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PERCOLATION AND CONNECTIVITY IN *AB* RANDOM GEOMETRIC GRAPHS

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Abstract

Given two independent Poisson point processes $\Phi^{(1)}$, $\Phi^{(2)}$ in $\mathbb{R}^d$, the *AB* Poisson Boolean model is the graph with the points of $\Phi^{(1)}$ as vertices and with edges between any pair of points for which the intersection of balls of radius $2r$ centered at these points contains at least one point of $\Phi^{(2)}$. This is a generalization of the *AB* percolation model on discrete lattices. We show the existence of percolation for all $d \geq 2$ and derive bounds for a critical intensity. We also provide a characterization for this critical intensity when $d = 2$. To study the connectivity problem, we consider independent Poisson point processes of intensities n and τn in the unit cube. The *AB* random geometric graph is defined as above but with balls of radius r. We derive a weak law result for the largest nearest-neighbor distance and almost-sure asymptotic bounds for the connectivity threshold.

Keywords: Random geometric graph; percolation; connectivity; wireless network; secure communication

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1. Introduction

The Bernoulli (site) percolation model on a graph $G := (V, E)$ is defined as follows. Each vertex $v \in V$ of the graph is retained with a probability p or removed with probability $1 - p$, along with all the edges incident to that vertex, independently of other vertices. The model is said to percolate if the random subgraph resulting from the deletion procedure contains an infinite connected component. The classical percolation model is the Bernoulli bond percolation model with the difference being that the deletion procedure is applied to the edges instead of the vertices. Grimmett (1999) is an excellent source for the rich theory on this classical percolation model. A variant of the Bernoulli site percolation model that has been of interest is the *AB* percolation model. This model was first studied in Halley (1980), Halley (1983), and Sevšek *et al*. (1983). The model is as follows. Given a graph G, each vertex is marked independently of other vertices as either A or B. Edges between vertices with similar marks (A or B) are removed. The resulting random subgraph is the *AB* graph model. If the *AB* graph contains an infinite connected component with positive probability, we say that the model percolates. An infinite connected component in the *AB* graph is equivalent to an infinite path of vertices in G with marks alternating between A and B. This model has been studied on lattices and

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some related graphs. The *AB* percolation model behaves quite differently as compared to the Bernoulli percolation model. For example, it is known that *AB* percolation does not occur in $\mathbb{Z}^2$ (see Appel and Wierman (1987)), but occurs on the planar triangular lattice (see Wierman and Appel (1987)), some periodic two-dimensional graphs (see Scheinerman and Wierman (1987)), and the half close-packed graph of $\mathbb{Z}^2$ (see Wu and Popov (2003)). It is also known that the *AB* bond percolation does not occur in $\mathbb{Z}^2$ for $p = \frac{1}{2}$ (see Wu and Popov (2003)). See Wu and Popov (2003) and Grimmett (1999) for further references.

The following generalization of the discrete *AB* percolation model has been studied on various graphs by Kesten *et al.* (see Benjamini and Kesten (1995), Kesten *et al*. (1998), and Kesten *et al*. (2001)). Mark each vertex or site of a graph G independently as either 0 or 1 with probability p and $1 - p$, respectively. Given any infinite sequence (referred to as a word) $w \in \{0, 1\}^\infty$, the question is whether w occurs in the graph G or not. The sentences $(1, 0, 1, 0, \dots)$, $(0, 1, 0, 1, \dots)$ correspond to *AB* percolation and the sequence $(1, 1, 1, \dots)$ corresponds to Bernoulli percolation. More generally, Kesten *et al.* answered the question of whether all (or almost all) infinite sequences (words) occur or not. The graphs for which the answer is known in the affirmative are $\mathbb{Z}^d$ for large d, the triangular lattice, and $\mathbb{Z}^2_{\text{cp}}$, the close-packed graph of $\mathbb{Z}^2$. Our results provide partial answers to these questions in the continuum.

Our aim is to study a generalization of the discrete *AB* percolation model to the continuum. We study the problem of percolation and connectivity in such models. For the percolation problem, the vertex set of the graph will be a homogeneous Poisson point process in $\mathbb{R}^d$. For the connectivity problem, we will consider a sequence of graphs whose vertex sets will be homogeneous Poisson point processes of intensity n in $[0, 1]^d$. We consider different models while studying percolation and connectivity so as to be consistent with the literature. This allows for easy comparison with, as well as the use of, existing results from the literature. We will refer to our graphs, in the percolation context, as the *AB* Poisson Boolean model, and as the *AB* random geometric graph while investigating the connectivity problem. The Poisson Boolean model and random geometric graphs where the nodes are of the same type are the topics of the monographs in Meester and Roy (1996) and Penrose (2003), respectively.

Our motivation for the study of *AB* random geometric graphs comes from applications to wireless communication. In models of ad hoc wireless networks, the nodes are assumed to be communicating entities that are distributed randomly in space. Edges between any two nodes in the graph represent the ability of the two nodes to communicate effectively with each other. In one of the widely used models, a pair of nodes share an edge if the distance between the nodes is less than a certain cutoff radius $r > 0$ that is determined by the transmission power. Percolation and connectivity thresholds for such a model have been used to derive, for example, the capacity of wireless networks (see Franceschetti *et al*. (2007) and Gupta and Kumar (2000)). Consider a transmission scheme called the frequency division half-duplex, where each node transmits at a frequency f_1 and receives at frequency f_2 or vice versa (see Tse and Vishwanath (2005)). Thus, nodes with transmission–reception frequency pair (f_1, f_2) can communicate only with nodes that have transmission–reception frequency pair (f_2, f_1) that are located within the cutoff distance r. Another example where such a model would be applicable is in communication between communicating units deployed at two different levels, for example, surface (or underwater) and in air. Units in a level can communicate only with those at the other level that are within a certain range. A third example is in secure communication in wireless sensor networks with two types of node, tagged and normal. Upon deployment, each tagged node broadcasts a key over a predetermined secure channel, which is received by all normal nodes that are within transmission range. Two normal nodes can then communicate, provided there is a tagged node from which both these normal nodes have received a key, that is, the tagged node is within the transmission range of both the normal nodes. A somewhat complimentary model has been proposed in this context, first in Haenggi (2008) and further developed in Pinto and Win (2010). It is a model for secure communication in the presence of malicious nodes. Legitimate nodes that have malicious nodes in their vicinity cannot communicate, or the edge set will be determined by some information-theoretic criterion. The problem then is to obtain the critical intensity of malicious nodes above which the network does not percolate.

The rest of the paper is organized as follows. Sections 2 and 3 provide definitions and statements of our main theorems on percolation and connectivity, respectively. Sections 4 and 5 contain the proofs of these results.

