamma'_0(\mathfrak{a})\backslash\Gamma/\Gamma_\infty$.

Let

$$v : \mathbf{H}^3 \to \mathbf{R}_+^\times; \qquad x + y\mathbf{i} + t\mathbf{k} \mapsto t$$

and

$$S(A) = \{w \in \mathbf{H}^3 \mid v(w) > A\}.$$

This is preserved by Γ_∞.

We shall need:

Proposition 2.1. (i) *If* $\gamma \in \Gamma$ *and*

$$\gamma S(1) \cap S(1) = \varnothing$$

then $\gamma \in \Gamma_\infty$.

(ii) *For any* $A > 0$ *the number of right* Γ_∞ *classes in*

$$\{\gamma \in \Gamma \mid \gamma S(A) \cap S(A) = \varnothing\}$$

is finite.

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(iii) *One has*

$$\bigcup_{\gamma\in\Gamma/\Gamma_\infty} \gamma\overline{S(\sqrt{3}/2)} = \mathbf{H}^3.$$

(iv) *Let for each cusp p of $\Gamma_0'(\mathfrak{a})$ the element $\sigma_p \in \Gamma$ represent p. Then*

$$\bigcup_{\gamma\in\Gamma_0'(\mathfrak{a})} \gamma\Big(\bigcup_p \sigma_p\overline{S(\sqrt{3}/2)}\Big) = \mathbf{H}^3.$$

Proof. This is standard. In (iii) one has to observe that $\overline{S(\sqrt{3}/2)}$ contains the fundamental domain of Γ described in [**10**].

Note that a Siegel domain for Γ is a fundamental domain for Γ_∞ on $S(A)$.
Another standard fact which we shall need is

PROPOSITION 2.2. *For any $\varepsilon > 0$ one has*

$$[\Gamma' : \Gamma_0'(\mathfrak{a})] \ll N(\mathfrak{a})^{1+\varepsilon}.$$

Proof. This follows from the formulae for the index (cf. [**9**, p. IV–5]) and a standard argument (cf. [**3**, Theorem 315]).

We shall from time to time regard $\mathbf{H}^3$ as $\mathbf{C}\times\mathbf{R}_+^\times$ via

$$(x + y\mathbf{i} + v\mathbf{k}) \mapsto (x + yi, v).$$

On $\mathbf{H}^3$ we can define an invariant measure. Let m be the standard Lebesgue measure on $\mathbf{C}$. Then on $\mathbf{C}\times\mathbf{R}_+^\times$ the invariant measure is

$$d\sigma(z, v) = \frac{dm(z)\,dv}{v^3}.$$

3. Eisenstein series. The Eisenstein series which we shall use here are

$$E(w, s, \chi, \mathfrak{a}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0'(\mathfrak{a})} \bar{\kappa}(\gamma)\bar{\chi}(\gamma)v(\gamma(w))^s$$

where $\operatorname{Re}(s) > 2$, $w \in \mathbf{H}^3$ and χ is a Dirichlet character to the modulus $\mathfrak{a}$. We shall assume that $(\mathfrak{a}, \sqrt{-3}) = 1$ for simplicity.

As usual

$$\int E(w + z, s, \chi, \mathfrak{a})\,dm(z),$$

has the form

$$Vv(w)^s + \frac{\pi}{s-1}\varphi(s, \chi, \mathfrak{a})v(w)^{2-s}$$

if the integration is taken over a fundamental domain for the action of Γ_∞' on $\mathbf{C}$ and m is the usual Lebesgue measure. By a computation similar to that of [**10**, §5] one can verify the following proposition.

PROPOSITION 3.1. *One has that* $\varphi(s, \chi, \mathfrak{a}) = 0$ *if* χ^3 *is not principal. If* χ^3 *is principal then*

$$\varphi(s, \chi, \mathfrak{a}) = 2 \,.\, N(c_*)^{-s} \prod_{\mathfrak{p}|\mathfrak{a}} \left\{ \frac{1 - N\mathfrak{p}^{-1}}{1 - N\mathfrak{p}^{2-3s}} \right\} \frac{\zeta_k(3s-3)}{\zeta_k(3s-2)}$$

where ζ_k *is the Dedekind zeta function of* $\mathbf{Q}(\omega)$ *and* c_* *is an integer in* $\mathbb{Z}[\omega]$ *with minimal norm satisfying*

a) $c_* \equiv 0 \pmod{3\mathfrak{a}}$

b) $d \mapsto \left(\frac{c_*}{d}\right)\bar{\chi}(d)$ *is principal.*

Next we need the basic analytic properties of the Eisenstein series. These we can excerpt from [**5**, II.1, I.2] and we shall merely quote what we need.

PROPOSITION 3.2. *Let* $L_k(s, \chi)$ *denote the Dedekind function over* k *associated with a primitive Dirichlet character* χ. *Then the function*

$$s \mapsto L_k(3s-2, (\chi^3)_1) E(w, s, \chi, \mathfrak{a}),$$

where $(\chi^3)_1$ *is the primitive character extending* χ^3, *has an analytic continuation to a meromorphic function in the entire plane. If* χ^3 *is principal it has a pole at* $s = 4/3$; *apart from this there are no poles in* $\mathrm{Re}(s) > 2/3$. *As a function of* w *this function is real-analytic at any* s *which is not a pole.*

An immediate consequence of this is the possibility to expand $E(w, s, \chi, \mathfrak{a})$ in a Fourier series about any cusp.

