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MINIMAL ARCS IN METRIC SPACES

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In this paper we discuss the existence of an arc of minimal length joining two arbitrary, yet fixed points in a complete metric space, where the metric is restricted only by the properties (A) and (B) given below. It is shown that under these conditions an arc of least length joining any two fixed points exists, and is unique. In addition, its length is shown to be equal to the metric distance between the points.

The additional restricting properties on the metric: ρ, are the following:

PROPERTY (A). *Given any pair of points,* p, q *with* $\rho(p,q) = a$, *for any* $b \geqq 0$, $c \geqq 0$, $b + c = a$, and any $e > 0$, *there is a point* r *such that* $\rho(p,r) \leqq b + e$ *and* $\rho(r,q) \leqq c + e$.

We use the notations: $N_p(\alpha) \equiv \{x \mid \rho(p,x) < \alpha\}$ and $\operatorname{diam}[S] = \operatorname{lub}_{x,y \in S}\rho(x,y)$ here and subsequently throughout the paper.

PROPERTY (B). *Given any pair of points* p, q *and* $\alpha > 0$ *such that* $q \notin N_p(\alpha)$, *for any* $e > 0$ *there is a* $\beta > 0$ *such that* (i) $N_p(\alpha) \cap N_q(\beta) \neq \emptyset$, (ii) $\operatorname{diam}[N_p(\alpha) \cap N_k(\beta)] < e$.

We remark that the octant surface of a sphere with the Euclidean metric in 3-space has property (B) but not property (A). A surface of a sphere with the same metric also fails to have property (B), as may be seen by considering p, q to be any pair of opposite poles. As an example of a space with property (A) and not (B), we may consider the Cartesian plane with $\rho((x, y), (p, q)) = \max\{|x-p|, |y-q|\}$.

In the following theorems, for notational convenience, we refer the unit interval $I \equiv [0, 1]$ to a binary representation. We shall also, on occasion, mix ordinary fractions with this representation; for example $0.11 + \frac{3}{8}$ which is $\frac{9}{8}$, or 1.001 in binary notation.

Theorem 1 illustrates that a complete metric space with a metric function having property (A) is arcwise connected.

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THEOREM 1. *If* (R,ρ) *is a complete metric space, and the metric* ρ *has properties* (*A*), *then* (R,ρ) *is arcwise connected.*

PROOF. Let $\varepsilon > 0$ be given, and p_0, p_1 be any two points of R, with $\rho(p_0, p_1) = d$. Set up the correspondence $p_0 = f(0)$, $p_1 = f(1)$. With $a = d$, $b = c$, $p = p_0$, $q = p_1$, $e = \varepsilon/2^2$, property (A) permits the selection of a point $p_{0.1}$ such that

$$\rho(p_0, p_{0.1}) < \frac{d}{2} + \frac{\varepsilon}{2^2} < \frac{d}{2} + \frac{\varepsilon}{2}, \text{ and } \rho(p_{0.1}, p_1) < \frac{d}{2} + \frac{\varepsilon}{2}.$$

Set up the correspondence $p_{0.1} = f(0.1)$. Then, with $a = d/2 + \varepsilon/2^2$, $b = c$, $e = \varepsilon/2^4$, again applying property (A) find $p_{0.01}, p_{0.11}$ such that $\rho(p_{0.00}, p_{0.01})$, $\rho(p_{0.01}, p_{0.10})$, $\rho(p_{0.10}, p_{0.11})$,

$$\rho(p_{0.11}, p_{1.00}) < \frac{d}{4} + \frac{\varepsilon}{2^3} + \frac{\varepsilon}{2^4} < \frac{d}{4} + \frac{\varepsilon}{2^2},$$