2. Percolation in the *AB* Poisson Boolean model

2.1. Model definition

Let $\Phi^{(1)} = \{X_i\}_{i\geq 1}$ and $\Phi^{(2)} = \{Y_i\}_{i\geq 1}$ be independent Poisson point processes in $\mathbb{R}^d$, $d \geq 2$, with intensities λ and μ, respectively. Let the Lebesuge measure and the Euclidean metric on $\mathbb{R}^d$ be denoted by $\|\cdot\|$ and $|\cdot|$, respectively. Let $B_x(r)$ denote the closed ball of radius r centered at $x \in \mathbb{R}^d$.

By percolation in a graph, we mean the existence of an infinite connected component in the graph. The standard continuum percolation model (introduced in Gilbert (1961)), also called the Poisson Boolean model or Gilbert disk graph, is defined as follows.

Definition 2.1. Define the Poisson Boolean model $\tilde{G}(\lambda, r) := (\Phi^{(1)}, \tilde{E}(\lambda, r))$ to be the graph with vertex set $\Phi^{(1)}$ and edge set

$$\tilde{E}(\lambda, r) = \{\langle X_i, X_j\rangle : X_i, X_j \in \Phi^{(1)}, |X_i - X_j| \leq 2r\}.$$

For fixed $r > 0$, define the critical intensity of the Poisson Boolean model as follows:

$$\lambda_c(r) := \sup\{\lambda > 0 : \mathrm{P}(\tilde{G}(\lambda, r) \text{ percolates}) = 0\}. \tag{2.1}$$

The edges in all the graphs that we consider are undirected, that is, $\langle X_i, X_j\rangle \equiv \langle X_j, X_i\rangle$. We will use the notation $X_i \sim X_j$ to denote existence of an edge between X_i and X_j when the underlying graph is unambiguous. For the Poisson Boolean model (see Meester and Roy (1996)), it is known that $0 < \lambda_c(r) < \infty$. Topologically, percolation in the Poisson Boolean model is equivalent to existence of an unbounded connected subset in $\bigcup_{X\in\Phi^{(1)}} B_X(r)$. Also, by the zero-one law, we can deduce that the probability of percolation is either 0 or 1.

A natural analogue of this model to the *AB* setup would be to consider a graph with vertex set $\Phi^{(1)}$ where each vertex is independently marked either A or B. We will consider a more general model from which results for the above model will follow as a corollary.

Definition 2.2. The *AB* Poisson Boolean model $G(\lambda, \mu, r) := (\Phi^{(1)}, E(\lambda, \mu, r))$ is the graph with vertex set $\Phi^{(1)}$ and edge set

$$E(\lambda, \mu, r) := \{\langle X_i, X_j\rangle : X_i, X_j \in \Phi^{(1)}, |X_i - Y| \leq 2r, |X_j - Y| \leq 2r, \text{ for some } Y \in \Phi^{(2)}\}.$$

Let $\theta(\lambda, \mu, r) = \mathrm{P}(G(\lambda, \mu, r) \text{ percolates})$. For a fixed $\lambda, r > 0$, define the critical intensity $\mu_c(\lambda, r)$ by

$$\mu_c(\lambda, r) := \sup\{\mu > 0 : \theta(\lambda, \mu, r) = 0\}. \tag{2.2}$$

It follows from the zero-one law that $\theta(\lambda, \mu, r) \in \{0, 1\}$. We are interested in characterizing the region formed by (λ, μ, r) for which $\theta(\lambda, \mu, r) = 1$.

2.2. Main results

The *AB* Poisson Boolean model $G(\lambda, \mu, r)$ is a subgraph of the Poisson Boolean model $\tilde{G}(\lambda + \mu, r)$ with the vertex set $\Phi^{(1)} \cup \Phi^{(2)}$. This simple coupling will be used to obtain the following lower bounds for the critical intensity $\mu_c(\lambda, r)$.

Proposition 2.1. *Fix* $\lambda, r > 0$. *Let* $\lambda_c(r)$ *and* $\mu_c(\lambda, r)$ *be the critical intensities as in (2.1) and (2.2), respectively. Then*

1. $\mu_c(\lambda, r) \geq \lambda_c(r) - \lambda$ *if* $\lambda_c(2r) < \lambda < \lambda_c(r)$, *and*
2. $\mu_c(\lambda, r) = \infty$ *if* $\lambda < \lambda_c(2r)$.

The second part of Proposition 2.1 holds for $\lambda = \lambda_c(2r)$ provided that $\tilde{G}(\lambda_c(2r), 2r)$ does not percolate. This has been proven for $d = 2$ (see Meester and Roy (1996, Theorem 4.5)) and for all but at most finitely many d (see Tanemura (1996)). The next question is whether $\mu_c(\lambda, r) < \infty$ if $\lambda > \lambda_c(2r)$. We answer this in the affirmative for $d = 2$.

Theorem 2.1. *Let* $d = 2$ *and* $r > 0$ *be fixed. Then, for any* $\lambda > \lambda_c(2r)$, *we have* $\mu_c(\lambda, r) < \infty$.

The proof of Theorem 2.1 will adapt the idea used in Dousse *et al.* (2006) of coupling the continuum percolation model to a 1-dependent discrete percolation model on $\mathbb{Z}^2$. We will use Peierl's argument to show that the discrete percolation model percolates and coupling will yield the percolation of the corresponding *AB* Poisson Boolean model.

The *AB* Poisson Boolean model thus exhibits a *phase transition* in the plane. However, Theorem 2.1 does not tell us how to choose a μ for a given $\lambda > \lambda_c(2r)$ for $d = 2$ such that *AB* percolation happens, or if indeed there is a phase transition for $d \geq 3$. We obtain an upper bound for $\mu_c(\lambda, r)$ as a special case of a more general result which is the continuum analog of word percolation on discrete lattices described in Section 1. In order to state this result, we need some notation.

Definition 2.3. For each $d \geq 2$, define the critical probabilities $p_c(d)$ and the functions $a(d, r)$ as follows.

1. For $d = 2$, consider the triangular lattice $\mathbb{T}$ (part of which is shown in Figure 1) with edge length $r/2$. Let $p_c(2)$ be the critical probability for the Bernoulli site percolation on this lattice. Around each vertex place a 'flower' formed by the intersection (see Figure 1) of the six circles, each of radius $r/2$ and centered at the midpoints of the six edges incident on the vertex. Let $a(2, r)$ be the area of a flower.

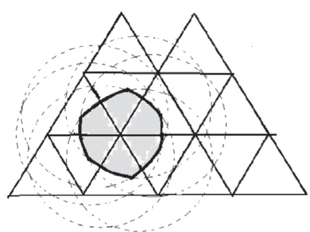

FIGURE 1: A piece of the triangular lattice and a flower (*shaded area*) in $\mathbb{R}^2$ with area $a(2, r)$. The arcs forming the flower have been darkened while the dotted circles are of radius $r/2$ and centered at the midpoints of the six edges incident on the vertex.