Let

$$\begin{aligned} v_T^s(w) &= v(w)^s \quad \text{if} \quad v(w) \le T, \\ &= 0 \qquad\quad \text{if} \quad v(w) > T. \end{aligned}$$

We define for $T > 1$

$$E^T(w, s, \chi, \mathfrak{a}) = \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma'_0(\mathfrak{a})} \bar{\kappa}(\gamma) \bar{\chi}(\gamma) v_T^s(\gamma w)$$

and

$$E_*^T(w, s, \chi, \mathfrak{a}) = E(w, s, \chi, \mathfrak{a}) - E^T(w, s, \chi, \mathfrak{a}).$$

One has

$$\begin{aligned} E_*^T(w, s, \chi, \mathfrak{a}) &= v(w)^s \quad \text{for} \quad w \in S(T), \\ &= 0 \qquad\quad \text{for} \quad w \in \mathbf{H}^3 - \bigcup_{\gamma \in \Gamma'_0(\mathfrak{a})} \gamma S(T). \end{aligned}$$

Now one has the following special case of the first Maass–Selberg relation:

PROPOSITION 3.3. *If* $\mathrm{Re}(s) > 1$ *then* $E^T(\cdot, s, \chi, \mathfrak{a})$ *is square-integrable on* $\Gamma'_0(\mathfrak{a}) \backslash \mathbf{H}^3$. *If* χ^3 *is not principal then*

$$\int_{\Gamma'_0(\mathfrak{a}) \backslash \mathbf{H}^3} |E^T(w, s, \chi, \mathfrak{a})|^2 \, d\sigma(w) = V \frac{T^{s+\bar{s}-2}}{s + \bar{s} - 2}.$$

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If χ^3 *is principal then*

$$\int_{\Gamma_0'(\mathfrak{a})\backslash\mathbf{H}^3} |E^T(w, s, \chi, \mathfrak{a})|^2\, d\sigma(w) = V\frac{T^{s+\bar{s}-2}}{s+\bar{s}-2} - \frac{T^{\bar{s}-s}}{s-\bar{s}}\frac{\pi}{s-1}\varphi(s, \chi, \mathfrak{a}) + \frac{T^{s-\bar{s}}}{s-\bar{s}}\frac{\pi}{\bar{s}-1}\varphi(\bar{s}, \chi, \mathfrak{a})$$

if $\mathrm{Im}(s) \neq 0$ *whereas if* $\mathrm{Im}(s) = 0$, $s \neq 4/3$ *it is*

$$V\frac{T^{2s-2}}{2s-2} + \left(\frac{2\pi}{s-1}\log T + \frac{\pi}{(s-1)^2}\right)\varphi(s, \chi, \mathfrak{a}) + \frac{2\pi}{s-1}\varphi'(s, \chi, \mathfrak{a}).$$

This follows by standard arguments from Proposition 3.1; see [**10**, (3.10)]. From it we derive the following estimate.

COROLLARY 3.4. *For any* $\varepsilon > 0$ *one has in* $\mathrm{Re}(s) \geq 1 + \varepsilon$

a) *if* χ^3 *is not principal then*

$$\int |E^T(w, s, \chi, \mathfrak{a})|^2\, d\sigma(w) \ll T^{2\,\mathrm{Re}(s-1)}$$

b) *if* χ^3 *is principal then*

$$\int |(s-4/3)E^T(w, s, \chi, \mathfrak{a})|^2\, d\sigma(w) \ll |s|^2 T^{2\,\mathrm{Re}(s-1)}.$$

The implied constants depend only on ε.

Proof. This is clear from Proposition 3.4 if χ^3 is not principal. If χ^3 is principal one remarks that from Proposition 3.1 $\varphi(s, \chi, \mathfrak{a})(s-4/3)$ is bounded by

$$2\,|\zeta_k(3s-3)(s-4/3)|\,.\,\zeta_k(3\,\mathrm{Re}(s)-2)\zeta_k(6\,\mathrm{Re}(s)-4)^{-1}.$$

The usual Phragmén–Lindelöf bound for $\zeta_k(3s-3)$ shows that it is $O(\mathrm{Im}(s)^{2-3\varepsilon})$ in the region in question if $|\mathrm{Im}(s)| \geq 1$. From Cauchy's inequality one derives the same bound for the derivative. If $|\mathrm{Im}(s)| \leq 1$ both the function and its derivative are bounded. From these remarks the quoted estimate follows.

It is worth remarking that these estimates can be sharpened but this is inessential for us. Our objective is to convert this L^2 estimate into a point-wise one. For this we need a formula essentially going back to A. Selberg but first published by H. Neuenhöffer [**8**]. We need the notion of the resolvent kernel.

Let

$$L(w, w') = (|z-z'|^2 + (v+v')^2)/vv'$$

where $w = (z, v)$, $w' = (z', v')$ are two points of $\mathbf{H}^3$. This is the standard point-pair invariant in the sense of Selberg. Let r_t be the function defined on $[4, \infty[$ by

$$r_t\left(\frac{(v+1)^2}{v}\right) = \frac{1}{2\pi}\frac{v^{1-t}}{v-v^{-1}} \qquad (v > 1).$$

Then $r_t(L(w, w'))$ is the resolvent kernel for the Laplace operator on $\mathbf{H}^3$, [**2**, Ch I, §5]. Let

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us form

$$R(w, w'; t) = \sum_{\gamma \in \Gamma_0'(\mathfrak{a})} \bar{\kappa}(\gamma)\bar{\chi}(\gamma) r_t(L(\gamma w, w'))$$

for $\mathrm{Re}(t) > 2$.

