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CLOSED SYMMETRIC OVERGROUPS OF S_n IN O_n

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ABSTRACT. A norm on $\mathbb{R}^n$ is said to be *permutation invariant* if its value is preserved under permutation of the coordinates of a vector. The isometry group of such a norm must be closed, contain S_n and $-I$, and be conjugate to a subgroup of O_n, the orthogonal group. Motivated by this, we are interested in classifying all closed groups G such that $\langle -I, S_n \rangle < G < O_n$. We use the theory of Lie groups to classify all possible infinite groups G, and use the theory of finite reflection groups to classify all possible finite groups G. In keeping with the original motivation, all groups arising are shown to be isometry groups. This completes the work of Gordon and Lewis, who studied the same problem and obtained the results for $n \geq 13$.

1. **Introduction.** Let S_n be the group of $n \times n$ permutation matrices. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is *permutation invariant* if $\|Px\| = \|x\|$ for any $x \in \mathbb{R}^n$ and for all $P \in S_n$. A standard example of such a norm is the ℓ_p norm, $p \geq 1$, defined by

$$\ell_p(x) = \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p}.$$

This norm is also *absolute*, *i.e.*, $\ell_p\big((x_1, \ldots, x_n)^t\big) = \ell_p\big((|x_1|, \ldots, |x_n|)^t\big)$ for all x. An example of a permutation invariant norm which is not absolute is the N_k norm, $1 < k < n$, defined by

$$N_k(x) = \max\{|x_{i_1} + \cdots + x_{i_k}| : 1 \leq i_1 < \cdots < i_k \leq n\}.$$

It is known (see [LM]) that permutation invariant norms are very useful in the study of other classes of norms on matrix spaces, and their basic properties are quite well-studied. Moreover, the isometries for the N_k norms and other permutation invariant norms have been characterized in [LM]. The purpose of this paper is to determine all possible isometry groups of a given permutation invariant norm.

Let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{R}^n$, and let $e = \sum_{i=1}^{n} e_i$. Denote by $\mathrm{GL}_n(\mathbb{R})$, or simply GL_n, the group of real $n \times n$ invertible matrices, and O_n the group of orthogonal matrices in GL_n. Evidently, a norm $\|\cdot\|$ on $\mathbb{R}^n$ is permutation invariant if and only if the isometry group G of $\|\cdot\|$ satisfies $S_n < G$. Note that for any norm $\|\cdot\|$ on $\mathbb{R}^n$, its isometry group G is closed and bounded. Further, $\|-x\| = \|x\|$ for all $x \in \mathbb{R}^n$, so that

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G must satisfy $-I \in G$. Then for the isometry group G of a permutation invariant norm, we have $\pm S_n < G$, where $\pm S_n = \langle -I, S_n \rangle$.

Let J_n, or simply J, be the $n \times n$ matrix of all 1's. We may then make use of the following theorem (see [LM, Proposition 8.1]).

THEOREM 1.1. *Let G be a bounded subgroup of* GL_n. *Then G is conjugate to a subgroup of O_n. Moreover, if $S_n < G$, then $S^{-1}GS < O_n$ for some $S = \alpha I + \beta J \in \mathrm{GL}_n$.*

Note that if $S = \alpha I + \beta J \in \mathrm{GL}_n$, then $SP = PS$ for all $P \in \pm S_n$. It follows that $S^{-1}(\pm S_n)S = \pm S_n$. Therefore, to characterize the isometry groups G of a permutation invariant norm $\|\cdot\|$ up to conjugation by matrices $S = \alpha I + \beta J \in \mathrm{GL}_n$, it is sufficient to consider groups G such that $\pm S_n < G < O_n$. Furthermore, we have the following theorem [GLo, Theorem 3.1] showing that a finite group G is the isometry group of a permutation invariant norm if and only if $\pm S_n < S^{-1}GS < O_n$ for some $S = \alpha I + \beta J \in \mathrm{GL}_n$.

THEOREM 1.2. *Let $G < O_n$ be a finite group containing $-I$. Then there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ whose isometry group is G.*

Motivated by the study of the isometry groups of permutation invariant norms, we are interested in determining all closed groups G satisfying $\pm S_n < G < O_n$. In doing this, we consider the case of infinite groups G and finite groups G separately in the next two sections. This problem has been studied by Gordon and Lewis in [GLe], and results have been obtained for $n \geq 13$. In Section 2, we use a different method to reprove the result of Gordon and Lewis for the infinite case. In Section 3, we use the theory of finite reflection groups to study the finite case. In Section 4, we compare our results with those of Gordon and Lewis, and discuss some related problems.

It is possible that some of our results can be deduced from advanced theory of Lie groups and reflection groups. The proofs presented in this paper depend only on basic results of the two subjects and elementary computations.

The standard basis of $\mathbb{R}^{n\times n}$ will be denoted by $\{E_{11}, E_{12}, \ldots, E_{nn}\}$.

2. **Infinite closed groups G satisfying $\pm S_n < G < O_n$.** Let G be an infinite closed group satisfying $\pm S_n < G < O_n$. If $n = 2$, then $G = O_2$. Thus we always assume $n \geq 3$ in this section. Some basic theory of Lie groups will be used to determine G. Note that O_n is a compact Lie group and so does G. Since $G < O_n$, the Lie algebra g of G is a subalgebra of the Lie algebra o_n of O_n, where o_n is the linear space of all $n \times n$ skew-symmetric matrices. Furthermore, since $S_n < G$, S_n acts on G by conjugation, and so g is a S_n-module under the action of conjugation, *i.e.*, $(P, A) \longmapsto P^tAP$ for any $P \in S_n$ and $A \in g$. The question of classifying G may then be approached by finding the subalgebras of o_n which are S_n-modules. To this end, we first establish the following result.

THEOREM 2.1. *Let $n \geq 3$, and let V be a non-trivial vector subspace of o_n which is a S_n-module under the action of conjugation. Then V is one of the following:*

(a) o_n;

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(b) $o_n^0 = \{S \in o_n : Se = 0\}$, *which is isomorphic to* o_{n-1};
(c) $W = \{S = (s_{ij}) \in o_n : \ s_{ij} = s_{ik} + s_{kj}, \ 1 \leq i, j, k \leq n\}$.