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2. For $d \geq 3$, let $p_c(d)$ be the critical probability for the Bernoulli site percolation on $\mathbb{Z}^{*d} := (\mathbb{Z}^d, \mathbb{E}^{*d} := \{\langle z, z_1 \rangle : |z - z_1|_\infty = 1\})$, where $|\cdot|_\infty$ stands for the l_∞-norm. Define $a(d, r) = (r/2\sqrt{d})^d$.

It is known that $p_c(2) = \frac{1}{2}$ and $p_c(d) < 1$ for $d \geq 3$ (see Grimmett (1999)).

Definition 2.4. For $i = 1, \dots, k$, let $\Phi^{(i)}$ be independent Poisson point processes in $\mathbb{R}^d$ of intensities $\lambda_i > 0$. Fix $(r_1, \dots, r_k) \in \mathbb{R}_+^k$. A word $\omega := \{w_i\}_{i \geq 1} \in \{1, 2, \dots, k\}^{\mathbb{N}}$ is said to occur if there exists a sequence $\{X_i\}_{i \geq 1}$ of distinct elements such that $X_i \in \Phi^{(w_i)}$, and $|X_i - X_{i+1}| \leq r_{w_i} + r_{w_{i+1}}$, for $i \geq 1$.

We will define a suitable coupling of the *AB* Poisson Boolean model with an independent percolation model on the triangular lattice in $d = 2$ and the Euclidean lattice in $d \geq 3$ such that percolation in these latter models imply percolation of the *AB* Poisson Boolean model. Once this coupling is obtained (as we will in the proofs), the next proposition and the following two corollaries will follow easily.

Proposition 2.2. *For any $d \geq 2$, let $p_c(d)$ and $a(d, r)$ be as in Definition 2.3. Fix $k \in \mathbb{N}$, and let $(r_1, \dots, r_k) \in \mathbb{R}_+^k$. Also, for $i = 1, \dots, k$, let $\Phi^{(i)}$ be independent Poisson point processes in $\mathbb{R}^d$ of intensities $\lambda_i > 0$. Set $r_0 = \inf_{1 \leq i, j \leq k}\{r_i + r_j\}$. If $\prod_{i=1}^{k}(1 - \mathrm{e}^{-\lambda_i a(d, r_0)}) > p_c(d)$ then, almost surely, every word occurs.*

The following corollary, the proof of which is given in Section 4, gives an upper bound for $\mu_c(\lambda, r)$ for large λ.

Corollary 2.1. *Suppose that $d \geq 2$, $r > 0$, and $\lambda > 0$ satisfies*

$$\lambda > -\frac{\log(1 - p_c(d))}{a(d, 2r)},$$

where $p_c(d)$ and $a(d, r)$ are as in Definition 2.3. Let $\mu_c(\lambda, r)$ be the critical intensity as in (2.2). Then

$$\mu_c(\lambda, r) \leq -\frac{1}{a(d, 2r)} \log\left[1 - \left(\frac{p_c(d)}{1 - \mathrm{e}^{-\lambda a(d, 2r)}}\right)\right]. \tag{2.3}$$

Though Corollary 2.1 and Proposition 2.1 together give upper and lower bounds for $\mu_c(\lambda, r)$, we believe that these bounds can be improved.

Remark 2.1. A simple calculation (see Meester and Roy (1996, p. 88)) gives $a(2, 2) \simeq 0.8227$ and

$$-(a(2, 2))^{-1} \log(1 - p_c(2)) \simeq 0.843.$$

Using these, it follows from Corollary 2.1 that $\mu_c(0.85, 1) < 6.2001$.

Remark 2.2. It can be shown that the number of infinite components in the *AB* Boolean model is at most 1, almost surely. The proof of this fact follows along the same lines as the proof in the Poisson Boolean model (see Meester and Roy (1996, Propositions 3.3 and 3.6)), since it relies on the ergodic theorem and the topology of infinite components, but not on the specific nature of the infinite components.

Proposition 2.2 can be used to show the existence of *AB* percolation in the natural analogue of the discrete *AB* percolation model (refer to the two sentences above Definition 2.2). Recall that $\Phi^{(1)}$ is a Poisson point process in $\mathbb{R}^d$ of intensity $\lambda > 0$. Let $\{m_i\}_{i\geq 1}$ be a sequence of independent and identically distributed (i.i.d.) marks distributed as $m \in \{A, B\}$, with $\mathrm{P}(m = A) = p = 1 - \mathrm{P}(m = B)$. Define the point processes Φ^A and Φ^B as

$$\Phi^A := \{X_i \in \Phi^{(1)} : m_i = A\}, \qquad \Phi^B := \Phi^{(1)} \setminus \Phi^A.$$

Definition 2.5. For any $\lambda, r > 0$ and $p \in (0, 1)$, let Φ^A and Φ^B be as defined above. Let $\widehat{G}(\lambda, p, r) := (\Phi^A, \widehat{E}(\lambda, p, r))$ be the graph with vertex set Φ^A and edge set

$$\widehat{E}(\lambda, p, r) := \{\langle X_i, X_j\rangle : X_i, X_j \in \Phi^A,\ |X_i - Y| \leq 2r,\ |X_j - Y| \leq 2r,\ \text{for some } Y \in \Phi^B\}.$$

Corollary 2.2. *Let $\widehat{\theta}(\lambda, p, r) := \mathrm{P}(\widehat{G}(\lambda, p, r)$ percolates). Then, for any λ satisfying*

$$\lambda > -\frac{2\log(1 - \sqrt{p_c(d)})}{a(d, 2r)},$$

there exists a $p(\lambda) < \frac{1}{2}$ such that $\widehat{\theta}(\lambda, p, r) = 1$ for all $p \in (p(\lambda), 1 - p(\lambda))$.

3. Connectivity in *AB* random geometric graphs

3.1. Model definition

The setup for the study of connectivity in *AB* random geometric graphs is as follows. Let $\{\mathcal{P}_n^{(1)}\}_{n\geq 1}$, $\{\mathcal{P}_n^{(2)}\}_{n\geq 1}$ be independent sequences of homogeneous Poisson point processes in $U = [0, 1]^d$, $d \geq 2$. The processes $\mathcal{P}_n^{(i)}$, $i = 1, 2$, have intensity n. We also nullify some of the technical complications arising out of boundary effects by choosing to work with the toroidal metric on the unit cube, defined as

$$d(x, y) := \inf\{|x - y + z| : z \in \mathbb{Z}^d\}, \qquad x, y \in U. \tag{3.1}$$

Definition 3.1. For any $m, n \geq 1$, the *AB* random geometric graph $G_n(m, r)$ is the graph with vertex set $\mathcal{P}_n^{(1)}$ and edge set

$$E_n(m, r) := \{\langle X_i, X_j\rangle : X_i, X_j \in \mathcal{P}_n^{(1)},\ d(X_i, Y) \leq r,\ d(X_j, Y) \leq r,\ \text{for some } Y \in \mathcal{P}_m^{(2)}\}.$$

Our goal in this section is to study the *connectivity threshold* in the sequence of graphs $G_n(\tau n, r)$ as $n \to \infty$ for $\tau > 0$. The constant τ can be thought of as a measure of the relative denseness or sparseness of $\mathcal{P}_n^{(1)}$ with respect to $\mathcal{P}_{\tau n}^{(2)}$ (see Remark 3.1 below). We will also prove a distributional convergence result for the critical radius required to eliminate isolated nodes. To this end, we introduce the following definition.