and set up the correspondence $p_{0.01} = f(0.01)$, $p_{0.11} = f(0.11)$. Proceeding inductively, $(n = 1, 2, 3, \cdots)$ suppose that, with α, β equal to 0 or 1, $p_{\alpha_0.\alpha_1\ldots\alpha_n}$, $p_{\beta_0.\beta_1\cdots\beta_n}$ have been found, where $\beta_0.\beta_1\beta_2\cdots\beta_n - \alpha_0.\alpha_1\alpha_2\cdots a_n = 2^{-n}$. Then, with

$$a = \frac{d}{2^n} + \frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2^{n+2}} + \cdots + \frac{\varepsilon}{2^{2n}},\ b = c,\ e = \varepsilon/2^{2n+2},$$

property (A) permits the selection of $p_{\alpha_0.\alpha_1\cdots\alpha_n}$, such that

$$\rho(p_{\alpha_0.\alpha_1\cdots\alpha_n}, p_{\alpha_0.\alpha_1\cdots\alpha_n 1}) < \frac{d}{2^{n+1}} + \frac{\varepsilon}{2^{n+1}} + \cdots + \frac{\varepsilon}{2^{2n+1}} + \frac{\varepsilon}{2^{2n+2}} < \frac{d}{2^{n+1}} + \frac{\varepsilon}{2^n}.$$

The correspondence $p_{\alpha_0.\alpha_0\cdots\alpha_n 1} = f(\alpha_0.\alpha_1\ldots\alpha_n 1)$ is made.

We show that if $p_{a_0.a_1\cdots a_k}, p_{b_0.b_1\cdots b_l}$, $(a, b = 0 \text{ or } 1;\ k, l < \infty)$ are two points with $b_0.b_1\cdots b_l \geqq a_0.a_1\cdots a_k$ then

$$\rho(p_{b_0.b_1\cdots b_l}, p_{a_0.a_1\cdots a_k}) < (b_0.b_1\cdots b_l - a_0.a_1\cdots a_k)(d+\varepsilon).$$

Let $\sigma = |l-k|$ and consider the sequence of numbers

$$\{a_0 \cdot a_1 \cdots a_k + j/2^\sigma \mid j = 0, 1, 2, \cdots, s;\ s = [b_0 \cdot b_1 \cdots b_l - a_0.a_1\cdots a_k]2^\sigma\}$$

with their associated images $p_{a_0.a_1\cdots a_k + j/2^\sigma}$. Then:

$$\begin{aligned} \rho(p_{b_0.b_1\cdots b_l}, p_{a_0.a_1\cdots a_k}) &\leqq \sum_{j=0}^{s-1} \rho(p_{a_0.a_1\cdots a_k + j/2^\sigma}, \\ p_{a_0.a_1\cdots a_k + (j+1)/2^\sigma}) &\leqq \sum_{j=0}^{s-1} \left(\frac{d}{2^\sigma} + \frac{\varepsilon}{2^{\sigma+1}} + \cdots + \frac{\varepsilon}{2^{2\sigma}}\right) \\ < \sum_{j=0}^{s-1}\left(\frac{d}{2^\sigma} + \frac{\varepsilon}{2^\sigma}\right) &= \frac{s}{2^\sigma}(d+\varepsilon) = (b_0.b_1\cdots b_l - a_0 \cdot a_1 \cdots a_k)(d+\varepsilon). \end{aligned}$$

The function ρ satisfies $\rho(p(x), \rho(y)) < |x-y|(d+\varepsilon)$ for all x and y in a dense

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subset K of I. Thus ρ is uniformly continuous on K and has a unique continuous extension to I.

We adopt the following definition of arc length, that will be used in subsequent theorems.

DEFINITION. *If* $S \equiv f[I]$ *is an arc, its length is defined as*

$$l(S) = \sup_{\{x_i\}} \sum_{i=0}^{N-1} \rho(f(x_i), f(x_{i+1}))$$

where $\{x_i \mid 0 = x_0 < x_1 < \cdots < x_N = 1\}$ *is any partition of* I. If $l(S) < \infty$, then S has finite length; otherwise $l(S)$ is infinite.