PROOF. First note that these three subsets of o_n are vector subspaces of o_n and S_n-modules. Now suppose that $V \not\subseteq W$. Then there exists $A \in V, A = (a_{ij})$, such that for some i, j, k, $1 \leq i, j, k \leq n$, $a_{ij} \neq a_{ik} + a_{kj}$. Since V is an S_n-module, we can permute the rows and columns of A as necessary and assume that $a_{12} \neq a_{13} - a_{23}$. We seek to simplify A as much as possible, to show that it generates under the action of S_n a basis for o_n^0.

Now consider $B = A - P^{-1}AP = (b_{ij})$, where $P \in S_n$ is obtained from I by interchanging its first and second rows. Then $B \in V$; $b_{12} = 2a_{12}$, and for $i, j \geq 3$, $b_{1i} = a_{1i} - a_{2i}$, $b_{2i} = a_{2i} - a_{1i}$, and $b_{ij} = 0$. Let $C = -B + P^{-1}BP + Q^{-1}BQ = (c_{ij})$, where $P = E_{13} + E_{31} + \sum_{i \neq 1,3} E_{ii}$, $Q = E_{23} + E_{32} + \sum_{i \neq 2,3} E_{ii} \in S_n$. Then $C \in V$; $c_{ij} = 0$ for i or $j \geq 3$; $c_{12} = c_{23} = -c_{13} = 2b_{13} - 2a_{12}$. Note that $2b_{13} - 2a_{12} = 2(a_{13} - a_{23} - a_{12}) \neq 0$. Scaling C and permuting its rows and columns appropriately, we get $C_0 = (E_{12} - E_{21}) + (E_{31} - E_{13}) + (E_{23} - E_{32}) \in o_n^0$. It is not hard to check that the orbit of C_0 under the action of S_n spans o_n^0. So $o_n^0 \subseteq V$.

If $V \neq o_n^0$, then there exists $A \in V$ such that $Ae \neq 0$, *i.e.*, some row sum of A is non-zero. We show that A generates a basis for o_n. Denote by μ_i the ith row sum of A. Choose $B = (b_{ij}) \in o_n^0$ such that $b_{ij} = a_{ij}$ for all $1 \leq i, j \leq n-1$. Let $C = A - B = (c_{ij})$. Then $C \in V$. For $1 \leq i, j \leq n-1$, $c_{ij} = 0$ and $c_{in} = \mu_i$. Since $\sum_{i=1}^n \mu_i = 0$ and for some i, $\mu_i \neq 0$, there exist non-zero μ_a, μ_b such that $\mu_a \neq \mu_b$. Permuting the rows and columns of C appropriately, we may assume that $a = 1, b = 2$. Then, as previously, let $D = C - P^{-1}CP = (d_{ij})$, where $P \in S_n$ is obtained from I by interchanging the first and second rows. Then $D \in V$ with $d_{ij} = 0$, except $d_{1n} = d_{n2} = -d_{2n} = -d_{n1} = \mu_1 - \mu_2 \neq 0$. Scaling appropriately, we may take $d_{1n} = -1$. Let $S = E_{12} - E_{21}$. Then $D + S \in o_n^0$, and hence $S \in V$. Since the orbit of S under the action of S_n spans o_n, we see that $V = o_n$.

Thus if $V \not\subseteq W$, then $V = o_n^0$ or $V = o_n$. Now consider $V \subseteq W$. Let $A = (a_{ij}) \in V$, $A \neq 0$; by permuting the rows and columns of A appropriately, we assume that $a_{12} \neq 0$. As before, let $B = A - P^{-1}AP = (b_{ij})$, where $P \in S_n$ is obtained from I by interchanging its first and second rows. Then for $i, j \geq 3$, $b_{12} = 2a_{12}$, $b_{1i} = -b_{2i} = -b_{i1} = b_{i2} = a_{12}$, and $b_{ij} = 0$. Now an element of W is determined entirely by its first column, *i.e.*, W is isomorphic to $\mathbb{R}^{n-1}$. But we see that the first columns of the matrices in the orbit of B under S_n generates a basis for $\mathbb{R}^{n-1}$. Therefore $V = W$. ■

As pointed out by the referee, Theorem 2.1 is equivalent to the assertion that o_n admits the (unique) decomposition $o_n = o_n^0 \oplus W$ into simple S_n-modules. This result may be of independent interest. In addition, this observation makes the construction of W more natural; one need consider the complement of o_n^0 in o_n.

Although there are 3 possible linear subspaces of o_n that are invariant under the action of S_n as shown in the previous theorem, we are interested only in subalgebras of o_n which are S_n-modules. The question arises, then, as which of the three vector spaces of Theorem 2.2 are closed under the Lie product. The following proposition gives an answer to this question.

PROPOSITION 2.2. *Of the three vector spaces o_n, o_n^0 and W in Theorem 2.1, only the first two are Lie algebras.*

PROOF. We know that o_n is a Lie algebra. To see that o_n^0 is closed under the Lie product, let $X, Y \in o_n^0$. Then,

$$[X, Y]e = (XY - YX)e = X(Ye) - Y(Xe) = X0 - Y0 = 0.$$

So $[X, Y] \in o_n^0$, and o_n^0 is a Lie algebra.

To see that W is not a Lie algebra, it suffices to show that $[A, B] \notin W$ for an appropriate choice of $A, B \in W$. Direct computation shows that this is the case for A with first row $(0, 1, \ldots, 1)$ and B with first row $(0, 2, 1, \ldots, 1)$. ∎

Thus we have classified the possible Lie algebras of a closed, infinite group G satisfying $\pm S_n < G < O_n$. In all cases, $o_n^0 < g$, so $\exp(o_n^0) < G$, where $\exp\colon o_n \longrightarrow O_n$ is the exponential map. Since $\pm S_n < G$, we see that G contains the subgroup

$$O_n^e = \langle \exp(o_n^0), \pm S_n \rangle = \{T \in O_n : Te = \pm e\}.$$

By the following result, we see that G is either O_n^e or O_n.

THEOREM 2.3. *Let $n \geq 3$. The group O_n^e is maximal in O_n.*

PROOF. Let G be a group such that $O_n^e < G < O_n$, with $G \neq O_n^e$. We show constructively that $G = O_n$.

