*J. Aust. Math. Soc.* **88** (2010), 405–412
doi:10.1017/S1446788710000170

# A NOTE ON SCHRÖDINGER MAXIMAL OPERATORS WITH A COMPLEX PARAMETER

**PER SJÖLIN and FERNANDO SORIA** ✉

(Received 23 April 2009; accepted 26 January 2010)

Communicated by A. M. Hassell

**Abstract**

Extending previous results of the first author, some new estimates are obtained for maximal operators of Schrödinger type with a complex parameter.

2000 *Mathematics subject classification*: primary 42B15; secondary 42B25.

*Keywords and phrases*: maximal function, Schrödinger equation, Sobolev spaces.

## 1. Introduction

For $f$ belonging to the Schwartz class $\mathcal{S}(\mathbb{R})$, we set

$$S_t f(x) = \int_{\mathbb{R}} e^{ix\xi} e^{it\xi^2} \widehat{f}(\xi)\, d\xi \quad \forall x \in \mathbb{R}.$$

Here $t$ is a complex number such that $\operatorname{Im} t \geq 0$, and $\widehat{f}$ denotes the Fourier transform of the function $f$, defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

If we set $U(x, t) = (2\pi)^{-1} S_t f(x)$, where $x \in \mathbb{R}$ and $t \in \mathbb{R}$, then it follows that $U(x, 0) = f(x)$ for all $x$ and further that $U$ satisfies the Schrödinger equation $i\partial U/\partial t = \partial^2 U/\partial x^2$. On the other hand, if we take $t = iu$, where $u > 0$, then $U$ is, modulo a constant, the solution to the usual heat equation with initial value $f$ with respect to the 'time variable' $u$.

We define the maximal function $S^* f$ by

$$S^* f(x) = \sup_{0<t<1} |S_t f(x)| \quad \forall x \in \mathbb{R},$$

The second author was partially supported by grants MTM2007-60952 and PR2009-0084.

and also define Sobolev spaces $H_s$ for all real $s$ by setting

$$H_s = \{f \in \mathcal{S}' : \|f\|_{H_s} < \infty\},$$

where

$$\|f\|_{H_s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi\right)^{1/2}.$$

It is well known that the estimate

$$\|S^* f\|_2 \le C\|f\|_{H_s}$$

holds if $s > 1/2$ and does not hold if $s < 1/2$ (see [1]). Here $\|S^* f\|_2$ denotes the norm of $S^* f$ in the space $L^2(\mathbb{R})$, and $C$ denotes a constant that varies from place to place.

When $0 < \gamma < \infty$ and $u > 0$, we set

$$P_u f(x) = S_{u+iu^\gamma} f(x) = \int_{\mathbb{R}} e^{ix\xi} e^{iu\xi^2} e^{-u^\gamma \xi^2} \widehat{f}(\xi) \, d\xi \quad \forall x \in \mathbb{R},$$

and

$$P^* f(x) = \sup_{0<u<1} |P_u f(x)| \quad \forall x \in \mathbb{R}.$$

In Sjölin [3] the inequality

$$\|P^* f\|_2 \le C\|f\|_{H_s} \tag{1.1}$$

was studied for various values of $\gamma$ and the following results were obtained.

**THEOREM A.**

(i) *When* $0 < \gamma \le 1$, *(1.1) holds if and only if* $s \ge 0$.
(ii) *When* $\gamma = 2$, *(1.1) holds if and only if* $s \ge 1/4$.
(iii) *When* $\gamma \ge 4$, *if (1.1) holds then* $s \ge 1/2 - 1/\gamma$.

When $\gamma > 0$, we denote by $E_\gamma$ the set of all $s$ such that (1.1) holds, and set

$$s(\gamma) = \inf E_\gamma.$$

It was proved in [3] that $s$ is a nondecreasing function on the interval $(0, \infty)$, and that $0 \le s(\gamma) \le 1/2$ when $0 < \gamma < \infty$.

The results in Theorem A can be stated in the following way.

**THEOREM B.**

(i) *When* $0 < \gamma \le 1$, $s(\gamma) = 0$.
(ii) $s(2) = 1/4$.
(iii) *When* $\gamma > 4$, $1/2 - 1/\gamma \le s(\gamma) \le 1/2$ *and hence*

$$\lim_{\gamma \to \infty} s(\gamma) = 1/2.$$

We give here the following improvement of the above results.

https://doi.org/10.1017/S1446788710000170 Published online by Cambridge University Press

THEOREM 1.1. *If* $\gamma > 1$ *and* $s > 1/2 - 1/(2\gamma)$, *then* (1.1) *holds.*

The result in Theorem 1.1 is new when $1 < \gamma < 2$ and $\gamma > 2$, and allows us to extend Theorem B in the following way.

