ON POLYNOMIAL EXTENSIONS OF RINGS

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Let A be a commutative ring with unit element, and let $A[x]$ be a ring of polynomials in an indeterminate x with coefficients in A. There are a number of well-known properties which A shares with $A[x]$. We shall state one of them in the following.

THEOREM. *If A is an integrally closed integral domain, then so also is $A[x]$.*

In an earlier volume of this journal, Messrs. Butts, Hall and Mann **(1)** gave a proof of the theorem. The purpose of the present note is to give a simpler elementary proof and another valuation-theoretic one.

First proof. Let K be the quotient field of A. At first we assume that K is algebraically closed. If $f(x) \in K(x)$ is integral over $A[x]$, then $f(x) \in K[x]$, since $K[x]$ is integrally closed. Since $f(x)$ satisfies a monic equation with coefficients in $A[x]$, for any element ξ in A, $f(\xi)$ is integral over A, that is, $f(\xi) \in A$. Set

$$f(x) = a_0+a_1 x+\ldots+a_m x^m \qquad (a_i \in K),$$

and take $m+1$ distinct elements ξ_j $(0 \leqslant j \leqslant m)$ in A. Then

$$a_0 + a_1\xi_j + \ldots + a_m\xi_j^m = \zeta_j, \quad \text{with } \zeta_j \in A \ (0 \leqslant j \leqslant m).$$

We solve these equations with respect to $a_0, a_1, \ldots, a_m$ and obtain

$$a_i = \frac{\eta_i}{D}, \text{ where } \eta_i \in A, D = \prod_{i<j} (\xi_i - \xi_j).$$

Here we notice that ξ_j $(0 \leqslant j \leqslant m)$ can be chosen such that $D = 1$, since any monic equation of the form $(x-\xi_1)(x-\xi_2)\ldots(x-\xi_k) = 1$, with $\xi_i \in A$, has a solution in A.

We now turn to the case in which K is not algebraically closed. Let $\bar{K}$ be the algebraic closure of K and let $\bar{A}$ be the integral closure of A in $\bar{K}$. Then the integral closure of $A[x]$ in $K(x)$ is contained in $K(x) \cap \bar{A}[x] = A[x]$.

Second proof. We first recall the following fact. Let ν be a valuation of a field K, then ν can be extended to a valuation $\bar{\nu}$ of $K(x)$ by setting, for any polynomial

$$f(x) = a_0+a_1 x+\ldots+a_m x^m$$

in $K[x]$,

$$\bar{\nu}[f(x)] = \min_{0 \leqslant i \leqslant m} \nu(a_i).$$

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Since A is integrally closed, A can be represented as an intersection of a set of valuation rings of K: $A = \bigcap R_\iota$. Denote by $\bar{R}_\iota$ the valuation ring of $K(x)$, which is uniquely deduced from R_ι in the manner described just above. Then we have obviously

$$A[x] = \bigcap \bar{R}_\iota \cap K[x].$$

Since $\bar{R}_\iota$ and $K[x]$ are integrally closed, so is $A[x]$.

Remark. An element a of K is said to be *almost integral over* A, if there exists an element $b \neq 0$ of A such that $ba^n \in A$ for all n. If any element of K which is almost integral over A is contained in A, A is said to be *fully integrally closed*. We note that in our theorem the phrase "integrally closed" can be replaced by "fully integrally closed." This can be proved as follows.

Let $f(x) \in K(x)$ be almost integral over $A[x]$, so that $f(x) \in K[x]$. We shall show that $f(x) \in A[x]$, by induction with respect to the degree of $f(x)$. By the definition of "almost integral," there exists a non-zero polynomial $g(x)$ in $A[x]$, such that

$$g(x)\, f(x)^\nu \in A[x]$$

for any positive integer ν. Let b, a be the leading coefficients of $g(x)$ and $f(x)$ respectively, and put $f(x) = ax^m + f_1(x)$. Then $ba^\nu \in A$, hence $a \in A$; consequently

$$g(x)\, f_1(x)^\nu = g(x)[f(x) - ax^m]^\nu \in A[x],$$

whence by the induction assumption, $f_1(x) \in A[x]$, hence $f(x) \in A[x]$.

References

1. H. Butts, M. Hall and H. B. Mann, *On integral closure*, Can. J. Math., *6* (1954), 471–473.

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