First consider $A \in G \setminus O_n^e$, and let $f = e/\sqrt{n}$. Then $Af = \cos\theta f + \sin\theta u$, for some $\theta \neq k\pi$, $k \in \mathbb{Z}$, and for some $u \in f^\perp$, $\ell_2(u) = 1$. Since $\langle Af, f\rangle = \langle f, A^{-1}f\rangle$, and since $A \in O_n$, $A^{-1}f = \cos\theta f + \sin\theta v$, for some $v \in f^\perp$ with $\ell_2(v) = 1$. Since O_n^e acts transitively on $f^\perp$, there exists $T \in O_n^e$ such that $Tu = -v$. Then $A_1 = AT \in G$, and A_1 satisfies $A_1 f = \cos\theta f + \sin\theta u$ and $A_1^{-1} f = \cos\theta f - \sin\theta u$. As $f = A_1 A_1^{-1} f = \cos\theta(\cos\theta f + \sin\theta u) - \sin\theta A_1 u$, we have $A_1 u = -\sin\theta f + \cos\theta u$. Therefore A_1 has $\operatorname{span}\{f, u\}$ as an invariant subspace. Choose $S \in O_n^e$ such that $A_2 = A_1 S$ is the identity on $\{f, u\}^\perp$; then $A_2 \in G$.

Given $B \in O_n \setminus O_n^e$ with $Bf = \cos\theta f + \sin\theta v$, for some $v \in f^\perp$, $\ell_2(v) = 1$, we may construct the analogous map B_2. We see that $A_2 = S^{-1} B_2 S$ for appropriate $S \in O_n^e$, where $Su = v$ and S is the identity on $\{u, v\}^\perp$. Hence $B_2 \in G$, and since the construction of B_2 uses only elements of O_n^e, $B \in G$.

Let K be the collection of real numbers $x \in [-1, 1]$ such that there exists $D \in G$ satisfying $\langle Df, f\rangle = x$. By the discussion in the previous paragraphs, we see that $G = O_n$ if $K = [-1, 1]$.

Note that K is symmetric with respect to the origin, as if $D \in G$, then $-D \in G$. So we need only show that $[0, 1] \subseteq K$. We have from above that $A_2 \in G$, where $\langle A_2 f, f\rangle = \cos\theta$. Assume that $\cos\theta \geq 0$; if not, consider $-A_2$. Then we will show that $[\cos 2\theta, 1] \subseteq K$. Inductively, $[\cos 2^k\theta, 1] \subseteq K$, for all $k \in \mathbb{N}$. But for some k, $\cos 2^k\theta < 0$, so $[0, 1] \subseteq K$.

Now, given A_2 as above, with $A_2 f = \cos\theta f + \sin\theta u$, let $v \in \{f, u\}^\perp$, $\ell_2(v) = 1$. Let $B = S^{-1} A_2 S$, where $S \in O_n^e$, $Sv = u$ and S is the identity on $\{u, v\}^\perp$. Then $B^{-1} f =$

$\cos\theta f - \sin\theta v$. Let $\phi \in [0, 2\pi)$, and let $C = TA_2$, where $T \in O_n^e$, $Tu = \cos\phi u + \sin\phi v$, $Tv = -\sin\phi u + \cos\phi v$, and T is the identity on $\{u, v\}^\perp$. Then $Cf = \cos\theta f + \sin\theta\cos\phi u + \sin\theta\sin\phi v$. So $B, C \in G$.

Now $BC \in G$, and let $\mu = \langle BCf, f\rangle$. Then $\mu = \langle Cf, B^{-1}f\rangle$, as $B \in O_n$. So $\mu = \cos^2\theta - \sin^2\theta\sin\phi$, and $\mu \in K$. As $\sin\phi$ varies from 1 to -1, μ varies from $\cos 2\theta$ to 1. Hence, $[\cos 2\theta, 1] \subseteq K$. Therefore, $G = O_n$ as asserted. ∎

Thus we have proven:

THEOREM 2.4. *Let $n \geq 3$. If G is a closed, infinite group satisfying $\pm S_n < G < O_n$, then $G = O_n$ or $G = O_n^e$.*

Since Theorem 1.2 applies only to finite groups, it remains of interest whether there exists norms whose isometry groups are O_n and O_n^e. The Euclidean norm ℓ_2 has O_n as its isometry group.

To construct a norm whose isometry group is O_n^e, define $\|x\| = \ell_2(x) + |\langle e, x\rangle|$. It is easy to check that $\|\cdot\|$ is a norm, and its isometry group G contains O_n^e as a subgroup. By Theorem 2.4, we know that G is either O_n^e or O_n. To show that G is not O_n, it suffices to choose $T \in O_n \setminus O_n^e$, and $x \in \mathbb{R}^n$ such that $\|Tx\| \neq \|x\|$. Direct computation shows that this holds for $x = e$ and T such that $Te = (\sqrt{n}, 0, \ldots, 0)^t$.

3. **Finite groups G satisfying $\pm S_n < G < O_n$.** Suppose G is a finite group such that $\pm S_n < G < O_n$. A matrix of the form $L_x = I - 2xx^t \in G$, where $x \in \mathbb{R}^n$ satisfies $x^t x = 1$, is called a *reflection*, and x is a *reflection point* or a *root* of G. Clearly, $L_x(v) = v - 2(x^t v)x$ for all $x \in \mathbb{R}^n$. Geometrically, this corresponds to reflecting through the $(n-1)$-dimensional hyperplane $x^\perp$.

Let $R = \{x \in \mathbb{R}^n : \langle x, x\rangle = 1, L_x \in G\}$. Then G acts on R: if $A \in G$, and $r \in R$, then $Ar \in R$, for $AL_rA^{-1} = L_{Ar}$, as may be directly verified. Since $S_n < G$, all the vectors of the form $\pm(e_i - e_j)/\sqrt{2}$, $1 \leq i < j \leq n$, are roots of G. Denote by H the reflection subgroup of G, *i.e.*, $H = \langle R\rangle$, the group generated by the reflections in G. We shall determine all possible G by studying H.