Definition 3.2. For each $n \geq 1$, let $W_n(r)$ be the number of isolated nodes, that is, vertices with degree 0 in $G_n(\tau n, r)$, and define the *largest nearest-neighbor radius* as

$$M_n := \sup\{r \geq 0 : W_n(r) > 0\}.$$

3.2. Main results

Let $\theta_d := \|B_O(1)\|$ be the volume of the d-dimensional unit closed ball centered at the origin. For any $\beta > 0$ and $n \geq 1$, define the sequence of cutoff functions,

$$r_n(\tau, \beta) = \left(\frac{\log(n/\beta)}{\tau n \theta_d}\right)^{1/d}, \tag{3.2}$$

and let

$$r_n(\tau) = r_n(\tau, 1). \tag{3.3}$$

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Let $e_1 := (1, 0, \ldots, 0) \in \mathbb{R}^d$ be the unit vector in the first coordinate direction. For $d \geq 2$ and $u, s > 0$, define

$$\eta(u, s) := \frac{\|B_O(u^{1/d}) \cap B_{s^{1/d} e_1}(u^{1/d})\|}{\theta_d u}. \tag{3.4}$$

For $s \leq 2u$, we have (see Goldstein and Penrose (2010, Equation (7.5)) and Moran (1973, Equation (6)))

$$\eta(u, s) = 1 - \frac{\theta_{d-1}}{\theta_d} \int_0^{(s/u)^{1/d}} \left(1 - \frac{t^2}{4}\right)^{(d-1)/d} \mathrm{d}t. \tag{3.5}$$

If $s \geq 2u$ then $\eta(u, s) = 0$. Since, the intersection $B_O(u^{1/d}) \cap B_{s^{1/d} e_1}(u^{1/d})$ always contains a ball of diameter $(2u^{1/d} - s^{1/d})$, we obtain the following lower bound:

$$\eta(u, s) \geq \left(1 - \frac{1}{2}\left(\frac{s}{u}\right)^{1/d}\right)^d. \tag{3.6}$$

The next theorem gives asymptotic bounds for a strong connectivity threshold in *AB* random geometric graphs. Asymptotics for the strong connectivity threshold was one of the more difficult problems in the theory of random geometric graphs. Since the strong connectivity threshold is insensitive to the parameter β, it will be more convenient to take $\beta = 1$ in (3.2) and work with the cutoff functions $r_n(\tau)$ as defined in (3.3). Subsequent to this result we will consider the critical radius required to eliminate isolated nodes for which we will revert back to an arbitrary β. Define the function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$\alpha(\tau) := \inf\{a\colon a\eta(a, \tau) > 1\}. \tag{3.7}$$

From (3.5), it is clear that, for fixed $\tau > 0$, $\eta(a, \tau)$ is increasing in a for $a > \tau/2$ and converges to 1 as $a \to \infty$ and, hence, $a(\tau) < \infty$. From the bound (3.6), we obtain $(1 + \tau^{1/d}/2)^d \times \eta((1 + \tau^{1/d}/2)^d, \tau) \geq 1$ for $d \geq 2$. Thus, we have the bound $\alpha(\tau) \leq (1 + \tau^{1/d}/2)^d$ for $d \geq 2$.

Theorem 3.1. *Let $\alpha(\tau)$ be as defined in (3.7) and $r_n(\tau)$ be as defined in (3.3). Define $\alpha_n^*(\tau) :=$ $\inf\{a\colon G_n(\tau n, a^{1/d} r_n(\tau))$ is connected$\}$. Then, for any $\tau > 0$, almost surely,*

$$1 \leq \liminf_{n\to\infty} \alpha_n^*(\tau) \leq \limsup_{n\to\infty} \alpha_n^*(\tau) \leq \alpha(\tau). \tag{3.8}$$

As is obvious, the bounds are tight for small enough τ. We derive the lower bound by covering the space with disjoint balls and showing that, almost surely, in the limit, at least one of these balls contains an isolated node. For the upper bound, we show that, almost surely, asymptotically the *AB* random geometric graph contains a random geometric graph with a certain radius R_n. By means of such a coupling and ensuring that R_n is greater than the connectivity threshold for the random geometric graph (see Theorem 5.1), we conclude that, almost surely, the *AB* random geometric graph is connected eventually.

In order to derive the asymptotic distribution of the critical radius required to eliminate isolated nodes, we need to first find conditions on the parameters τ and β in (3.2) so that the expected number of isolated nodes will stabilize in the limit. This is the content of Lemma 3.1 below.

Set $\eta(s) := \eta(1, s)$, and note that $\eta(u, s) = \eta(s/u)$ by (3.5). Define the constant τ_0 as

$$\tau_0 := \begin{cases} \sup\left\{\tau : \eta(\tau) + \dfrac{1}{\tau} > 1\right\} & \text{if } d = 2, \\ 1 & \text{if } d \geq 3. \end{cases} \tag{3.9}$$

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From (3.5), it is clear that $\eta(\tau) + 1/\tau$ is decreasing in τ. Hence, $1 < \tau_0 < 4$ for $d = 2$ as $\eta(1) > 0$ and $\eta(4) = 0$. The first part of Lemma 3.1 shows that, for $\tau < \tau_0$, the above choice of radius stabilizes the expected number of isolated nodes in $G_n(\tau n, r_n(\tau, \beta))$ as $n \to \infty$. The second part shows that the assumption $\tau < \tau_0$ is not merely technical. The lemma also suggests a *phase transition* at some $\tilde{\tau} \in [1, 2^d]$, in the sense that the expected number of isolated nodes in $G_n(\tau n, r_n(\tau, \beta))$ converges to a finite limit for $\tau < \tilde{\tau}$ and diverges for $\tau > \tilde{\tau}$.