THEOREM 2. *If* (R, ρ), *a complete metric space, has properties* (A) *and* (B), *and* $p, q \in R$ *with* $\rho(p,q) = d$, *then there is an arc* $f(I)$ *joining* p *and* q *such that* $\rho(f(x), f(y)) = d\,|x-y|$ *for all* $x, y \in I$.

PROOF. Let p_0, p_1 be the two points of R. We shall use the same notations as in the proof of theorem 1. Let $f(0) = p_0$, $f(1) = p_1$. The procedure for establishing the correspondence $f\colon I \to S$ is similar to that of theorem 1, with some modification. Suppose that $p_{\alpha_0.\alpha_1\cdots\alpha_n}, p_{\beta_0.\beta_1\cdots\beta_n}$ have been found, and that $\beta_0\cdot\beta_1\cdots\beta_n - \alpha_0\cdot\alpha_1\cdots\alpha_n = 2^{-n}$. Using properties (A) and (B), putting $a = d/2^n$, $b = c$, letting $\varepsilon_\gamma = 2^{-\gamma}$, $(\gamma = 1,2,\cdots)$, and choosing $\varepsilon' > 0$ sufficiently small, pick

$$p_\gamma \in Np_{\alpha_0.\alpha_1\cdots\alpha_n}(b+\varepsilon') \cap Np_{\beta_0.\beta_1\cdots\beta_n}(c+\varepsilon') \text{ with:}$$

$$\operatorname{diam}[Np_{\alpha_0.\alpha_1\cdots\alpha_n}(b+\varepsilon') \cap Np_{\beta_0.\beta_1\cdots\beta_n}(c+\varepsilon')] < \varepsilon_\gamma$$

If μ, $\lambda \geqq \gamma$, then $\rho(p_\mu, p_\lambda) < \varepsilon_\gamma$. By completeness the sequence $\{p_\gamma\}$ has a limit which we denote as: $p_{\alpha_0.\alpha_1\cdots\alpha_n 1}$. We make the correspondence $p_{\alpha_0.\alpha_1\cdots\alpha_n 1} = f(\alpha_0\cdot\alpha_1\cdots\alpha_{n1})$.

In a metric space $\rho(q_\gamma, r) \to \rho(q_0, r)$ when $q_\gamma \to q_0$, $(\gamma = 1,2,\cdots)$. With $q_\gamma = p_\gamma$, $q_0 = p_{\alpha_0.\alpha_1\cdots\alpha_n 1}$ and r alternately equal to $p_{\alpha_0.\alpha_1\cdots\alpha_n}$ and $p_{\beta_0.\beta_1\cdots\beta_n}$, it follows that $\rho(p_{\alpha_0.\alpha_1\cdots\alpha n1}, r) = b = c = a/2 = d/2^{n+1}$.

The construction of S and the proof of the continuity of f follow the technique of theorem 1.

We consider two points $p_{a_0.a_1\cdots a_k}$ and $p_{b_0.\cdots b_l}$ which correspond to terminating binary decimals:

$$b_0\cdot b_1\cdots b_l \geqq a_0\cdot a_1\cdots a_k.$$

We show $\rho(p_{b_0.b_1\cdots b_l}, p_{a_0.a_1\cdots a_k}) = (b_0\cdot b_1\cdots\ b_l - a_0\cdot a_1\cdots a_k)d$. Let $\sigma = |l-k|$ and consider the sequence of numbers:

$$\{a_0.a_1\cdots a_k + j/2^\sigma \mid j = 0,1,2,\cdots,s;\ s = (b_0\cdot b_1\cdots b_l - a_0\cdot a_1\cdots a_k)2^\sigma\}$$

with their associated images $p_{a_0.a_1\cdots a_k + j/2^\sigma}$. Then, as above:

$$\rho(p_{a_0.a_1\cdots a_k+j/2^\sigma}, p_{a_0\ a_1\cdots a_k+(j+1)/2^\sigma}) = d/2^\sigma$$