The theory of finite reflection groups in O_n has been quite well-developed (*e.g.*, see [BG] and [Bo]). It is known that if $\tilde{H}$ is a finite irreducible reflection group with the standard root systems (see [BG, p. 76 Table 5.2]), then $\tilde{H}$ is one of the following groups:

$$\mathcal{A}_n,\ \mathcal{B}_n,\ \mathcal{D}_n,\ \mathcal{J}_3,\ \mathcal{J}_4,\ \mathcal{F}_4,\ \mathcal{E}_8,\ \mathcal{E}_7,\ \mathcal{E}_6,\ \mathcal{H}_2^m (\text{with } m \geq 5).$$

With the standard root systems, $\mathcal{A}_n = S_{n+1}$ acting on $\mathbb{R}_0^{n+1} \simeq \mathbb{R}^n$ where $\mathbb{R}_0^{n+1}$ is the set of vectors in $\mathbb{R}^{n+1}$ with sum of entries equal to zero, $\mathcal{B}_n$ is the group of *signed* or *generalized permutation* matrices, *i.e.*, matrices of the form DP for some diagonal $D \in O_n$ and some $P \in S_n$, $\mathcal{D}_n$ is the group of signed permutation matrices having even number of negative entries, and $\mathcal{H}_2^m$ is the dihedral group in O_2 with $2m$ elements. Note that $\mathcal{H}_2^3$ and $\mathcal{H}_2^4$ are just $\mathcal{A}_2$ and $\mathcal{B}_2$, respectively. For our purpose, there is no need to list $\mathcal{H}_2^6$ as $\mathcal{G}_2$ as in standard references of finite reflection groups.

If $n = 2$, then G must be the dihedral group containing $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *i.e.*, G is the dihedral group with $2k$ elements for some positive integer k. We shall assume $n > 2$ in the following.

Note that $\mathbb{R}^n$ decomposes into the irreducible subspaces $\text{span}\{e\}$ and $\mathbb{R}^n_0 = e^\perp$ under the action of S_n. Thus H is either irreducible, or reducible with $\text{span}\{e\}$ and $\mathbb{R}^n_0 = e^\perp$ as invariant subspaces. It turns out that G is irreducible if and only if H is irreducible. We have the following two results characterizing G in the irreducible and reducible cases, respectively.

THEOREM 3.1. *Suppose $n \geq 3$ and $G < O_n$ is a finite irreducible group containing $\pm S_n$. Then $G = T^t K T$, where K is one of the following groups:*

$$\pm\mathcal{A}_n,\ \mathcal{B}_n,\ \mathcal{D}_n \text{ when } n \text{ is even},\ \pm\mathcal{E}_6,\ \mathcal{E}_7,\ \mathcal{E}_8,\ \mathcal{J}_3,\ \mathcal{J}_4,\ \mathcal{F}_4,\ \langle \mathcal{D}_4, I_4 - J_4/2\rangle,$$
$$\text{or } \langle \mathcal{F}_4, Y\rangle \text{ with } Y = \frac{1}{\sqrt{2}}\left\{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right\},$$

and T is an orthogonal transformation satisfying the following condition.

(a) *If $K = \pm\mathcal{A}_n$, then $T \in \mathbb{R}^{(n+1)\times n}$ is the linear map from $\mathbb{R}^{n+1}_0$ to $\mathbb{R}^n$ such that*

$$T(e_1 - \gamma e) = \tilde{e}_1 - \tilde{e}_{n+1} \text{ with } \gamma = \frac{1 \pm \sqrt{n+1}}{n},$$
$$\text{and } T(e_1 - e_j) = \tilde{e}_1 - \tilde{e}_j$$

for $2 \leq j \leq n$, where $\{\tilde{e}_1, \ldots, \tilde{e}_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$.

(b) *If $K = \mathcal{B}_n$, $\mathcal{D}_n$, $\mathcal{E}_8$, or $\langle \mathcal{D}_4, I_4 - J_4/2\rangle$, then $T = I$ or $I - 2J/n$.*

(c) *If $K = \mathcal{J}_3$, then $T \in \mathbb{R}^{3\times 3}$ is the linear operator on $\mathbb{R}^3$ such that*

$$T(e_1 - e_2) = \sqrt{2}e_1, \qquad T(e_1 - e_3) = \sqrt{2}\beta(2\alpha, 2\alpha + 1, -1)^t,$$
$$T(e) = \pm 2(0, \beta, \alpha)^t,$$

where $\alpha = (1 + \sqrt{5})/4$ and $\beta = (-1 + \sqrt{5})/4$.

(d) *If $K = \mathcal{J}_4$, then $T = Y$.*

(e) *If $K = \mathcal{E}_7$, then $T \in \mathbb{R}^{8\times 7}$ is the linear map from $\mathbb{R}^7$ to $W_1 := (e_8 - e/2)^\perp$ in $\mathbb{R}^8$ such that the i-th column of T equals*

$$v_i = e_i - a\Big(\sum_{j=1}^{7} e_j\Big) + be_8 \in \mathbb{R}^8 \text{ for } 1 \leq i \leq 7,$$

where $(a, b) = (4 + \sqrt{2}, -7\sqrt{2})/28$ or $(4 - \sqrt{2}, 7\sqrt{2})/28$.

(f) *If $K = \pm\mathcal{E}_6$, then $T \in \mathbb{R}^{8\times 6}$ is the linear map from $\mathbb{R}^6$ to $W_2 := \{e_8 - e/2, e_7 - e_8\}^\perp$ in $\mathbb{R}^8$ such that the i-th column of T equals*

$$v_i = e_i - \frac{1}{6}\Big(\sum_{j=1}^{6} e_j\Big) \pm \frac{1}{\sqrt{12}}(e_7 + e_8) \in \mathbb{R}^8 \text{ for } 1 \leq i \leq 6.$$

(g) *If* $K = \mathcal{F}_4$ *or* $\langle \mathcal{F}_4, Y\rangle$, *then* $T = I_4$.

THEOREM 3.2. *Suppose* $n \geq 3$ *and* $G < O_n$ *is a finite reducible group containing* $\pm S_n$. *Then* $G = T^t H_0 T \oplus G_1$, *where* $G_1 = \{I\}$ *or* $G_1 = \langle I - 2J/n\rangle$ *acts on* $\langle e\rangle$, $T^t H_0 T$ *acts on* $e^{\perp}$ *with* H_0 *equal to one of the following groups:*

$$\pm\mathcal{A}_{n-1},\ \mathcal{E}_{n-1} \text{ with } 8 \leq n \leq 9, \text{ or } \mathcal{H}_2^{6k} \text{ for some positive integer } k \text{ when } n = 3,$$

and T *is an orthogonal transformation satisfying the following condition.*

(a) *If* $H_0 = \pm\mathcal{A}_{n-1}$, *then* T *is the identity map on* $\mathbb{R}_0^n$.