Lemma 3.1. *For any $\beta, \tau > 0$, let $r_n(\tau, \beta)$ be as defined in (3.2), and let $W_n(r_n(\tau, \beta))$ be the number of isolated nodes in $G_n(\tau n, r_n(\tau, \beta))$. Let τ_0 be as defined in (3.9). Then, as $n \to \infty$,*

1. $\mathrm{E}(W_n(r_n(\tau, \beta))) \to \beta$ *for $\tau < \tau_0$, and*
2. $\mathrm{E}(W_n(r_n(\tau, \beta))) \to \infty$ *for $\tau > 2^d$.*

The second part of Lemma 3.1 follows by a coupling with the standard random geometric graph, similar to that described for percolation at the beginning of Section 2.2. A node of $\mathcal{P}_n^{(1)}$ in the *AB* random geometric graph is isolated because there is no neighboring node from $\mathcal{P}_{\tau n}^{(2)}$ or the neighboring $\mathcal{P}_{\tau n}^{(2)}$ nodes do not have any other neighbors from $\mathcal{P}_n^{(1)}$. Thus, the number of isolated nodes is upper bounded by the sum of $\mathcal{P}_n^{(1)}$ nodes with no neighboring $\mathcal{P}_{\tau n}^{(2)}$ node and $\mathcal{P}_{\tau n}^{(2)}$ nodes with at most one neighboring $\mathcal{P}_n^{(1)}$ node. This bound is good enough to give the convergence of $\mathrm{E}(W_n(r_n(\tau, \beta)))$ for $\tau < 1$ in all dimensions. For $\tau > 1$, we use the closed-form expression obtained (by Palm calculus) for the above expectation as a Laplace transform of the covered region of the Poisson Boolean model driven by $\mathcal{P}_n^{(1)}$. However, to show convergence of the expectation, we need to obtain bounds on the probability of the covered region not covering the whole space and this estimate has so far been carried out only in two dimensions. This explains why we have improved convergence results in $d = 2$.

For $\tau < \tau_0$, having found the radius that stabilizes the mean number of isolated nodes, we use the Stein–Chen method of Poisson approximation for locally dependent Bernoulli random variables to show that the number of isolated nodes converge in distribution to $\mathrm{Po}(\beta)$, a Poisson random variable with mean β. Furthermore, we can conclude that the largest nearest-neighbor radius in $G_n(\tau n, r_n(\tau, \beta))$ converges in distribution as $n \to \infty$. Let '$\xrightarrow{\mathrm{D}}$' denote convergence in distribution.

Theorem 3.2. *Let $r_n(\tau, \beta)$ be as defined in (3.2) with $\beta > 0$ and $0 < \tau < \tau_0$. Then, as $n \to \infty$,*

$$W_n(r_n(\tau, \beta)) \xrightarrow{\mathrm{D}} \mathrm{Po}(\beta), \tag{3.10}$$

$$\mathrm{P}(M_n \le r_n(\tau, \beta)) \to \mathrm{e}^{-\beta}. \tag{3.11}$$

Remark 3.1. For any locally finite point process $\mathcal{X}$ (for example, $\mathcal{P}_n^{(1)}$ or $\mathcal{P}_n^{(2)}$), we denote the number of points of $\mathcal{X}$ in A, $A \subset \mathbb{R}^d$, by $\mathcal{X}(A)$. Define

$$W_n^0(\tau, r) = \sum_{Y_i \in \mathcal{P}_{\tau n}^{(2)}} \mathbf{1}\{\mathcal{P}_n^{(1)}(B_{Y_i}(r)) = 0\},$$

that is, $W_n^0(\tau, r)$ is the number of $\mathcal{P}_{\tau n}^{(2)}$ nodes isolated from $\mathcal{P}_n^{(1)}$ nodes. From Palm calculus for Poisson point processes (see Theorem 1.6 of Penrose (2003)) and the fact that the metric is toroidal, we have

$$\mathrm{E}(W_n^0(\tau, r_n(\tau, \beta))) = \tau n \int_U \mathrm{P}(\mathcal{P}_n^{(1)}(B_x(r)) = 0)\,\mathrm{d}x = \tau n \exp(-n\theta_d r_n(\tau, \beta)^d).$$

Substituting for r_n using (3.2) we obtain

$$\lim_{n\to\infty} \mathrm{E}(W_n^0(\tau, r_n(\tau,\beta))) = \begin{cases} 0 & \text{if } \tau < 1, \\ \beta & \text{if } \tau = 1, \\ \infty & \text{if } \tau > 1. \end{cases}$$

Thus, there is a trade-off between the relative density of the nodes and the radius required to stabilize the expected number of isolated nodes.

4. Proofs for Section 2

Proof of Proposition 2.1. 1. Recall from Definition 2.2 the graph $G(\lambda, \mu, r)$ with vertex set $\Phi^{(1)}$ and edge set $E(\lambda, \mu, r)$. Consider the graph $\tilde{G}(\lambda + \mu, r)$ (see Definition 2.1), where the vertex set is taken to be $\Phi^{(1)} \cup \Phi^{(2)}$, and let the edge set of this graph be denoted by $\tilde{E}(\lambda + \mu, r)$.

If $\langle X_i, X_j \rangle \in E(\lambda, \mu, r)$, there exists a $Y \in \Phi^{(2)}$ such that $\langle X_i, Y \rangle, \langle X_j, Y \rangle \in \tilde{E}(\lambda + \mu, r)$. It follows that $G(\lambda, \mu, r)$ has an infinite component only if $\tilde{G}(\lambda + \mu, r)$ has an infinite component. Consequently, for any $\mu > \mu_c(\lambda, r)$, we have $\mu + \lambda > \lambda_c(r)$, and, hence, $\mu_c(\lambda, r) + \lambda \geq \lambda_c(r)$. Thus, for any $\lambda < \lambda_c(r)$, we obtain the (nontrivial) lower bound $\mu_c(\lambda, r) \geq \lambda_c(r) - \lambda$.

2. Again, $\langle X_i, X_j \rangle \in E(\lambda, \mu, r)$ implies that $|X_i - X_j| \leq 4r$. Hence, $G(\lambda, \mu, r)$ has an infinite component only if $\tilde{G}(\lambda, 2r)$ has an infinite component. Thus, $\mu_c(\lambda, r) = \infty$ if $\lambda \leq \lambda_c(2r)$.

Proof of Theorem 2.1. Fix $\lambda > \lambda_c(2r)$. For $l > 0$, let $l\mathbb{L}^2$ be the graph with vertex set $l\mathbb{Z}^2$, the expanded two-dimensional integer lattice, and endowed with the usual graph structure, that is, $x, y \in l\mathbb{Z}^2$ share an edge if $|x - y| = l$. Denote the edge set by $l\mathbb{E}^2$. For any edge $e \in l\mathbb{E}^2$, denote the midpoint of e by (x_e, y_e). For every horizontal edge e, define three rectangles R_{ei}, $i = 1, 2, 3$, as follows: R_{e1} is the rectangle $[x_e - 3l/4, x_e - l/4] \times [y_e - l/4, y_e + l/4]$, R_{e2} is the rectangle $[x_e - l/4, x_e + l/4] \times [y_e - l/4, y_e + l/4]$, and R_{e3} is the rectangle $[x_e + l/4, x_e + 3l/4] \times [y_e - l/4, y_e - l/4]$. Let $R_e = \bigcup_i R_{ei}$. The corresponding rectangles for vertical edges are defined similarly. See Figure 2.