One has now:

PROPOSITION 3.5. *If* $\mathrm{Re}(s) > 1$, $\mathrm{Re}(t) > 2$ *we have*

$$\int_{\Gamma_0'(\mathfrak{a})\backslash \mathbf{H}^3} R(w, w'; t) E^T(w', s, \chi, \mathfrak{a})\, d\sigma(w')$$
$$= \frac{1}{s(2-s) - t(2-t)} E^T(w, s, \chi, \mathfrak{a}) + \frac{1}{2} \frac{T^{s-t}}{(t-1)(s-t)} E^T(w, t, \chi, \mathfrak{a}) + \frac{1}{2} \frac{T^{s+t-2}}{(t-1)(s+t-2)} E_*^T(w, 2-t, \chi, \mathfrak{a}).$$

Proof. (Cf. [**8**, §5].) This is obtained by summing from the identity

$$\int_{\mathbf{H}^3} r_t(L(w, w')) v_T^s(w')\, d\sigma(w')$$
$$= \frac{1}{s(2-s) - t(2-t)} v_T^s(w) + \frac{1}{2} v_T^t(w) \frac{T^{s-t}}{(t-1)(s-t)} + \frac{1}{2}(v^{2-t}(w) - v_T^{2-t}(w)) \frac{T^{s+t-2}}{(t-1)}(s+t-2)$$

which is easily proved by carrying out the integrations. This argument is valid if $\mathrm{Re}(s) > 2$; the general case follows by analytic continuation.

Another approach is by way of Stokes' theorem.

We now need an estimate for $R(w, w'; t)$.

PROPOSITION 3.6. *Suppose that* $K \subset]2, \infty[$ *is a compact set. Let p be a cusp of* $\Gamma_0'(\mathfrak{a})$ *and let* $\sigma_p \in \Gamma$ *represent it. Let* $A > 0$ *be given. Then one has for t with* $\mathrm{Re}(t) \in K$, $w \in S(A)$

$$\int_{\Gamma_0'(\mathfrak{a})\backslash \mathbf{H}^3} |R(\sigma_p w, w'; t)|^2\, d\sigma(w') \ll v(w)^2$$

for $w \in S(A)$. *The implied constant depends on K, A but not on* $\mathfrak{a}$ *or p.*

Proof. One has that

$$\int R(w_1, w', t)\overline{R(w_2, w', t)}\, d\sigma(w') = \sum_{\gamma \in \Gamma_0'(\mathfrak{a})} \bar{\kappa}(\gamma)\bar{\chi}(\gamma) r_t^{(2)}(L(\gamma w_1, w_2))$$

where

$$r_t^{(2)}(L(w_1, w_2)) = \int r_t(L(w_1, w')) r_{\bar{t}}(L(w', w_2))\, d\sigma(w').$$

This can be given easily with the aid of the Hilbert resolvent equation. If $t \neq \bar{t}$ then

$$r_t^{(2)}(L) = (t - \bar{t})^{-1}(2 - t - \bar{t})^{-1}(r_t(L) - r_{\bar{t}}(L)).$$

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This we estimate by the mean value theorem and we see that with $\tau=\mathrm{Re}(t)$

$$r_t^{(2)}\left(\frac{(v+1)^2}{v}\right)\leq\frac{v^{1-\tau}}{2\tau-2}\cdot\frac{1}{2\pi}\frac{\log v}{v-v^{-1}}\qquad(v>1).$$

Define ρ_τ by

$$\rho_\tau\left(\frac{(v+1)^2}{v}\right)=\frac{v^{1-\tau}}{2\tau-2}\cdot\frac{1}{2\pi}\cdot\frac{\log v}{v-v^{-1}}\qquad(v>1).$$

It follows that we can majorize the integral to be estimated by

$$\sum_{\gamma\in\Gamma}\rho_\tau(L(w,\gamma w))\qquad(w\in S(A)).$$

We split this into two parts, the sum over Γ_∞ and that over $\Gamma-\Gamma_\infty$. The latter is

$$O\left(\sum_{\gamma\in\Gamma_\infty\backslash(\Gamma-\Gamma_\infty)/\Gamma_\infty}\int_{\mathbf{C}}\int_{\mathbf{C}}\rho_\tau(w+z,\gamma(w+z'))\,dm(z)\cdot dm(z')\right)$$

since $\rho_\tau(w_1,w_2)$ varies by at most a constant multiple if w_1 or w_2 runs over a hyperbolic sphere of given centre and radius. Then this expression is easily simplified and estimated. One obtains $o(v(w)^{-2})$.

The sum over Γ_∞ is with $v=v(w)$

$$\sum_{n\in\mathbb{Z}[\omega]}\rho_\tau\left(\frac{|n|^2+4v^2}{v^2}\right)=\left\{\sum_{n\in\mathbb{Z}[\omega]}\rho_\tau\left(\frac{|n|^2}{v^2}+4\right)\frac{1}{v^2}\right\}v^2.$$

As $v\to\infty$ the inner sum approximates the integral

$$\frac{2}{\sqrt{3}}\int_{\mathbf{C}}\rho_\tau(|z|^2+4)\,dm(z),$$

a continuous function of τ. Since we can assume that $A\gg1$ the conclusion follows from this.

We now come to theorem which is the objective of this section.

THEOREM 3.7. *Suppose that $\varepsilon>0$ is given. Then if p is a cusp of $\Gamma_0'(\mathfrak{a})$ and if σ_p represents it one has for $w\in S(\sqrt{3}/2)$, and s with $1+\varepsilon\leq\mathrm{Re}(s)\leq2-\varepsilon$*

$$E^1(w,s,\chi,\mathfrak{a})\ll v(w)\cdot|s|,$$
$$E(\sigma_p(w),s,\chi,\mathfrak{a})\ll v(w)\cdot|s|\qquad(p\neq\infty)$$

if χ^3 is not principal. If χ^3 is principal the same estimates hold if the left-hand sides are multiplied by $\dfrac{s-4/3}{s}$. The implied constants depend only on ε.

Before proving this we remark that this is neither the most precise nor the most complete result which can be obtained by these methods. The virtue of the formulation given here is its simplicity; it suffices for our purposes.