(b) *If* $n = 9$ *and* $H_0 = \mathcal{E}_8$, *then* $T \in \mathbb{R}^{8\times 9}$ *is the linear map from* $\mathbb{R}_0^9$ *to* $\mathbb{R}^8$ *such that*

$$T(e_i - e_j) = \tilde{e}_i - \tilde{e}_j \text{ for } 1 \leq i < j \leq 8, \text{ and}$$
$$T(e_i - e_9) = \tilde{e}_i - \frac{1}{2}\Big(\sum_{j=1}^{8}\tilde{e}_j\Big) \text{ for } 1 \leq i \leq 8,$$

where $\{\tilde{e}_1, \ldots, \tilde{e}_8\}$ *is the standard basis of* $\mathbb{R}^8$.

(c) *If* $n = 8$ *and* $H_0 = \mathcal{E}_7$, *then* $T \in \mathbb{R}^{8\times 8}$ *is the linear map from* $\mathbb{R}_0^8$ *to* $W_1 = (e_8 - e/2)^{\perp}$ *in* $\mathbb{R}^8$ *such that*

$$T(e_i - e_j) = e_i - e_j \text{ for } 1 \leq i < j \leq 7, \text{ and}$$
$$T(e_i - e_8) = e_i + e_8 \text{ for } 1 \leq i \leq 7.$$

(d) *If* $H_0 = \mathcal{H}_2^m$, *then* $T \in \mathbb{R}^{2\times 3}$ *is the linear map from* $\mathbb{R}_0^3$ *to* $\mathbb{R}^2$ *such that*

$$T(e_1 - e_2) = \sqrt{2}(1,0)^t, \text{ and } T(e_1 - e_3) = (1, \sqrt{3})^t/2.$$

Theorems 3.1 and 3.2 together classify all the finite groups G satisfying $\pm S_n < G < O_n$. From Theorem 1.2, we know that each of these groups is realizable as the isometry group of a norm on $\mathbb{R}^n$.

As pointed out by the referee, Theorem 3.1 is closely related to an exercise in [Bo, Chapter 6, Problem 16]. Our statement is more specific and explicit.

Note that whenever there are two choices for T, say T_1 and T_2, in Theorem 3.1, then $T_1(I - 2J/n) = T_2$. This also follows from [GLe, Theorem 1.3].

The advanced theory developed in [Bo] may be used to prove Theorems 3.1 and 3.2. In the following, we present proofs that depend only on basic results on reflection groups and elementary computations.

PROOF OF THEOREM 3.1. In this subsection, we assume that the maximum reflection group H contained in G is irreducible. To determine G, we first determine the orthogonal transformations T that satisfy $H = T^{-1}\tilde{H}T$, where $\tilde{H}$ is a finite reflection group in O_n with the standard root system $\tilde{R}$ (*e.g.*, see [BG, Theorem 5.1.2 and Table 5.2]). Since we assume $n > 2$, $\tilde{H}$ must be one of the following groups:

$$\mathcal{A}_n,\ \mathcal{B}_n,\ \mathcal{D}_n,\ \mathcal{J}_3,\ \mathcal{J}_4,\ \mathcal{F}_4,\ \mathcal{E}_8,\ \mathcal{E}_7,\ \mathcal{E}_6.$$

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In each case, since T will map the roots of H to the roots of $\tilde{R}$, and since $\pm(e_i - e_j)/\sqrt{2}$ with $1 \leq i < j \leq n$ are roots of H, we have $\pm T(e_i - e_j)/\sqrt{2} \in \tilde{R}$ for all $1 \leq i < j \leq n$. Using this fact, one can determine the structure of T, and then characterize H and G. Actually, except for the case when $\tilde{H}$ is $\mathcal{D}_4$ or $\mathcal{F}_4$, we always have $G = \pm H = \langle -I, H \rangle$ (cf. [Bo, Chapter 6, Problem 16].)

We shall sketch the proof of Theorem 3.1. The details for several cases, including the exceptional cases mentioned in [Bo, Chapter 6, Problem 16] will be worked out.

Suppose $\tilde{H} = \mathcal{A}_n$. Then $\tilde{H}$ has standard root system $\tilde{R} = \{\pm(\tilde{e}_i - \tilde{e}_j)/\sqrt{2} : 1 \leq i \leq j \leq n+1\}$, where $\{\tilde{e}_1, \ldots, \tilde{e}_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$, and $\tilde{H} = S_{n+1}$ acts on $\mathbb{R}_0^{n+1}$. Let T be an orthogonal transformation mapping the roots of H to the roots of $\tilde{H}$. First, we may assume $T(e_1 - e_2) = P(\tilde{e}_1 - \tilde{e}_2)$ for some $P \in \mathcal{A}_n$. Since T is orthogonal, we have $\langle T(e_1 - e_2), T(e_1 - e_j)\rangle = \langle (e_1 - e_2), (e_1 - e_j)\rangle = 1$ for all $3 \leq j \leq n$, we may assume that $T(e_1 - e_j) = P(\tilde{e}_1 - \tilde{e}_j)$ for $2 \leq j \leq n$, by a suitable modification of P. Now, consider $u \in \mathbb{R}^n$ such that $Tu = P(\tilde{e}_1 - \tilde{e}_{n+1})$. Since $\langle u, (e_1 - e_j)\rangle = \langle (\tilde{e}_1 - \tilde{e}_{n+1}), (\tilde{e}_1 - \tilde{e}_j)\rangle = 1$ for $1 \leq j \leq n$, we see that $u = e_1 - \gamma(\sum_{i=1}^n e_i)$ with $\gamma = (1 \pm \sqrt{n+1})/n$. Since $P\mathcal{H}_n P^t = \mathcal{A}_n$, we have $H = T^t \mathcal{A}_n T = T^t P \mathcal{A}_n P^t T$. Hence, we may replace T by $P^t T$ and obtain condition (a).

Next, we show that $G = \pm H$. Let T satisfy $T^t \mathcal{A}_n T = H$ as determined in the preceding paragraph. Suppose $T^t \tilde{G} T = G$. The result will follow once we show that $\tilde{G} = \pm \mathcal{A}_n$. Clearly, $\pm \mathcal{A}_n \subseteq \tilde{G}$ by our assumption. To prove the reverse inclusion, let $L \in \tilde{G}$ act on $\mathbb{R}_0^n$, and let $d_i = \tilde{e}_1 - \tilde{e}_i$ for $1 \leq i \leq n+1$. Then the map L is determined by its action on $\{d_i : 2 \leq i \leq n+1\}$. Since L is orthogonal and maps the root system of $\mathcal{A}_n$ onto itself, one easily deduces that $Ld_i = Ad_i$ for $1 \leq i \leq n+1$, for some suitable $A \in \pm\mathcal{A}_n$. Thus $L \in \pm\mathcal{A}_n$.