THEOREM 1.2.

(i) *When* $0 < \gamma \le 1$, $s(\gamma) = 0$.
(ii) *When* $1 < \gamma < 2$, $0 \le s(\gamma) \le 1/2 - 1/(2\gamma)$.
(iii) $s(2) = 1/4$.
(iv) *When* $2 < \gamma \le 4$, $1/4 \le s(\gamma) \le 1/2 - 1/(2\gamma)$.
(v) *When* $\gamma > 4$, $1/2 - 1/\gamma \le s(\gamma) \le 1/2 - 1/(2\gamma)$.

## 2. Proof of the theorems

For the proof of the above results we shall use the following lemmas.

LEMMA 2.1. *Assume that* $a > 1$, $1/2 \le s < 1$ *and* $\mu \in C_0^\infty(\mathbb{R})$. *Then*

$$\left| \int_{\mathbb{R}} e^{ix\xi + it|\xi|^a} |\xi|^{-s} \mu(\xi/N)\, d\xi \right| \le C \frac{1}{|x|^{1-s}} \quad \forall x \in \mathbb{R}\setminus\{0\},$$

*when* $t \in \mathbb{R}$ *and* $N = 1, 2, 3, \ldots$. *Here the constant* $C$ *may depend on* $s$ *and* $a$ *but not on* $x$, $t$ *or* $N$.

A proof of Lemma 2.1 can be found in [2].

LEMMA 2.2. *Assume that* $1/2 \le \alpha < 1$ *and* $0 < d_1, d_2 < 1$, *and also that* $\mu \in C_0^\infty(\mathbb{R})$ *is even and real-valued. Then*

$$\left| \int_{\mathbb{R}} \exp(i(d_1 - d_2)\xi^2 - ix\xi)(1 + \xi^2)^{-\alpha/2} \exp(-(d_1^2 + d_2^2)\xi^2) \mu(\xi/N)\, d\xi \right| \le K(x) \quad \forall x \in \mathbb{R}$$

*when* $N = 1, 2, 3, \ldots$, *where* $K \in L^1(\mathbb{R})$. *Here* $K$ *is independent of* $d_1$, $d_2$ *and* $N$.

Lemma 2.2 is proved in [3].

We also need two new lemmas.

LEMMA 2.3. *Assume that* $1 < \gamma < 2$, $(\gamma - 1)/\gamma < \alpha < 1/2$, $0 < d_1, d_2 < 1$, *and* $\mu$ *is as in Lemma* 2.2. *Then*

$$\left| \int_{\mathbb{R}} \exp(i(d_1 - d_2)\xi^2 - ix\xi)(1 + \xi^2)^{-\alpha/2} \exp(-(d_1^\gamma + d_2^\gamma)\xi^2) \mu(\xi/N)\, d\xi \right| \le K(x) \quad \forall x \in \mathbb{R}$$

*when* $N = 1, 2, 3, \ldots$, *where* $K \in L^1(\mathbb{R})$. *Here* $K$ *is independent of* $d_1$, $d_2$ *and* $N$.

https://doi.org/10.1017/S1446788710000170 Published online by Cambridge University Press

LEMMA 2.4. *Assume that* $\gamma > 2$, $(\gamma - 1)/\gamma < \alpha < 1$, $0 < d_1, d_2 < 1$, *and* $\mu$ *is as in Lemma 2.2. Then*

$$\left| \int_{\mathbb{R}} \exp(i(d_1 - d_2)\xi^2 - ix\xi)(1+\xi^2)^{-\alpha/2} \exp(-(d_1^\gamma + d_2^\gamma)\xi^2)\mu(\xi/N)\, d\xi \right| \\ \le K(x) \quad \forall x \in \mathbb{R}$$

*when* $N = 1, 2, 3, \ldots$, *where* $K \in L^1(\mathbb{R})$. *Here* $K$ *is independent of* $d_1$, $d_2$ *and* $N$.

We now give the proofs of Lemmas 2.4 and 2.3.

PROOF OF LEMMA 2.4. Let $C_0$ denote a large constant. Since $1/2 < \alpha < 1$, in the case where $|x| \le C_0$ we can use the proof in [2] of Lemma 2.1 to conclude that the estimate in Lemma 2.4 holds when $K(x) = C|x|^{\alpha-1}$. To obtain this, we have to use the observation (see [3]) that if $h(\xi) = h_\epsilon(\xi) = e^{-\epsilon\xi^2}$ where $0 < \epsilon < 2$, then

$$|h'(\xi)| \le C\,\frac{1}{\xi} \quad \forall \xi \in [1/2, \infty),$$

where $C$ is independent of $\epsilon$.