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Proof. We use the formula of Proposition 3.5 in the form

$$E^T(w, s, \chi, \mathfrak{a}) = (s(2-s) - t(2-t)) \int R(w, w'; t) E^T(w', s, \chi, \mathfrak{a})\, d\sigma(w')$$
$$-\frac{1}{2}\frac{T^{s-t}(2-s-t)}{(t-1)} E^T(w, t, \chi, \mathfrak{a}) - \frac{1}{2}\frac{T^{s+t-2}(s-t)}{(1-t)} E^T_*(w, 2-t, \chi, \mathfrak{a}).$$

We take $t = 2 + \varepsilon + i\,.\,\mathrm{Im}(s)$. To investigate the case when $w \in S(\sqrt{3}/2)$ we take $T = 1$. By the Cauchy–Schwarz inequality the first term on the right-hand side is

$$O(|s| \cdot v(w))$$

with the natural modification if χ^3 is principal. Here we have used Corollary 3.4 and Proposition 3.6. The second term is $O(v^{2-\mathrm{Re}(t)})$ and the third term is $O(|s| \cdot v^{2-\mathrm{Re}(t)})$. From these results the first estimate follows. The estimate used here for $E^T(w, t, \chi, \mathfrak{a})$ is

$$E^T(w, t, \chi, \mathfrak{a}) \le \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma'_0(\mathfrak{a})} v_T^{\mathrm{Re}(t)}(\gamma(w))$$

and the left-hand side is a standard, well-understood Eisenstein series.

Now we consider the case $p \neq \infty$. The same method, with the same choice for t and T yields

$$E(\sigma_p(w), s, \chi, \mathfrak{a}) \ll |s|\,.\, v(w) + v(w)^{2-\mathrm{Re}(t)}$$

where we have noted that E^1_* is zero on $\sigma_p(S(\sqrt{3}/2))$ and have used the same type of estimate as above for $E^1(\sigma_p w, t, \chi, \mathfrak{a})$,

$$E^1(\sigma_p w, t, \chi, \mathfrak{a}) \le \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma'_0(\mathfrak{a})} v_1^{\mathrm{Re}(t)}(\gamma\sigma_p(w))$$
$$\le \sum_{\gamma \in \Gamma_\infty \backslash (\Gamma - \sigma_p^{-1}\Gamma_\infty \sigma_p)} v_1^{\mathrm{Re}(t)}(\gamma\sigma_p(w)).$$

This is again a standard Eisenstein series and we obtain the quoted estimate as before.

4. A Hecke integral. We shall now use the results just proved to gain information on the Dirichlet series in which we are interested.

Let D be square-free and a non-square and let A be an integer of $\mathbb{Z}[\omega]$. We form the following order in $\mathbf{Q}(\omega)(\sqrt{D})$

$$R(A\sqrt{D}) = \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}[\omega], \qquad b \equiv 0(3A), \qquad a \equiv 0, 1, 2(3)\}.$$

Write K_D for $\mathbf{Q}(\omega)(\sqrt{D})$. Let $\mathfrak{a} = (3A)$.

We begin with the following result.

Proposition 4.1. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'_0(\mathfrak{a})$ *and let D be as above. Let* ${}_D(-)$ *be the cubic Legendre symbol in* K_D. *Let* $\alpha = c\sqrt{D} + d$. *Let * denote the non-trivial automorphism of*

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K_D/k. Then one has $\alpha \in R(A\sqrt{D})$ and

$$\kappa\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = {}_D\left(\frac{\alpha^*}{\alpha}\right)^{-1} \cdot \left(\frac{2}{N_{K_D/k}(\alpha)}\right)\left(\frac{D}{d}\right)$$

if $(\alpha, 2) = 1$. Moreover suppose $\alpha' \in R(A\sqrt{D})$ is given, $\alpha' = c'\sqrt{D} + d'$. Then there exists $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(\mathfrak{a})$ if and only if there is no element $y \in \mathbb{Z}[\omega]$ so that α' is divisible by y in $R(\sqrt{D})$.

If this last condition is satisfied we shall say that *α' has no factor in $\mathbb{Z}[\omega]$.*

Proof. The only part that needs proof is the formula for $\kappa\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$. The formula is trivial if $c = 0$ so we may assume that $c \neq 0$. Then

$$\begin{aligned} \kappa\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \left(\frac{c}{d}\right) \\ &= \left(\frac{c^2D - d^2}{d}\right)^{-1} \cdot \left(\frac{D}{d}\right) \\ &= \left(\frac{d}{N_{K_D/k}(\alpha)}\right)^{-1} \cdot \left(\frac{D}{d}\right) \end{aligned}$$

by the reciprocity law. By the "transfer property" this is

$$\begin{aligned} {}_D\left(\frac{d}{\alpha}\right)^{-1} \cdot \left(\frac{D}{d}\right) &= {}_D\left(\frac{(\alpha + \alpha^*)/2}{\alpha}\right)^{-1}\left(\frac{D}{d}\right) \\ &= {}_D\left(\frac{\alpha^*}{\alpha}\right)^{-1} \cdot \left(\frac{2}{N_{K_D/k}(\alpha)}\right) \cdot \left(\frac{D}{d}\right) \end{aligned}$$

as required.

Note that the requirement $(2, \alpha) = 1$ was only required to justify the final step.