The proof of the case $\tilde{H} = \mathcal{B}_n$ is similar to the previous one. Note that $\mathcal{B}_n$ has standard root system $\tilde{R} = \{\pm e_i : 1 \leq i \leq n\} \cup \{(\pm e_i \pm e_j)/\sqrt{2} : 1 \leq i < j \leq n\}$. If $T \in O_n$ satisfies $T(R) = \tilde{R}$, one can show that $T(e_1 - e_j) = P(e_1 - e_j)$ for some suitable $P \in \mathcal{B}_n$, and either $T(e_1) = Pe_1$ or $T(e_1 - 2e/n) = e_1$. It follows that $T = I_n$ or $I - 2J_n/n$. Suppose $T = I_n$. Then $H = \mathcal{B}_n < G$. If $L \in G$, one can show that there exists $A \in \mathcal{B}_n$ such that $L(e_i) = Ae_i$ for all $1 \leq i \leq n$. It follows that $G < H$, and hence $G = H$. One can get the conclusion by similar arguments if $T = I_n - 2J_n/n$. Note that the conclusion can also be deduced from the results in [DLR] and the fact that G cannot contain $\mathcal{F}_4$ by our assumption.

Suppose $\tilde{H} = \mathcal{D}_n$. Clearly, n is even, otherwise, $\mathcal{B}_n = \pm\mathcal{D}_n < \tilde{H}$, which is impossible. Note that $\mathcal{D}_n$ has standard root system $\tilde{R} = \{(\pm e_i \pm e_j)/\sqrt{2} : i < j\}$. One can show that $H = \mathcal{D}_n$ or $(I - 2J/n)\mathcal{D}_n(I - 2J/n)$ as before. If $n \geq 6$, we can show that $G = H$. Now suppose $n = 4$, and $T = I_4$ for simplicity. (The case of $T = I - 2J/4$ can be treated similarly.) One readily checks that $\langle \mathcal{D}_4, I - J/4\rangle = \mathcal{D}_4 \cup \{P(I - J/2)Q : P, Q \in \mathcal{D}_4\}$, $|\langle \mathcal{D}_4, I - J/4\rangle| = 3|\mathcal{D}_4|$, and $\mathcal{D}_4$ is a maximal subgroup of $\langle \mathcal{D}_4, I - J/4\rangle$.

If $L \in G$, then L is orthogonal and maps the root system of $\mathcal{D}_4$ onto itself. One easily verifies that there exists $A \in \mathcal{D}_4$ such that either

(i) $Lr = Ar$ for $r = e_1 - e_2, e_1 - e_3, e_1 - e_4, e_1 + e_4$, and hence $L \in \mathcal{D}_4$, or

(ii) $Lr = Ar$ for $r = e_1 - e_2, e_1 - e_3, e_1 - e_4$, $L(e_1 + e_4) = -(e_2 + e_3)$, and hence $L = P(I - J/2)Q$ for some $P, Q \in \mathcal{D}_4$. As a result, $\mathcal{D}_4 < G < \langle \mathcal{D}_4, I - J/2\rangle$, and thus $G = \mathcal{D}_4$ or $G = \langle \mathcal{D}_4, I - J/2\rangle$.

Suppose $\tilde{H} = \mathcal{J}_3$. Then $\tilde{R}$ consists of $\pm e_i$ with $1 \le i \le 3$, $\beta\big(\pm(2\alpha + 1), \pm 1, \pm 2\alpha\big)^t$ and all even permutations of the coordinates of these vectors, where $\alpha = (1 + \sqrt{5})/4$ and $\beta = (-1 + \sqrt{5})/4$. Suppose $THT^{-1} = \tilde{H}$. Since $\mathcal{J}_3$ is transitive on the roots (see [BG, pp. 78–79]), there exists $P \in \mathcal{J}_3$ such that $PT(e_1 - e_2) = \sqrt{2}e_1$. Note that $\langle e_1 - e_2, e_1 - e_3\rangle = 1$, we see that $PT(e_1 - e_3)/\sqrt{2}$ must be of the form $\beta\big(2\alpha, \pm(2\alpha + 1), \pm 1\big)^t$. Since $e_i \in \tilde{R}$ for $1 \le i \le 3$, $\mathcal{J}_3$ contains all diagonal orthogonal matrices. Thus we may adjust P and assume that $PT(e_1 - e_3)/\sqrt{2} = \beta(2\alpha, 2\alpha + 1, -1)^t$. Finally, consider $PT(e)$. Since $\langle e, e_1 - e_2\rangle = 0$ and $\langle e, e_1 - e_3\rangle = 0$, one sees that $PT(e)$ is of the form $(0, x, y)^t$ such that $(2\alpha + 1)x - y = 0$ and $x^2 + y^2 = 3$. There are two solutions for this system of equations, and hence there are two possible choices of T. Direct computation shows that condition (c) holds, and one can then show that $TGT^{-1} = \tilde{H} = \mathcal{J}_3$ by arguments similar to the previous cases.

Suppose $\tilde{H} = \mathcal{J}_4$, whose root system $\tilde{R}$ consists of $\pm e_i$ with $1 \le i \le 4$, $\sum_{j=1}^4 \mu_j e_i$ with $\mu_j = \pm 1$, $\beta\big(\pm 2\alpha, 0, \pm(2\alpha + 1), \pm 1\big)^t$ and all even permutations of the coordinates of these vectors, where $\alpha = (1 + \sqrt{5})/4$ and $\beta = (-1 + \sqrt{5})/4$. Suppose T is orthogonal such that $THT^{-1} = \tilde{H}$. One can show that there exists $P \in \mathcal{J}_4$ such that

$$PT(e_1 - e_2) = \sqrt{2}e_1, \qquad PT(e_1 - e_3) = \frac{e}{\sqrt{2}},$$
$$PT(e_1 - e_4) = \frac{e - 2e_3}{\sqrt{2}}, \qquad PT(e) = \sqrt{2}(e_2 - e_3).$$

Thus T satisfies condition (d). One can then show that $G = T^{-1}\mathcal{J}_4 T$.