Owing to the continuity of $\lambda_c(2r)$ (see Meester and Roy (1996, Theorem 3.7)), there exists $r_1 < r$ such that $\lambda > \lambda_c(2r_1)$. We will now define some random variables associated with horizontal edges, the corresponding definitions for vertical edges are similar. Let A_e be the indicator random variable for the event that there exists a left–right crossing of R_e, and top–down crossings of R_{e1} and R_{e3} by a component of $\tilde{G}(\lambda, 2r_1)$. Let C_e be the indicator random

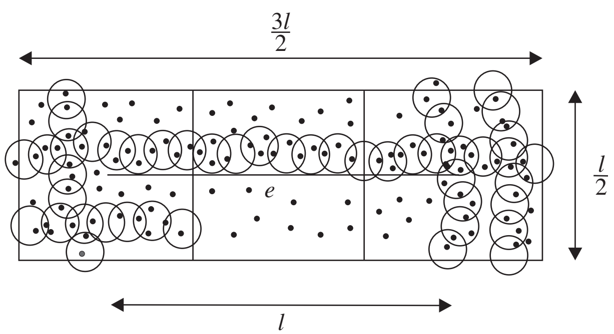

FIGURE 2: A horizontal edge e that satisfies the condition for $B_e = 1$. The balls are of radius $2r$, centered at points of $\Phi^{(1)}$ and the adjacent centers are of at most distance r_1. The dots are the points of $\Phi^{(2)}$.

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variable of the event that $\Phi^{(2)} \cap B_X(2r) \cap B_Y(2r) \neq \varnothing$ for all $X, Y \in \Phi^{(1)} \cap R_e$ such that $B_X(2r_1) \cap B_Y(2r_1) \neq \varnothing$. Let $B_e := \mathbf{1}\{A_e C_e = 1\}$ (see Figure 2). Declare an edge $e \in l\mathbb{E}^2$ to be open if $B_e = 1$. We first show that, for $\lambda > \lambda_c(2r)$, there exist a μ and l such that $l\mathbb{L}^2$ percolates (step 1). The next step is to show that this implies percolation in the continuum model $G(\lambda, \mu, r)$ (step 2).

Step 1. The random variables $\{B_e\}_{e \in l\mathbb{E}^2}$ are 1-dependent, that is, B_es indexed by two nonadjacent edges (edges that do not share a common vertex) are independent. Hence, given edges $e_1, \dots, e_n \in l\mathbb{E}^2$, there exists $\{k_j\}_{j=1}^m \subset \{1, \dots, n\}$ with $m \geq n/4$ such that $\{B_{e_{k_j}}\}_{1 \leq j \leq m}$ are i.i.d. Bernoulli random variables. Hence,

$$\mathrm{P}(B_{e_i} = 0,\ 1 \leq i \leq n) \leq \mathrm{P}(B_{e_{k_j}} = 0,\ 1 \leq j \leq m) \leq \mathrm{P}(B_e = 0)^{n/4}. \tag{4.1}$$

We need to show that, for a given $\varepsilon > 0$, there exist l and μ for which $\mathrm{P}(B_e = 0) < \varepsilon$ for any $e \in l\mathbb{E}^2$. Fix an edge e. Observe that

$$\begin{aligned} \mathrm{P}(B_e = 0) &= \mathrm{P}(A_e = 0) + \mathrm{P}(B_e = 0 \mid A_e = 1)\,\mathrm{P}(A_e = 1) \\ &\leq \mathrm{P}(A_e = 0) + \mathrm{P}(B_e = 0 \mid A_e = 1). \end{aligned} \tag{4.2}$$

Since $\lambda > \lambda_c(2r_1)$, $\tilde{G}(\lambda, 2r_1)$ percolates. Hence, by Meester and Roy (1996, Corollary 4.1) we can and do choose a large enough l so that

$$\mathrm{P}(A_e = 0) < \frac{\varepsilon}{2}. \tag{4.3}$$

Now consider the second term on the right-hand side of (4.2). Given $A_e = 1$, there exist crossings as specified in the definition of A_e in $\tilde{G}(\lambda, 2r_1)$. Draw balls of radius $2r (> 2r_1)$ around each vertex. Any two vertices that share an edge in $\tilde{G}(\lambda, 2r_1)$ are centered at a distance of at most $4r_1$. The width of the lens of intersection of two balls of radius $2r$ whose centers are at most $4r_1 (< 4r)$ apart is bounded below by a constant, say $b(r, r_1) > 0$. Hence, if we cover R_e with disjoint squares of diagonal length $b(r, r_1)/3$ then every lens of intersection will contain at least one such square. Let S_j, $j = 1, \dots, N(b)$, be the disjoint squares of diagonal length $b(r, r_1)/3$ that cover R_e. Note that

$$\begin{aligned} \mathrm{P}(B_e = 1 \mid A_e = 1) &\geq \mathrm{P}(\Phi^{(2)} \cap S_j \neq \varnothing,\ 1 \leq j \leq N(b)) \\ &= \bigg(1 - \exp\bigg(-\frac{\mu b(r, r_1)^2}{18}\bigg)\bigg)^{N(b)} \\ &\to 1 \quad \text{as } \mu \to \infty. \end{aligned}$$

Thus, for the choice of l satisfying (4.3), we can choose a large enough μ so that

$$\mathrm{P}(B_e = 0 \mid A_e = 1) < \frac{\varepsilon}{2}. \tag{4.4}$$

From (4.2)–(4.4), we obtain $\mathrm{P}(B_e = 0) < \varepsilon$. Hence, given any $\varepsilon > 0$, it follows from (4.1) that there exist l and μ large enough so that $\mathrm{P}(B_{e_i} = 0,\ 1 \leq i \leq n) \leq \varepsilon^{n/4}$. That $l\mathbb{L}^2$ percolates now follows from a standard Peierl's argument, as in Grimmett (1999, pp. 17, 18).

Step 2. By step 1, choose l and μ so that $l\mathbb{L}^2$ percolates. Consider any infinite component in $l\mathbb{L}^2$. Let e and f be any two adjacent edges in the infinite component. In particular, $B_e = B_f = 1$. This has two implications, the first being that there exist crossings I_e and

I_f of R_e and R_f, respectively, in $\tilde{G}(\lambda, 2r_1)$. Since e and f are adjacent, $R_{ei} = R_{fj}$ for some $i, j \in \{1, 3\}$. Hence, there exists a crossing J of R_{ei} in $\tilde{G}(\lambda, 2r_1)$ that intersects both I_e and I_f. Draw balls of radius $2r$ around each vertex of the crossings J, I_e, and I_f. The second implication is that every pairwise intersection of these balls will contain at least one point of $\Phi^{(2)}$. This implies that I_e and I_f belong to the same AB component in $G(\lambda, \mu, r)$. Therefore, $G(\lambda, \mu, r)$ percolates when $l\mathbb{L}^2$ does.