We now consider the case where $|x| > C_0$. To that end, we shall modify the proof in [3] of our Lemma 2.2.

We may assume that $d_2 < d_1$ and set $d = d_1 - d_2$ and $\epsilon = d_1^\gamma + d_2^\gamma$, so that $0 < d < 1$ and $0 < \epsilon < 2$. Also set $\rho = |x|/(2d)$ and

$$\psi(\xi) = (1+\xi^2)^{-\alpha/2} e^{-\epsilon\xi^2} \mu(\xi/N) \quad \forall \xi \in \mathbb{R}.$$

Choose an even function $\varphi_0 \in \mathcal{C}^\infty$ such that $\varphi_0(\xi) = 1$ if $|\xi| \le 1/2$ and $\varphi(\xi) = 0$ if $|\xi| \ge 1$. Set $\psi_0 = \psi\varphi_0$, so that supp $\psi_0 \subset [-1, 1]$. Then, for a large constant $K_1$, choose $\varphi_2 \in \mathcal{C}_0^\infty$ so that supp $\varphi_2 \subset [\rho/4, 2K_1\rho]$ and $\varphi_2(\xi) = 1$ if $\rho/2 \le \xi \le K_1\rho$. We may also assume that $|\varphi_2'(\xi)| \le C\xi^{-1}$ and $|\varphi_2''(\xi)| \le C\xi^{-2}$ if $\xi > 0$. We also set $\varphi_3 = (1 - \varphi_2)\chi_{[K_1\rho,\infty)}$ and $\varphi_1 = (1 - \varphi_2 - \varphi_0)\chi_{[0,\rho/2]}$.

Having defined the cutoff functions $\varphi_j$, where $j = 0, 1, 2, 3$, it is clear that it is sufficient to estimate the integrals

$$\mathcal{J}_j = \int e^{iF}\psi_j\, d\xi$$

where $F(\xi) = d\xi^2 - x\xi$ and $\psi_j(\xi) = \psi(\xi)\varphi_j(\xi)$. (A similar argument works for the functions $\psi(\xi)\varphi_j(-\xi)$.) A double integration by parts easily shows the estimate $|\mathcal{J}_0| \le C/|x|^2$ (see [3]). Now observe that when $j = 1, 2, 3$ and $\xi > 1/2$, the pointwise estimates

$$|\psi_j(\xi)| \le C\frac{1}{(1+\xi^2)^{\alpha/2}},$$
$$|\psi_j'(\xi)| \le C\frac{1}{(1+\xi^2)^{\alpha/2}\xi},$$

https://doi.org/10.1017/S1446788710000170 Published online by Cambridge University Press

and

$$|\psi_j''(\xi)| \le C\frac{1}{(1+\xi^2)^{\alpha/2}\xi^2}$$

hold. Using the same arguments as in [3], we obtain the estimate $\mathcal{O}(|x|^{-2})$ for $\mathcal{J}_1$ and $\mathcal{J}_3$.

To estimate $\mathcal{J}_2$, we use van der Corput's lemma and deduce that

$$\begin{aligned}|\mathcal{J}_2| &\le Cd^{-1/2}\rho^{-\alpha}\exp(-c\epsilon\rho^2)\\ &\le Cd^{-1/2}\left(\frac{|x|}{d}\right)^{-\alpha}\exp(-c(d_1^\gamma+d_2^\gamma)|x|^2/d^2)\\ &\le Cd^{\alpha-1/2}|x|^{-\alpha}\exp(-c(d_1+d_2)^\gamma|x|^2/d^2)\\ &\le Cd^{\alpha-1/2}|x|^{-\alpha}\exp(-cd^{\gamma-2}|x|^2),\end{aligned}$$

where we have used the fact that $d_1+d_2\ge d$. Here $c$ denotes possibly different positive constants.