Now suppose $\tilde{H} = \mathcal{F}_4$. Note that (*e.g.*, see [DLR]) $\mathcal{F}_4 = \langle \mathcal{B}_4, I_4 - J_4/2\rangle$, and the normalizer of $\mathcal{F}_4$ in O_4 equals $\langle \mathcal{B}_4, Y\rangle$, where Y is defined as in the statement of the theorem. Now suppose $THT^{-1} = \tilde{H}$. One can use arguments similar to those in the case of $\tilde{H} = \mathcal{D}_4$ to show that $T = I$, and $G = \mathcal{F}_4$ or $\langle \mathcal{F}_4, Y\rangle$ (*cf.* [DLR]).

Next suppose $\tilde{H} = \mathcal{E}_8$. Then $\tilde{R} = \{(\pm e_i \pm e_j)/\sqrt{2} : 1 \le i < j \le 8\} \cup \{\sum_{i=1}^8 \varepsilon_i e_i/\sqrt{8} : \varepsilon_i = \pm 1, \Pi_{i=1}^8 \varepsilon_i = -1\}$. Let T be orthogonal such that $THT^{-1} = \tilde{H}$. One can then show that there exists $P \in \mathcal{E}_8$ such that $PT(e_1 - e_j) = e_1 - e_j$ for $2 \le j \le n$, and $PT(e) = \pm e$. Thus T satisfies condition (b). To show that $G = H$, let $A \in TGT^{-1}$. Then A permutes the roots of $\mathcal{E}_8$. Suppose $A(e_1 + e_2) = x$. Since $\tilde{H}$ is transitive on the root systems (*e.g.*, see [BG, pp. 78–79]), there exists $P \in H$ such that $Px = e_1 + e_2$ and hence $PA(e_1 + e_2) = e_1 + e_2$. Now consider $PA(e_1 - e_2) = y$. If $y \ne \pm(e_1 - e_2)$, then $y = \sum_{i=1}^8 \mu_i e_i$, where $\mu_1\mu_2 = -1$ and $\mu_1 \cdots \mu_8 = -1$. Note that the set $\{v \in \mathcal{R} : v^t z/(\|v\|\,\|z\|) = 1/2$ for $z = e_1 \pm e_2\} = \{e_1 \pm e_i : 3 \le i \le 8\}$ has 12 elements, but $\{v \in \mathcal{R} : v^t z/(\|v\|\,\|z\|) = 1/2$ for $z = e_1 + e_2, y\} = \{e_j + y_i e_i : e_j^t y = 1, 3 \le i \le 8\}$ has only 6 elements, which is a contradiction. Thus $PA(e_1 - e_2) = \pm(e_1 - e_2)$. Further, $PA(e_3 + e_4)$ must be orthogonal to $e_1 \pm e_2$, we can show that $PA(e_3 \pm e_4) = \pm(e_i \pm e_j)$ for some $i > j > 2$. One can get

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similar conclusions on $\pm(e_5 \pm e_6)$ and $\pm(e_7 \pm e_8)$. Consequently, one sees that $PA \in S_n$ and hence $TGT^{-1} = \mathcal{E}_8$.

Suppose $\tilde{H} = \mathcal{E}_7$. Then $\tilde{R} \subseteq \mathbb{R}^8$ consists of the roots of $\mathcal{E}_8$ lying in $W_1 = (e_8 - e/2)^{\perp}$. One can get the conclusion by arguments similar to the previous case.

Suppose $\tilde{H} = \mathcal{E}_6$. Then $\tilde{R} \subseteq \mathbb{R}^8$ consists of the roots of $\mathcal{E}_8$ lying in $W_2 = \{e_8 - e/2, e_7 - e_8\}^{\perp}$. One can first show that $H = T^t \mathcal{E}_6 T$ by argument similar to last two cases, where T satisfies condition (f). Since $-I \notin \mathcal{E}_6$, one will conclude that $\tilde{G} = \pm\mathcal{E}_6$ by arguments similar to those in the previous cases if $T\tilde{G}T^{-1} = G$.

Combining the above analysis, we get Theorem 3.1.

PROOF OF THEOREM 3.2. In this subsection, we assume that the maximum reflection subgroup H contained in G is reducible. Then $H = H_0 \oplus H_1$, where H_0 acts on $\mathbb{R}^n_0$ and H_1 acts on $\langle e \rangle$. If G has no other reflection points, then every $A \in G$ will maps the roots of H to themselves. Hence, $G = G_0 \oplus G_1$ with G_0 acting on $\mathbb{R}^n_0$ and G_1 acting trivially on $\langle e \rangle$. If e is a reflection point of G, then for every $A \in G$, $Ae = \pm e$. Again, $G = G_0 \oplus G_1$.

First, we use arguments similar to those in Theorem 3.1 to determine all the possible H_0. Let R_0 be the root systems of H_0. Then R_0 contains $\pm(e_i - e_j)/\sqrt{2}$, $1 \leq i < j \leq n$, and there is an orthogonal transformation T such that $H_0 = T^{-1}\tilde{H}T$, where $\tilde{H}$ is a finite reflection group in O_n with the standard root system $\tilde{R}$ (see [BG, p. 76, Table 5.2]), and must be one of the following:

$$\mathcal{A}_{n-1},\ \mathcal{B}_{n-1},\ \mathcal{D}_{n-1}\mathcal{J}_{n-1} \text{ with } n = 4 \text{ or } 5,$$
$$\mathcal{E}_{n-1} \text{ with } 7 \leq n \leq 9, \qquad \mathcal{H}_2^m \text{ (with } m \geq 5) \text{ when } n = 3.$$

Since T maps the roots of H_0 to those of $\tilde{H}$, and since $\pm(e_i - e_j)/\sqrt{2}$ with $1 \leq i < j \leq n$ are roots of H_0, $\pm T(e_i - e_j)/\sqrt{2} \in \tilde{R}$ for all $1 \leq i < j \leq n$. For each $\tilde{H}$ in the above list, we determine all possible orthogonal transformations T, which may not exist as will be seen, such that $\pm T(e_i - e_j)/\sqrt{2} \in \tilde{R}$ for all $1 \leq i < j \leq n$. Once this is done, it is not hard to determine H and G.

If $\tilde{H} = \mathcal{A}_{n-1}$, then $T(e_i - e_j) = P(e_i - e_j)$ for all $1 \leq i < j \leq n$, for some $P \in S_n$. Thus $H_0 = S_n = \mathcal{A}_{n-1}$.