Corollary 4.2. *If $\begin{pmatrix} d & cD \\ c & d \end{pmatrix} \in \Gamma_0'(\mathfrak{a})$ then*

$$\kappa\left(\begin{pmatrix} d & cD \\ c & d \end{pmatrix}\right) = 1.$$

Proof. Here $\alpha = c\sqrt{D} + d$ is a unit. As $\alpha\alpha^*$ is a unit of $\mathbb{Z}[\omega]$ and as $\alpha\alpha^* \equiv 1 (\bmod 3)$ we have $\alpha\alpha^* = 1$. Thus

$$\begin{aligned} \kappa\left(\begin{pmatrix} d & cD \\ c & d \end{pmatrix}\right) &= \left(\frac{D}{d}\right) \\ &= \left(\frac{D}{d^2}\right)^2. \end{aligned}$$

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But $d^2-c^2D=1$; so $d^2\equiv 1 \pmod{9D}$ and hence

$$\left(\frac{D}{d^2}\right)=1.$$

This proves the corollary.

Let χ be a Dirichlet character of $\mathbb{Z}[\omega]$ to the modulus A; then χ extends to $R(A\sqrt{D})$ by

$$\chi(a+b\sqrt{D})=\chi(a).$$

We form now the Dirichlet character φ of $R(A\sqrt{D})$ by

$$\varphi(a+b\sqrt{D})=\chi(a)^{-1}\left(\frac{N_{K_D/k}(\alpha)}{2}\right)^{-1}\left(\frac{a}{D}\right)^{-1}$$

defined for $a+b\sqrt{D}\in R(A\sqrt{D})$, $(a, A)=1$.

Let $U(A\sqrt{D})=R(A\sqrt{D})^{\times}$. We let $u\in U(A\sqrt{D})$ be $u=c\sqrt{D}+d$ and let $\gamma=\begin{pmatrix} d & cD \\ c & d\end{pmatrix}\in\Gamma_0'(\mathfrak{a})$. Let $g=\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix}\in\Gamma_0'(\mathfrak{a})$, $\alpha=c'\sqrt{D}+d'$. Then

$$g\gamma=\begin{pmatrix} a'' & b'' \\ c'' & d''\end{pmatrix}$$

where

$$c''\sqrt{D}+d''=\alpha u.$$

From $\kappa(g\gamma)=\kappa(g)\kappa(\gamma)$ and (Corollary 4.2) $\kappa(\gamma)=1$ we have

$${}_D\!\left(\frac{\alpha^* u^*}{\alpha u}\right)={}_D\!\left(\frac{\alpha^*}{\alpha}\right).$$

We assume now that χ is chosen so that $\varphi\mid U(A\sqrt{D})=1$. If $s\in\mathbf{C}$, $\mathrm{Re}(s)>1$ we define

$$\psi_{A\sqrt{D}}(s,\varphi)=\sum {}_D\!\left(\frac{\alpha^*}{\alpha}\right).\varphi(\alpha).N(\alpha)^{-s}$$

where we suppose that $2\mid A$ and α has no factor from $\mathbb{Z}[\omega]$; the sum is modulo $U(A\sqrt{D})$ multiplicatively. This clearly converges and is well-defined. The objective of this section is to prove the following theorem.

THEOREM 4.3. *The function $\psi_{A\sqrt{D}}(s,\varphi)$ has an analytic continuation as a meromorphic function of finite order to the entire complex plane. In the region $\mathrm{Re}(s)\geq\frac{1}{2}$ it is entire if φ^3 is not principal and has at most a simple pole at $s=2/3$ if φ^3 is principal. Let $\varepsilon>0$ be given. Then if $\mathrm{Re}(s)=\sigma$, $\frac{1}{2}+\varepsilon<\sigma<1+\varepsilon$ one has if φ^3 is not principal*

$$\psi_{A\sqrt{D}}(s,\varphi)\ll|s|^{3(1-\sigma+\varepsilon)}N(A^2D)^{(1-\sigma+2\varepsilon)}$$

and if φ^3 is principal

$$\psi_{A\sqrt{D}}(s,\varphi)(s-2/3)\ll|s|^{3(1-\sigma+\varepsilon)+1}N(A^2D)^{(1-\sigma+\varepsilon)}.$$

The implied constants depend only on ε.

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REMARK. One can show that $L_k(6s-2, \chi^3)\psi_{A\sqrt{D}}(s, \rho)$ satisfies a functional equation of the usual kind with gamma factor

$$(2\pi)^{-6s}\Gamma(s-1/3)\Gamma(s-1/6)\Gamma(s)^2\Gamma(s+1/6)\Gamma(s+1/3).$$

The form is rather complicated; the example which we have given, Theorem 1.1, may be proved by using the technique described here supplemented by [**10**, (3.13), (5.19), (5.20), (5.21), (5.24), Proposition 5.1].

Proof. This is based on an idea of Hecke's. Let

$$H_D = \frac{1}{(2\sqrt{D})^{1/2}}\begin{pmatrix} \sqrt{D} & -\sqrt{D} \\ 1 & 1 \end{pmatrix};$$

then

$$\{H_D(t\mathbf{k}) \mid t \in \mathbf{R}_+^\times\}$$

is the geodesic joining $\sqrt{D}$ and $-\sqrt{D}$ in $\mathbf{H}^3$. If $u = c\sqrt{D}+d \in U(A\sqrt{D})$ then

$$H_D\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}H_D^{-1} = \begin{pmatrix} d & cD \\ c & d \end{pmatrix}$$
$$\in \Gamma_0'(\mathfrak{a}).$$

Thus $U(A\sqrt{D})$ acts on the geodesic by

$$\begin{pmatrix} d & cD \\ c & d \end{pmatrix}H_D(t\mathbf{k}) = H_D(|u|^2 t\,.\,\mathbf{k}).$$

Hence the function

$$t \mapsto E(H_D(t\,.\,\mathbf{k}), s, \chi, \mathfrak{a})$$

is invariant under $t \mapsto |u|^2 t$ ($u \in U(A\sqrt{D})$). Let us regard $U(A\sqrt{D})$ as acting on $\mathbf{R}_+^\times$. Then

$$\int_{U(A\sqrt{D})\backslash\mathbf{R}_+^\times} E(H_D(t\,.\,\mathbf{k}), s, \chi, \mathfrak{a})t^{-1}\,dt$$

is defined. We shall show that it is equal to

$$\frac{\Gamma(s/2)^2}{\Gamma(s)}\,.\,2^{s-1}\,.\,|D|^{s/2}\,.\,\psi_{A\sqrt{D}}(s/2, \varphi),$$

when $\mathrm{Re}(s) > 2$.