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Therefore:

$$\rho(p_{a_0 \cdot a_1 \cdots a_k}, p_{b_0 \cdot b_1 \cdots b_l}) \leqq sd/2^\sigma = (b_0 \cdot b_1 \cdots b_l - a_0 \cdot a_1 \cdots a_k)d \tag{1}$$

By a similar argument, it follows that:

$$\rho(p_0, p_{a_0 . a_1 \cdots a_k}) \leqq (a_0 . a_1 \cdots a_k)d$$

$$\rho(p_{b_0 . b_1 \cdots b_l}, p_1) \leqq (1 - b_0 . b_1 \cdots b_l)d$$

But:

$$d = \rho(p_0, p_1) \leqq \rho(p_0, p_{a_0 . a_1 \cdots a_k}) + \rho(p_{a_0 . a_1 \cdots a_k}, p_{b_0 . b_1 \cdots b_l}) + \rho(p_{b_0 . b_1 \cdots b_l}, p_1)$$

$$\leqq d[a_0 \cdot a_1 \cdots a_k + (b_0 \cdot b_1 \cdots b_l - a_0 . a_1 \cdots a_k) + (1 - b_0 \cdot b_1 \cdots b_l)] = d$$

so that:

$$\rho(p_{a_0 . a_n \cdots a_k}, p_{b_0 . b_1 \cdots b_l}) =$$

$$= d - \rho(p_0, p_{a_0 . a_1 \cdots a_k}) - \rho(p_{b_0 . b_1 \cdots b_l}, p_1) \geqq d[1 - a_0 \cdot a_1 \cdots a_k - (1 - b_0 \cdot b_1 \cdots b_l)].$$

Combining this with equation (1) and noting that if $\{q_\gamma\}, \{r_\gamma\}$, $(\gamma = 1, 2, \cdots)$ are any two sequences of points with terminating representations that tend to q_a, q_b in the limit, then $\rho(q_a, q_b) = d\,|\,b - a\,|$, the stated conclusion follows.

COROLLARY (i). *The arc $f(I)$ in theorem 2 has minimum length and $l(f(I)) = \rho(p, q)$.*

PROOF. Suppose that $\{x_i \,|\, i = 0, 1, \cdots, N\}$ is any partition of I; then

$$\sum_{i=0}^{N-1} \rho(f(x_i), f(x_{i+1})) = d \sum_{i=0}^{N-1} (x_{i+1} - x_i) = d(x_N - x_0) = d.$$

Therefore $l(S) = d$. Since the length of any arc joining p_0 to p_1 is at least $\rho(p_0, p_1)$ we have proven the corollary.

COROLLARY (ii) *Let r be any third point on the arc $f(I)$ joining p and q, and let S_0 and S_1 be the subarcs of $f(I)$ from p to r, and r to q. Then S_0 and S_1 have minimum length and $l(S_0) = \rho(p, r)$ and $l(S_1) = \rho(r, b)$.*

PROOF. By Theorem 2 and Corollary (i) there is an arc T_0 of length $\rho(p, r)$ joining p to r; and there is an arc T_1, joining r to f of length $\rho(r, f)$. Also $l(T_0) \leqq l(S_0), l(T_1) \leqq l(S_1)$. Therefore:

$$\rho(p, r) + \rho(r, q) = l(T_0) + l(T_1) \leqq l(S_0) + l(S_1) = l(f(I)) = \rho(p, q).$$

But the triangle inequality implies

$$\rho(p, r) + \rho(r, q) \geqq \rho(p, q).$$

Therefore: $l(T_0) + l(T_1) = l(S_0) + l(S_1)$. Suppose $l(T_0) < l(S_0)$ then $l(T_1) > l(S_1)$ and there is a contradiction. Similarly if $l(T_1) < l(S_1)$.