We now invoke the inequality

$$e^{-y}\le C_\beta y^{-\beta}, \tag{2.1}$$

which holds whenever $y>0$ and $\beta>0$, to deduce that

$$\begin{aligned}|\mathcal{J}_2| &\le Cd^{\alpha-1/2}|x|^{-\alpha}\frac{1}{d^{(\gamma-2)\beta}|x|^{2\beta}}\\ &= C\frac{d^{\alpha-1/2}}{d^{\beta(\gamma-2)}}\frac{1}{|x|^{\alpha+2\beta}}.\end{aligned}$$

We now choose $\beta$ so that $\beta(\gamma-2)=\alpha-1/2$, that is,

$$\beta=\frac{\alpha-1/2}{\gamma-2}.$$

Since $\gamma>2$ and $1/2<\alpha<1$, it is clear that $\beta$ is positive. We obtain the inequality

$$|\mathcal{J}_2|\le C\frac{1}{|x|^{\alpha+2\beta}}.$$

Finally, using our assumption that $\alpha>(\gamma-1)/\gamma$, we get

$$\alpha+2\beta=\frac{\alpha\gamma-1}{\gamma-2}>\frac{\gamma-1-1}{\gamma-2}=1.$$

Hence the function $|x|^{-\alpha-2\beta}$ is integrable when $|x|>C_0$ and the proof of Lemma 2.4 is complete. □

Proof of Lemma 2.3. As before, we let $C_0$ denote a large constant. We first study the case where $|x|>C_0$. With the same notation as in the previous proof and the arguments in [3], the estimates for $\mathcal{J}_0$, $\mathcal{J}_1$ and $\mathcal{J}_3$ follow easily. (Observe that the condition $\alpha\ge 1/2$ was not used for these estimates.)

https://doi.org/10.1017/S1446788710000170 Published online by Cambridge University Press

To estimate $\mathcal{J}_2$ we use van der Corput's lemma again and deduce that

$$\begin{aligned}|\mathcal{J}_2| &\le C d^{-1/2} \rho^{-\alpha} e^{-c\epsilon\rho^2} \\ &\le C d^{\alpha-1/2} |x|^{-\alpha} e^{-cd^{\gamma-2}|x|^2}.\end{aligned}$$

Using inequality (2.1), we then obtain

$$\begin{aligned}|\mathcal{J}_2| &\le C d^{\alpha-1/2} |x|^{-\alpha} \frac{1}{d^{(\gamma-2)\beta} |x|^{2\beta}} \\ &= C \frac{d^{\beta(2-\gamma)}}{d^{1/2-\alpha}} \frac{1}{|x|^{\alpha+2\beta}}.\end{aligned}$$

Here $2-\gamma > 0$ and, therefore, $1/2 - \alpha > 0$. Choosing $\beta$ large, we conclude that

$$|\mathcal{J}_2| \le C \frac{1}{|x|^{\alpha+2\beta}} \le C \frac{1}{|x|^2}.$$

This completes the proof in the case where $|x| > C_0$. It remains to study the case where $|x| \le C_0$. To do so, we modify the arguments given in the proof of Lemma 2.1 (see [2]). Since $\alpha < 1/2$, we need a different argument to estimate

$$\int_{I_2} e^{iF} \psi \, d\xi,$$

where, for some constants $c_1$ small and $C_1$ large, $I_2$ denotes the interval

$$I_2 = \left\{ \xi \ge \frac{1}{|x|} : c_1 \frac{|x|}{d} \le \xi \le C_1 \frac{|x|}{d} \right\}.$$

Also,

$$\begin{aligned}F(\xi) &= -x\xi + d\xi^2, \\ \psi(\xi) &= (1+\xi^2)^{-\alpha/2} e^{-\epsilon\xi^2} \mu(\xi/N) \quad \forall \xi \in \mathbb{R},\end{aligned}$$

and $d = d_1 - d_2$, $\epsilon = d_1^\gamma + d_2^\gamma$. The rest of the proof is unchanged.

Set $\rho = |x|/(2d)$ as before. Arguing as in the proof of Lemma 2.2, we deduce that

$$|\psi| \le C \rho^{-\alpha} e^{-c\epsilon\rho^2}$$

on $I_2$, and

$$\int_{I_2} |\psi'| \, d\xi \le C \rho^{-\alpha} e^{-c\epsilon\rho^2}.$$

An application of van der Corput's lemma then yields

$$\left| \int_{I_2} e^{iF} \psi \, d\xi \right| \le C d^{-1/2} \rho^{-\alpha} e^{-c\epsilon\rho^2}.$$

https://doi.org/10.1017/S1446788710000170 Published online by Cambridge University Press

Arguing as in the previous case, we obtain the estimate

$$\left|\int_{I_2} e^{iF}\psi\, d\xi\right| \le C\frac{d^{\beta(2-\gamma)}}{d^{1/2-\alpha}}\frac{1}{|x|^{\alpha+2\beta}}.$$