To prove this we note that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0'(\mathfrak{a})$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}H_D = \frac{1}{(2\sqrt{D})^{1/2}}\begin{pmatrix} * & * \\ \alpha & \alpha^* \end{pmatrix}$$

where, as before, $\alpha = c\sqrt{D}+d$. Thus

$$v\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}H_D(t\mathbf{k})\right) = 2\,|D|^{1/2}\,t/(|\alpha|^2\,t^2+|\alpha^*|^2),$$

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and as $2 \mid A$

$$\kappa\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = {}_D\left(\frac{\alpha^*}{\alpha}\right)^{-1} \varphi(\alpha)^{-1}.$$

Hence using these expressions in the series definition of $E(H_D(t\mathbf{k}), s, \chi, \mathfrak{a})$ and grouping together the terms with $\alpha u (u \in U(A\sqrt{D}))$ we see that the original integral becomes

$$\sum_{\alpha} \int_0^\infty {}_D\left(\frac{\alpha^*}{\alpha}\right) . \varphi(\alpha) . (2\,|D|^{1/2}\, t/(|\alpha|^2\, t^2 + |\alpha^*|^2))^s t^{-1}\, dt$$

where the sum is over all $\alpha \in R(A\sqrt{D})$ with no factors from $\mathbb{Z}[\omega]$ modulo $U(A\sqrt{D})$. Now

$$\int_0^\infty (t/(|\alpha|^2\, t^2 + |\alpha^*|^2))^s t^{-1}\, dt = \tfrac{1}{2}\, |\alpha\alpha^*|^{-s}\Gamma(s/2)^2/\Gamma(s)$$

and

$$|\alpha\alpha^*| = N(\alpha)^{1/2}$$

so that the formula claimed above follows.

From this formula all the assertions concerning the analytic continuation and the position of the poles of $\psi_{A\sqrt{D}}(s, \varphi)$ made in the theorem follow. We have therefore only to prove the estimates.

We shall do this by estimating the integral

$$\int_{U(A\sqrt{D})\backslash \mathbf{R}_+^\times} E(H_D(t\mathbf{k}), s, \chi, \mathfrak{a}) t^{-1}\, dt$$

as

$$\operatorname{meas}(U(A\sqrt{D})\backslash \mathbf{R}_+^\times) . \sup |E(H_D(t\mathbf{k}), s, \chi, \mathfrak{a})|.$$

We note that

$$\begin{aligned}[U(\sqrt{D}) : U(A\sqrt{D})] &\leq [\Gamma' : \Gamma_0'(\mathfrak{a})] \\ &\ll_\varepsilon N(A)^{1+\varepsilon}\end{aligned}$$

by Proposition 2.2. Also one has that up to a constant $\operatorname{meas}(U(\sqrt{D})\backslash \mathbf{R}_+^\times)$ is the regulator of K_D. We can estimate this by

$$\operatorname{meas}(U(\sqrt{D})\backslash \mathbf{R}_+^\times) \ll N(D)^{1/2+\varepsilon}$$

by the simple part of the Brauer–Siegel theorem [**6**, XVI, §1].

We turn now to the estimation of $E(H_D(t\mathbf{k}), s, \chi, \mathfrak{a})$. Let $t \in \mathbf{R}_+^\times$ and find g so that $H_D(t\mathbf{k}) \in g^{-1}S(\sqrt{3}/2)$. Then $g(H_D(t\mathbf{k}))$ lies on the geodesic joining $g(\sqrt{D})$ and $g(-\sqrt{D})$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then

$$\begin{aligned}|g(\sqrt{D}) - g(-\sqrt{D})| &= \frac{2\,|D|^{1/2}}{|c^2 D - d^2|} \\ &\leq 2\,|D|^{1/2}.\end{aligned}$$

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Thus

$$v(g(H_D(t.\mathbf{k})))\leq 2\,|D|^{1/2},$$

and so, by Proposition 3.7,

$$\|E(H_D(t.\mathbf{k}),s,\chi,\mathfrak{a})\|\ll|D|^{1/2}\,|s|+|D|^{\mathrm{Re}(s)/2}$$

with the natural interpretation if χ^3 is principal.

Thus we see that

$$\int_{U(A\sqrt{D})\backslash\mathbf{R}_+^\times} E(H_D(t.\mathbf{k}),s,\chi,\mathfrak{a})t^{-1}\,dt$$

is bounded by

$$O(N(D)^{\mathrm{Re}(s)/2}\,.\,N(A^2D)^{1+\varepsilon})$$

on $\mathrm{Re}(s)=1+\varepsilon$. It follows that on the line $\mathrm{Re}(s)=\dfrac{1+\varepsilon}{2}$

$$\psi_{A\sqrt{D}}(s,\varphi)\ll_\varepsilon|s|^{3/2}\,N(A^2D)^{1/2+\varepsilon}.$$

One has however on $\mathrm{Re}(s)=1+\varepsilon/2$

$$\begin{aligned}\psi_{A\sqrt{D}}(s,\varphi)&\ll\sum_\alpha N(\alpha)^{-(1+\varepsilon/2)}\\&\leq\zeta_{K_D}(1+\varepsilon/2).\end{aligned}$$

This one can estimate by noting that $\zeta_{K_D}(s)$ can be written in the form $\zeta_k(s)\,.\,L_k(s,\chi_D)$ for a certain character χ_D from which one has

$$\begin{aligned}\zeta_{K_D}(1+\varepsilon/2)&<\zeta_k(1+\varepsilon/2)^2\\&\ll1.\end{aligned}$$

We can now apply Rademacher's form of the Phragmén–Lindelöf theorem [**12**, §33] to deduce that if $\frac{1}{2}+\frac{\varepsilon}{2}<\mathrm{Re}(s)<1+\frac{\varepsilon}{2}$, $\sigma=\mathrm{Re}(s)$,

$$\psi_{A\sqrt{D}}(s,\varphi)\ll_\varepsilon N(A^2D)^{(1-\sigma)+2\varepsilon}\,|s|^{3(1-\sigma)+3\varepsilon}$$

with the appropriate interpretation if φ^3 is principal. This easily yields the assertion of the theorem.