Proof of Proposition 2.2. Recall Definition 2.3. For $d = 2$, let $\mathbb{T}$ be the triangular lattice with edge length $r_0/2$, and let Q_z be the flower centered at $z \in \mathbb{T}$, as shown in Figure 1. For $d \geq 3$, let

$$\mathbb{Z}^{*d}_{r_0} := \left(\frac{r_0}{2\sqrt{d}}\mathbb{Z}^d, \left\{\langle z, z_1\rangle \in \left(\frac{r_0}{2\sqrt{d}}\mathbb{Z}^d\right) \times \left(\frac{r_0}{2\sqrt{d}}\mathbb{Z}^d\right) : \|z - z_1\| = \frac{r_0}{2\sqrt{d}}\right\}\right),$$

and let Q_z be the cube of side length $r_0/2\sqrt{d}$ centered at $z \in \mathbb{Z}^{*d}_{r_0}$. Note that the flowers or cubes are disjoint. We declare z open if $Q_z \cap \Phi^{(i)} \neq \varnothing$, $1 \leq i \leq k$. This is clearly a Bernoulli site percolation model on $\mathbb{T}$ ($d = 2$) or $\mathbb{Z}^{*d}_{r_0}$ ($d \geq 3$) with probability $\prod_{i=1}^{k}(1 - \mathrm{e}^{-\lambda_i a(d,r_0)})$ of z being open. By hypothesis, $\prod_{i=1}^{k}(1 - \mathrm{e}^{-\lambda_i a(d,r_0)}) > p_\mathrm{c}(d)$, the critical probability for Bernoulli site percolation on $\mathbb{T}$ ($d = 2$) or $\mathbb{Z}^{*d}_{r_0}$ ($d \geq 3$) and, hence, the corresponding graphs percolate. Let $\langle z_1, z_2, \ldots\rangle$ denote an infinite percolating path in $\mathbb{T}$ ($d = 2$) or $\mathbb{Z}^{*d}_{r_0}$ ($d \geq 3$). Since it is a percolating path, almost surely, for all $i \geq 1$ and every $j = 1, 2, \ldots, k$, $\Phi^{(j)}(Q_{z_i}) > 0$, that is, each (flower or cube) Q_{z_i} contains a point of each of $\Phi^{(1)}, \ldots, \Phi^{(k)}$. Hence, almost surely, for every word $\{w(i)\}_{i\geq 1}$, we can find a sequence $\{X_i\}_{i\geq 1}$ such that, for all $i \geq 1$, $X_i \in \Phi^{(w(i))} \cap Q_{z_i}$. Furthermore, $|X_i - X_{i+1}| \leq r_0 \leq r_{w(i)} + r_{w(i+1)}$. Thus, almost surely, every word occurs.

Proof of Corollary 2.1. Apply Proposition 2.2 with $k = 2$, $\lambda_1 = \lambda$, $\lambda_2 = \mu$, $r_1 = r_2 = r$, and so $r_0 = 2r$. It follows that, almost surely, every word occurs provided

$$(1 - \mathrm{e}^{-\lambda a(d,2r)})(1 - \mathrm{e}^{-\mu a(d,2r)}) > p_\mathrm{c}(d).$$

In particular, under the above condition, almost surely, the word $(1, 2, 1, 2, \ldots)$ occurs. This implies that there is a sequence $\{X_i\}_{i\geq 1}$ such that $X_{2j-1} \in \Phi^{(1)}$, $X_{2j} \in \Phi^{(2)}$, and $|X_{2j} - X_{2j-1}| \leq 2r$, for all $j \geq 1$. But this is equivalent to percolation in $G(\lambda, \mu, r)$. This proves the corollary once we note that there exists a $\mu < \infty$ satisfying the condition above only if $(1 - \mathrm{e}^{-\lambda a(d,2r)}) > p_\mathrm{c}(d)$, or, equivalently, $a(d, 2r)\lambda > \log(1/(1 - p_\mathrm{c}(d)))$ and the least such μ is given on the right-hand side of (2.3).

Proof of Corollary 2.2. By the given condition $(1 - \mathrm{e}^{-\lambda a(d,r)/2}) > \sqrt{p_\mathrm{c}(d)}$ and continuity, there exists a $\varepsilon > 0$ such that, for all $p \in (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$, we have $(1 - \mathrm{e}^{-\lambda p a(d,r)}) > \sqrt{p_\mathrm{c}(d)}$. Thus, for all $p \in (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$, we obtain

$$(1 - \mathrm{e}^{-\lambda p a(d,r)})(1 - \mathrm{e}^{-\lambda(1-p)a(d,r)}) > p_\mathrm{c}(d).$$

Hence, by invoking Proposition 2.2 as in the proof of Corollary 2.1 with $\lambda_1 = \lambda p$, $\lambda_2 = \lambda(1-p)$, and $r_1 = r_2 = r$, we obtain $\widehat{\theta}(\lambda, p, r) = 1$.

5. Proofs for Section 3

For the lower bound of the connectivity threshold, the following result analogous to Penrose (2003, Theorem 7.1) will suffice.

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Proposition 5.1. *Let M_n and $r_n(\tau)$ be as defined in Definition 3.2 and (3.3), respectively. Then, for any $\tau > 0$ and $a < 1$, $\mathrm{P}(M_n \le a^{1/d} r_n(\tau)$ i.o.$) = 0$, where i.o. stands for infinitely often.*

Proof. For $a < 1$, set $r_n = a^{1/d} r_n(\tau)$ and choose a $\varepsilon > 0$ such that

$$\varepsilon^{1/d} + a^{1/d} < (1-\varepsilon)^{1/d}.$$

For $x \in U$, define the events

$$A_n(x) := \{\mathcal{P}^{(2)}_{\tau n}(B_x((1-\varepsilon)^{1/d} r_n(\tau))) = 0\} \cap \{\mathcal{P}^{(1)}_n(B_x(\varepsilon^{1/d} r_n(\tau))) \ge 1\}.$$

Choose points $x_1^n, \ldots, x_{\sigma_n}^n$ in U of maximal cardinality such that the balls $B_{x_i^n}((1-\varepsilon)^{1/d} r_n(\tau))$, $1 \le i \le \sigma_n$, are disjoint. By Penrose (2003, Lemma 5.2) we can choose a constant $0 < \kappa < 1$ such that, for all large enough n,

$$\sigma_n > \kappa \frac{n}{\log n}. \tag{5.1}$$

If $A_n(x)$ occurs for some $x \in U$ then there exists a point $X \in \mathcal{P}^{(1)}_n \cap B_x(\varepsilon^{1/d} r_n(\tau))$ such that, for all $Y \in \mathcal{P}^{(2)}_{\tau n}$,

$$d(X, Y) \ge ((1-\varepsilon)^{1/d} - \varepsilon^{1/d}) r_n(\tau) > a^{1/d} r_n(\tau),$$

by the choice of ε. It follows that X is an isolated node in $G_n(\tau n, r_n)$ or, equivalently, $M_n > r_n$. Therefore,

$$\{M_n \le r_n\} \subset \bigg(\bigcup_{i=1}^{\sigma_n} A_n(x_i)\bigg)^c. \tag{5.2}$$