THEOREM 3. *If (R, ρ), a complete metric space, has properties* (A) *and* (B) *and if p_0, p_1 are any two points, then any arc of least length joining p_0 to p_1*

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intersects the locus of points $C_\alpha \equiv \{x \mid \rho(p_0, x) = \alpha, 0 \leqq \alpha \leqq \rho(p_0, p_1)\}$ *in exactly one point.*

PROOF. Since any arc of least length joining p_0 to p_1 is a continuum, it must have at least one point in common with C_α, $(0 \leqq \alpha \leqq \rho(p_0, p_1))$. Suppose there are two points q_1, q_2 common to S and C_α. Without loss of generality, let q_1 be the first point according to the mapping of the arc from the unit interval to S_0. If S_0 denotes the subarc of S from p_0 to q_1, then corollary (ii) of theorem 2 shows that $l(S_0) = \rho(p_0, p_1) = \alpha$. If S_1 denotes the subarc of S from q_2 to p_1, then the same corollary shows that $l(S_1) = \rho(p_0, p_1) - \alpha$. Hence, the subarc of S from q_1 to q_2 must have zero length; or $q_1 = q_2$.

THEOREM 4. *If* (R, ρ), *a complete metric space, has properties* (A) *and* (B) *then the minimizing arc joining any two points* p_0, p_1 *is unique up to parametric representation.*

PROOF. Let S be the arc of least length according to the construction of Theorem 2. Assume that T is any other arc with $l(T) = l(S)$. Let $C_\alpha \equiv \{x \mid \rho(p_0, x) = \alpha\}$. By theorem 3, $C_\alpha \cap S$ and $C_\alpha \cap T$ are singleton sets, for $0 \leqq \alpha \leqq \rho(p_0, p_1)$. If $C_\alpha \cap S \equiv C_\alpha \cap T$ for all $0 \leqq \alpha \leqq \rho(p_0, p_1)$, then T is merely a different parametric representation of S. Suppose therefore that for some α, $C_\alpha \cap S = q \neq r = C_\alpha \cap T$. Then $\rho(p, r) = k \neq 0$. Using the notation introduced in the definition of property (B), by property (B) there exists an $\varepsilon > 0$ such that:

$$\operatorname{diam}[N_{p_0}(\alpha) \cap N_{p_1}(d - \alpha + \varepsilon)] < k/2.$$

We note that $\operatorname{diam}[\overline{N_{p_0}(\alpha) \cap N_{p_1}(d - \alpha + \varepsilon)}] < k/2$. Now $q \in N_{p_1}(d - \alpha + \varepsilon)$ and since $q \in C_\alpha$, therefore $q \in \overline{N_{p_0}(\alpha)}$. By an elementary theorem of topology, if A, B are subsets of a topological space and A is open, then $A \cap \bar{B} \subseteq \overline{A \cap B}$ noting that $N_{p_1}(d - \alpha + \varepsilon)$ is open, it follows that:

$$q \in \overline{N_{p_0}(\alpha) \cap N_{p_1}(d - \alpha + \varepsilon)}$$

But $r \notin \overline{N_{p_0}(\alpha) \cap N_{p_1}(d - \alpha + \varepsilon)}$, since $\rho(q, r) > k/2$ and $\operatorname{diam}[\overline{N_{p_0}(\alpha) \cap N_{p_1}(d - \alpha + \varepsilon)}] < k/2$. However, $r \in \overline{N_{p_0}(\alpha)}$, so that by the above cited theorem, $r \notin N_{p_1}(d - \alpha + \varepsilon)$ Therefore $\rho(p_1, r) \geqq d - \alpha + \varepsilon$.

If T_0 is the subarc of T from p_0 to r, and T_1 is that from r to p_1, then:

$$d = l(T_0) + l(T_1) \geqq \rho(p_0, r) + \rho(r, p_1) = \alpha + \rho(p_1, r) \geqq \alpha + (d - \alpha + \varepsilon) = d + \varepsilon.$$

This is a contradiction, and hence S is the unique arc of minimal length joining p_0 to p_1.

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