5. Final results. We have to sharpen the results of the last section, so that these could be used in conjunction with sieve methods. We retain the notations of the last section. Let $\delta\in R(A\sqrt{D})(\delta,A\sqrt{D})=1$ be without factors from $\mathbb{Z}[\omega]$. Form

$$\psi_{A\sqrt{D}}(s,\varphi,\delta)=\sum_{\alpha\equiv0(\mathrm{mod}\,\delta)}\,{}_D\!\left(\frac{\alpha^*}{\alpha}\right)\varphi(\alpha)\,.\,N(\alpha)^{-s}$$

where the conditions of the summation are the same as before. We need the following generalization of Theorem 4.3.

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THEOREM 5.1. *The function $\psi_{A\sqrt{D}}(s,\varphi)$ has an analytic continuation as a meromorphic function of finite order to the entire complex plane. In the region* $\mathrm{Re}(s)\geq\frac{1}{2}$ *it is entire if φ^3 is not principal and has at most a simple pole at $s=2/3$ if φ^3 is principal. Let $\varepsilon>0$ be given. Then if* $\mathrm{Re}(s)=\sigma$, $\frac{1}{2}+\varepsilon<\sigma<1+\varepsilon$ *one has if φ^3 is not principal*

$$\psi_{A\sqrt{D}}(s,\varphi,\delta)\ll|s|^{3(1-\sigma+\varepsilon)}N(A^2D)^{(1-\sigma+2\varepsilon)}N(\delta)^{1-2\sigma+2\varepsilon}$$

and if φ^3 is principal

$$(s-\tfrac{2}{3})\psi_{A\sqrt{D}}(s,\varphi,\delta)\ll|s|^{3(1-\sigma+\varepsilon)+1}N(A^2D)^{(1-\sigma+2\varepsilon)}N(\delta)^{1-2\sigma+2\varepsilon}.$$

Again the implied constants depend only on ε.

Proof. This we prove by a modification of the previous argument, and we need only explain the necessary changes.

Let $\delta=r\sqrt{D}+s$ and choose $\begin{pmatrix}p & q\\ r & s\end{pmatrix}\in\Gamma_0'(\mathfrak{a})$. Instead of integrating over the geodesic $\{H_D(t\,.\,\mathbf{k})\mid t\in\mathbf{R}_+^\times\}$ (modulo units) we shall integrate over the geodesic

$$\begin{pmatrix}p & q\\ r & s\end{pmatrix}(\{H_D(t\,.\,\mathbf{k})\mid t\in\mathbf{R}_+^\times\}),$$

again modulo units.

The group which we shall use is $\Gamma_0'(\mathfrak{a}\Delta)$ where $\Delta=N_{K_D/k}(\delta)$. The subgroup fixing the geodesic is

$$\left\{\begin{pmatrix}p & q\\ r & s\end{pmatrix}\begin{pmatrix}\rho & \sigma D\\ \sigma & \rho\end{pmatrix}\begin{pmatrix}p & q\\ r & s\end{pmatrix}^{-1}\,\middle|\,\sigma\equiv0(\mathfrak{a})\right\}$$

since the 21-entry of the matrix here is $\sigma\Delta$. Thus $\rho+\sigma\sqrt{D}\in U(A\sqrt{D})$.

Let now $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0'(\mathfrak{a}\Delta)$. Then

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}p & q\\ r & s\end{pmatrix}=\begin{pmatrix}* & *\\ cp+dr & cq+ds\end{pmatrix}$$

and

$$\begin{aligned}(cp+dr)\sqrt{D}+(cq+ds)&\equiv d(r\sqrt{D}+s) &&\pmod{\Delta}\\ &\equiv d\delta &&\pmod{\Delta}.\end{aligned}$$

Write now

$$\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}p & q\\ r & s\end{pmatrix}.$$

It follows that

$$c'\sqrt{D}+d'\equiv0\qquad\pmod{\delta}.$$

Conversely if $c'\sqrt{D}+d'\in R(A\sqrt{D})$ without factors from $\mathbb{Z}[\omega]$, and $\delta\mid c'\sqrt{D}+d'$ then

$$(c'\sqrt{D}+d')\,.\,(-r\sqrt{D}+s)\equiv0\qquad\pmod{\Delta}.$$

This becomes

$$c's\equiv d'r\qquad\pmod{\Delta}.$$

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We can find a', b' so that $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0'(\mathfrak{a})$ and it follows that $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1} \in \Gamma_0'(\mathfrak{a}\Delta)$. Moreover

$$\begin{aligned}\kappa\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \kappa\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} . \kappa\begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1} \\ &= {}_D\!\left(\frac{\alpha^*}{\alpha}\right)\left(\frac{2}{N_{K_D/k}(\alpha)}\right)\left(\frac{D}{d'}\right).\end{aligned}$$

If χ is to the modulus $\mathfrak{a}$ then

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \chi\left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) . \chi\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right)^{-1}.$$