Choosing

$$\beta = \frac{1/2-\alpha}{2-\gamma},$$

it follows that

$$\left|\int_{I_2} e^{iF}\psi\, d\xi\right| \le C\frac{1}{|x|^{\alpha+2\beta}},$$

and using our assumption that $\alpha > (\gamma-1)/\gamma$, we get

$$\alpha + 2\beta = \frac{1-\alpha\gamma}{2-\gamma} < \frac{1-(\gamma-1)}{2-\gamma} = 1.$$

Hence, the function $x \mapsto |x|^{-\alpha-2\beta}$ is integrable in the interval $|x| \le C_0$ and the proof of Lemma 2.3 is complete. □

Finally, we give the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. As in [3, Theorem 1], we only need to prove that

$$\|T_N^* h\|_2 \le C\|h\|_2, \tag{2.2}$$

when $N = 1, 2, 3, \ldots$, where the operators $T_N^*$ are defined by

$$T_N^* h(\xi) = \rho_N(\xi)(1+\xi^2)^{-s/2}\int_{\mathbb{R}} e^{-ix\xi}e^{-iu(x)\xi^2}e^{-(u(x))^\gamma\xi^2}\chi_N(x)h(x)\,dx.$$

Here $\chi_N(x) = \chi(x/N)$, $\rho_N(\xi) = \rho(\xi/N)$ and $\chi, \rho \in C_0^\infty(\mathbb{R})$ are such that

$$\chi(x) = \rho(x) = \begin{cases} 1 & \text{when } |x| \le 1, \\ 0 & \text{when } |x| \ge 2, \end{cases}$$

and both $\chi$ and $\rho$ are even and real-valued. Further, $u$ is a measurable function on $\mathbb{R}$ such that $0 < u(x) < 1$. Invoking Lemmas 2.3 or 2.4, we then have

$$\begin{aligned} \|T_N^* h\|_2^2 &= \int T_N^* h(\xi)\overline{T_N^* h(\xi)}\,d\xi \\ &= \int \rho_N(\xi)^2(1+\xi^2)^{-s}\left(\int_{\mathbb{R}} e^{-ix\xi}e^{-iu(x)\xi^2}e^{-(u(x))^\gamma\xi^2}\chi_N(x)h(x)\,dx\right) \\ &\quad\times\left(\int_{\mathbb{R}} e^{iy\xi}e^{iu(y)\xi^2}e^{-(u(y))^\gamma\xi^2}\chi_N(y)\overline{h(y)}\,dy\right)d\xi. \end{aligned}$$

https://doi.org/10.1017/S1446788710000170 Published online by Cambridge University Press

Here, when $1 < \gamma < 2$ we have assumed, as we may, that $1/2 - 1/(2\gamma) < s < 1/4$. If $\alpha = 2s$ and $1 < \gamma < 2$, then

$$1 - 1/\gamma < \alpha < 1/2.$$

Also, if $\alpha = 2s$ and $\gamma > 2$, then we will assume that $1/2 - 1/(2\gamma) < s < 1/2$, so that

$$1 - 1/\gamma < \alpha < 1.$$

Hence, setting $\mu = \rho^2$ and applying Lemmas 2.3 and 2.4,

$$\begin{aligned}\|T_N^* h\|_2^2 = \iint \Big( \int & (1+\xi^2)^{-s} \exp(i(y-x)\xi) \exp(i(u(y)-u(x))\xi^2) \\ & \times \exp(-((u(y))^\gamma + (u(x))^\gamma)\xi^2)\mu(\xi/N)\, d\xi \Big) \\ & \times \chi_N(x)\chi_N(y)h(x)\overline{h(y)}\, dx\, dy \\ \leq C \iint & K(x-y)|h(x)|\,|h(y)|\, dx\, dy \leq C\|h\|_2^2.\end{aligned}$$

Hence (2.2) is proved, and the proof of Theorem 1.1 is complete. □

## References

[1] P. Sjölin, 'Global maximal estimates for solutions to the Schrödinger equation', *Studia Math.* **110** (1994), 105–114.

[2] P. Sjölin, 'Maximal estimates for solutions to the nonelliptic Schrödinger equation', *Bull. Lond. Math. Soc.* **39** (2007), 404–412.

[3] P. Sjölin, 'Maximal operators of Schrödinger type with a complex parameter', *Math. Scand.* **105** (2009), 121–133.

PER SJÖLIN, Department of Mathematics, Royal Institute of Technology,
S-100 44 Stockholm, Sweden
e-mail: pers@math.kth.se

FERNANDO SORIA, Departamento de Matemáticas,
Universidad Autónoma de Madrid and Instituto de Ciencias Matemáticas, ICMAT,
E-28049 Madrid, Spain
e-mail: fernando.soria@uam.es

https://doi.org/10.1017/S1446788710000170 Published online by Cambridge University Press