Next we show that it is impossible to have $\tilde{H}$ equal to any one of the groups :

$$\mathcal{B}_{n-1},\ \mathcal{D}_{n-1},\ \mathcal{J}_{n-1} \text{ (when } n = 4 \text{ or } 5),$$
$$\mathcal{F}_{n-1} \text{ (when } n = 5), \text{ or } \mathcal{E}_6 \text{ (when } n = 7).$$

First, note (*e.g.*, see [BG, p. 80]) that the order of $\mathcal{E}_6$ is $2^7 \cdot 3^4 \cdot 5$, which is not divisible by 7!. Thus, it cannot contain a subgroup isomorphic to S_7. For the other cases, consider $d_i = (e_1 - e_i)/\sqrt{2} \in R^n_0$ for $2 \leq i \leq n$. Then $\langle d_i, d_j \rangle = 1/2$ for $2 \leq i < j \leq n$, but there are no roots $r_2, \ldots, r_n$ of unit length in the root system of any one of the groups

$$\mathcal{B}_{n-1},\ \mathcal{D}_{n-1},\ \mathcal{J}_{n-1} \text{ (when } n = 4 \text{ or } 5),$$
$$\mathcal{F}_{n-1} \text{ (when } n = 5),$$

such that $\langle r_i, r_j \rangle = 1/2$ for $2 \le i < j \le n$. Thus $\tilde{H}$ cannot be any one of these groups.

In the following, we show that it is possible to have $\tilde{H} = \mathcal{E}_i$ for $7 \le i \le 8$, or $\tilde{H} = \mathcal{H}_2^m$ for some special m. Moreover, there is essentially only one orthogonal transformation T satisfying $H_0 = T^{-1}\tilde{H}T$ in each case.

Suppose $\tilde{H} = \mathcal{E}_i$ for $i = 8$ or 7. Let $\{\tilde{e}_1, \ldots, \tilde{e}_8\}$ be the standard basis of $\mathbb{R}^8$, and let $\tilde{e} = \sum_{i=1}^8 \tilde{e}_i$.

If $\tilde{H} = \mathcal{E}_8$, then $n = 9$. By arguments similar to those in the irreducible cases, we see that there exists $P \in \mathcal{D}_8$, which is a subgroup of the normalizer of $\mathcal{E}_8$ in O_8, such that the orthogonal transformation $T: \mathbb{R}_0^9 \longrightarrow \mathbb{R}^8$ satisfies $T(e_i - e_j) = P(\tilde{e}_i - \tilde{e}_j)$ for all $1 \le i < j \le 8$, where $\{\tilde{e}_1, \ldots, \tilde{e}_8\}$ is the standard basis of $\mathbb{R}^8$. Since T is orthogonal, it follows that $T(e_i - e_9) = P(\tilde{e}_i - (\sum_{i=j}^8 \tilde{e}_j)/2)$ for $1 \le i \le 8$. Thus $H_0 = T^{-1}\mathcal{E}_8 T$.

If $\tilde{H} = \mathcal{E}_7$, then $n = 8$. One can show that $T: \mathbb{R}^8 \longrightarrow W_1$, where $W_1 = (e_8 - e/2)^\perp$ in $\mathbb{R}^8$, satisfies $T(e_i - e_j) = P(e_i - e_j)$ for all $1 \le i < j \le 7$ and $T(e_i - e_8) = P(e_i + e_8)$ for $i = 1, \ldots, 7$, for some P in the normalizer of $\mathcal{E}_7$ in O_7. Thus $H_0 = T^{-1}\mathcal{E}_7 T$.

Finally, suppose $\tilde{H} = \mathcal{H}_2^m$, i.e., $n = 3$. Since $\mathcal{H}_2^m$ is transitive on its roots, we may assume that $T(e_1 - e_2)/\sqrt{2} = (1, 0)^t$. It follows that $T(e_1 - e_3)/\sqrt{2} = (1, \pm\sqrt{3})^t/2$. Thus m must be a multiple of 6, and we have $H_0 = T^{-1}H_2^m T$ with T satisfying condition (d).

After determining H_0, one can show that $G_0 = \pm H_0$ and therefore $G = \pm H_0 \oplus G_1$ by arguments similar to those in the irreducible case.

Combining the above analysis, we get Theorem 3.2.

4. **Remarks and open problems.** In [GLe], the authors consider the norms on $\mathbb{R}^n$ that are permutation invariant with respect a certain basis. In their setting, they only need to show that with a suitable choice of an inner product and an orthonormal basis, the isometry group must be of a certain form. In particular, they showed that for $n \ge 13$, if the isometry group of a permutation invariant norm is finite then with a suitable choice of orthonormal basis the group must be one of the following:

$$\pm\mathcal{A}_n,\ \mathcal{B}_n,\ \mathcal{D}_n \text{ when } n \text{ is even},\ \pm\ \mathcal{A}_{n-1} \text{ or } \langle \pm\mathcal{A}_{n-1}, I_n - 2J/n \rangle.$$

From our results, we see that the above conclusion actually holds for $n \ge 10$.

The referee pointed out that there are several results and references that are related to our work.

First, W. Burnside [Bu] classified the finite subgroups of $\mathrm{GL}_n(\mathbb{Q})$ which contain the symmetric group S_n. Burnside did not have root systems or the notion of reflection groups at his disposal, but nevertheless found the reflection groups of type $\mathcal{F}_4, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8$. It was also noted by the referee that:

(i) Burnside's list is not complete as mentioned in [Ba].
(ii) Since Burnside assumed that the ground field is $\mathbb{Q}$, our list is bigger.
(iii) Burnside's paper is, from present perspective, old fashioned and hard to read, and thus there is good reason to rework it.

Second, E. Bannai [Ba] classified subgroups of $GL_n(\mathbb{C})$ which contain the commutator subgroup $[S_n, S_n]$ - the alternating group on the variables - and are conjugate in $GL_n(\mathbb{C})$ to a subgroup of $GL_n(\mathbb{Q})$. Bannai found some additions to Burnside's list. The paper of Bannai, according to the referee, is very difficult to read, and does more than we do in that S_n is replaced by $[S_n, S_n]$. On the other hand, it does less than we do in that the ground field is $\mathbb{Q}$ not $\mathbb{R}$. Thus Bannai would not capture the dihedral groups of orders different from 4, 6, 8, 12 or the reflection groups of type $\mathcal{J}_3$ and $\mathcal{J}_4$, etc.

To conclude our paper, we list some related problems that deserve further research.

(a) Determine all possible isometry groups of a permutation invariant norm on $\mathbb{C}^n$.

(b) Determine all possible isometry groups of an absolute norm on $\mathbb{R}^n$. For the complex case, see [ST].

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