Thus proceeding as before we see that

$$\int_{U(A\sqrt{D})\backslash \mathbf{R}_+^\times} E\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix} H_D(t\mathbf{k}), s, \chi, \mathfrak{a}\Delta\right) t^{-1}\, dt . \kappa\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right)^{-1} \chi\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right)^{-1}$$

is equal to

$$\frac{\Gamma(s/2)^2}{\Gamma(s)} 2^{s-1} |D|^{s-1} \psi_{A\sqrt{D}}(s/2, \varphi, \delta).$$

Hence we can argue as before. In particular we again obtain the estimate

$$\psi_{A\sqrt{D}}(s, \varphi, \delta) \ll |s|^{3/2} . N(A^2D)^{1/2+\varepsilon}$$

on $\mathrm{Re}(s) = \frac{1}{2} + \varepsilon$. On the other hand on $\mathrm{Re}(s) = 1 + \varepsilon$ we have now

$$\psi_{A\sqrt{D}}(s, \varphi, \delta) \ll N(D)^{\varepsilon} N(\delta)^{-1-\varepsilon}.$$

Hence we obtain for $\mathrm{Re}(s) = \sigma$, $\frac{1}{2} + \varepsilon \le \sigma \le 1 + \varepsilon$ the estimates quoted.

From this we shall now deduce:

THEOREM 5.2. *With the notations above and $\varepsilon, k > 0$ given, then, subject to $X > kN(\delta^2 A^2 D)$ one has*

$$\sum_{\substack{N(\alpha) \le X \\ \alpha \equiv 0(\delta)}} {}_D\!\left(\frac{\alpha^*}{\alpha}\right) \varphi(\alpha) \ll X^{8/9+\varepsilon} N(\delta)^{-7/9+\varepsilon} N(A^2D)^{1/9+\varepsilon}$$

where the summation is taken over $\alpha \in R(A\sqrt{D})$ without factors from $\mathbb{Z}[\omega]$ and modulo $U(A\sqrt{D})$. The implied constant depends only on ε and k.

Proof. Let

$$a_n = \sum_{\substack{N(\alpha) = n \\ \alpha \equiv 0(\delta)}} {}_D\!\left(\frac{\alpha^*}{\alpha}\right) \varphi(\alpha).$$

Then $a_n = 0$ unless $n \equiv 0 \pmod{N(\delta)}$ and $a_{nN(\delta)} \ll n^{\varepsilon}$ (with ε as given). Moreover

$$\begin{aligned}\sum_{n \ge 1} a_{nN(\delta)} . n^{-s} &\ll \zeta_{K_D}(s) \\ &\ll (s-1)^{-1}\end{aligned}$$

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as $s\downarrow 1$. We can now apply [**14,** 3.12] to our situation. Without loss of generality we may suppose that $X\in\frac{1}{2}+\mathbf{N}$, $X>N(\delta)$. We obtain

$$\sum_{n\le X/N(\delta)} a_{nN(\delta)}=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\psi_{A\sqrt{D}}(s,\varphi,\delta)X^s\frac{ds}{s}$$
$$+O(((c-1)^{-1}(X/N(\delta))^c+(X/N(\delta))^{1+\varepsilon})T^{-1})$$

for $c>1$, $T>0$. The left-hand side is the expression we seek. We shall take $c=1+\varepsilon$ so that the error term becomes

$$O((X/N(\delta))^{1+\varepsilon}T^{-1}).$$

In particular, to obtain non-trivial results we have to have that $T\gg 1$. This we shall assume henceforth.

We replace the integral by one over the three segments $c-iT$ to $5/6-iT$, $5/6-iT$ to $5/6+iT$ and $5/6+iT$ to $c+iT$.

The first and last of these can be bounded by

$$O\left(T^{-1}\int_{5/6}^{c} T^{3(1-\sigma)+3\varepsilon}N(A^2D)^{(1-\sigma+2\varepsilon)}N(\delta)^{1-2\sigma+2\varepsilon}X^{\sigma}\,d\sigma\right)$$

Since the integrand, as a function of σ, is either increasing or decreasing it is bounded by the sum of the values at the endpoints. Hence these are bounded by

$$O(T^{-1+3\varepsilon}N(A^2D)^{2\varepsilon}N(\delta)^{-1+2\varepsilon}X^c)+O(T^{-1/2+3\varepsilon}N(A^2D)^{1/6+2\varepsilon}N(\delta)^{-2/3+2\varepsilon}X^{5/6}).$$

The integral over the vertical line is bounded by

$$O\left(\int_0^T(1+t)^{-1/2+3\varepsilon}dt\,.\,N(A^2D)^{1/6+2\varepsilon}N(\delta)^{-2/3+2\varepsilon}X^{5/6}\right)$$
$$=O\left(T^{1/2+3\varepsilon}N(A^2D)^{1/6+2\varepsilon}N(\delta)^{-2/3+2\varepsilon}X^{5/6}\right).$$

Combining these we obtain

$$\sum_{n\le X/N(\delta)} a_{n.N(\delta)}\ll T^{-1+3\varepsilon}N(A^2D)^{2\varepsilon}N(\delta)^{-1+2\varepsilon}X^{1+\varepsilon}$$
$$+T^{1/2+3\varepsilon}N(A^2D)^{1/6+2\varepsilon}N(\delta)^{-2/3+2\varepsilon}X^{5/6}.$$

We can now take

$$T=N(A^2D)^{-1/9}N(\delta)^{-2/9}X^{1/9}.$$

By assumption we have that $T\gg 1$. This yields for the right-hand side

$$O(X^{8/9+\varepsilon}N(\delta)^{-7/9+2\varepsilon}N(A^2D)^{1/9+2\varepsilon}).$$

The quoted result follows.

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