For all large enough n, we have

$$\mathrm{P}(\mathcal{P}^{(1)}_n(B_x(\varepsilon^{1/d} r_n(\tau))) \ge 1) = 1 - n^{-\varepsilon/\tau} \ge \kappa$$

and

$$\mathrm{P}(\mathcal{P}^{(2)}_{\tau n}(B_x((1-\varepsilon)^{1/d} r_n(\tau))) = 0) = n^{\varepsilon - 1}.$$

Since $\mathcal{P}^{(1)}_n$ and $\mathcal{P}^{(2)}_{\tau n}$ are independent, we obtain, for all large enough n,

$$\mathrm{P}(A_n(x_i^n)) \ge \kappa n^{\varepsilon-1}, \qquad 1 \le i \le \sigma_n.$$

By the above estimate, the independence of events $A_n(x_i^n)$, $1 \le i \le \sigma_n$, (5.1), and the inequality $1 - t \le \mathrm{e}^{-t}$, we obtain, for all large enough n,

$$\mathrm{P}\bigg(\bigg(\bigcup_{x \in \mathbb{R}^d} A_n(x)\bigg)^c\bigg) \le \mathrm{P}\bigg(\bigg(\bigcup_{i=1}^{\sigma_n} A_n(x_i^n)\bigg)^c\bigg) \le \exp(-\kappa \sigma_n n^{\varepsilon-1}) \le \exp\bigg(-\kappa^2 \frac{n^\varepsilon}{\log n}\bigg),$$

which is summable in n. It follows by the Borel–Cantelli lemma and (5.2) that, for $a < 1$, with probability 1, $M_n > r_n$ for all large enough n.

We now prove Theorem 3.1. In the second part of this proof we will couple our sequence of *AB* random geometric graphs with a sequence of random geometric graphs. By a random geometric graph, we mean the graph $\underline{G}_n(r)$ with vertex set $\mathcal{P}^{(1)}_n$ and edge set

$$\{\langle X_i, X_j \rangle : X_i, X_j \in \mathcal{P}^{(1)}_n,\ d(X_i, X_j) \le r\},$$

where d is the toroidal metric defined in (3.1). We will use the following well-known result regarding strong connectivity in the graphs $\underline{G}_n(r)$.

Theorem 5.1. (Theorem 13.2 of Penrose (2003).) *For $R_n(A_0) = (A_0 \log n / n\theta_d)^{1/d}$, almost surely, the sequence of graphs $\underline{G}_n(R_n(A_0))$ is connected eventually if and only if $A_0 > 1$.*

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Proof. Again, let $r_n = a^{1/d} r_n(\tau)$, where $r_n(\tau) = r_n(\tau, 1)$ is as defined in (3.3). It is enough to show the following for $c > 0$:

$$\text{for } a < 1, \quad \mathrm{P}(G_n(\tau n, r_n) \text{ is connected i.o.}) \le \mathrm{P}(M_n \le r_n \text{ i.o.}) = 0 \tag{5.3}$$

and

$$\text{for } a > \alpha(\tau), \quad \mathrm{P}(G_n(\tau n, r_n) \text{ is not connected i.o.}) = 0. \tag{5.4}$$

Equations (5.3) and (5.4) give the lower and upper bounds in (3.8), respectively. Equation (5.3) follows immediately from Proposition 5.1.

We now prove (5.4). Since $a > \alpha(\tau)$, by definition, $a\eta(a, \tau) > 1$. By continuity we can, and do, choose $A_0 > 1$ such that $a\eta(a, A_0\tau) > 1$. Choose $\varepsilon \in (0, 1)$ so that

$$(1 - \varepsilon)^2 a\eta(a, A_0\tau) > 1. \tag{5.5}$$

Let $R_n = R_n(A_0)$, where $R_n(A_0)$ is as defined in Theorem 5.1. For each $X_i \in \mathcal{P}_n^{(1)}$, define the event

$$A_i(n, m, r, R) := \{X_i \text{ connects to all points of } \mathcal{P}_n^{(1)} \cap B_{X_i}(R) \text{ in } G_n(m, r)\},$$

and let

$$B(n, m, r, R) = \bigcup_{X_i \in \mathcal{P}_n^{(1)}} A_i(n, m, r, R)^c.$$

We want to show that the event that every point of $\mathcal{P}_n^{(1)}$ is connected in $G_n(\tau n, r_n)$ to all points of $\mathcal{P}_n^{(1)}$ that fall within a distance $R_n(A_0)$ for all large enough n, happens almost surely, or, equivalently,

$$\mathrm{P}(B(n, \tau n, r_n, R_n) \text{ i.o.}) = 0.$$

We will use a subsequence argument and the Borel–Cantelli lemma to show this. Observe that $B(n, m, r, R) \subset B(n_1, m_1, r_1, R_1)$, provided $n \le n_1, m \ge m_1, r \ge r_1$, and $R \le R_1$. Let $n_j = j^b$ for some integer $b > 0$ that will be chosen later. Since $B(n, \tau n, r_n, R_n) \subset B(n_{j+1}, \tau n_j, r_{n_{j+1}}, R_{n_j})$, for $n_j \le n \le n_{j+1}$,

$$\bigcup_{n=n_j}^{n_{j+1}} B(n, \tau n, r_n, R_n) \subset B(n_{j+1}, \tau n_j, r_{n_{j+1}}, R_{n_j}). \tag{5.6}$$

Let $p_j = \mathrm{P}(A_i(n_{j+1}, \tau n_j, r_{n_{j+1}}, R_{n_j})^c)$. Let $N_n = \mathcal{P}_n^{(1)}([0, 1]^2)$. From (5.6) and the union bound, we obtain

$$\begin{aligned}
\mathrm{P}\bigg(\bigcup_{n=n_j}^{n_{j+1}} B(n, \tau n, r_n, R_n)\bigg) &\le \mathrm{P}(B(n_{j+1}, \tau n_j, r_{n_{j+1}}, R_{n_j})) \\
&\le \mathrm{P}\bigg(\bigcup_{i=1}^{N_{n_{j+1}}} A_i(n_{j+1}, \tau n_j, r_{n_{j+1}}, R_{n_j})^c\bigg) \\
&\le \sum_{i=1}^{n_{j+1}+n_{j+1}^{3/4}} \mathrm{P}(A_i(n_{j+1}, \tau n_j, r_{n_{j+1}}, R_{n_j})^c) \\
&\quad + \mathrm{P}(|N_{n_{j+1}} - n_{j+1}| > n_{j+1}^{3/4}) \\
&\le 2n_{j+1} p_j + \mathrm{P}(|N_{n_{j+1}} - n_{j+1}| > n_{j+1}^{3/4}).
\end{aligned} \tag{5.7}$$

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We now estimate p_j. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$. Conditioning on the number of points of $\mathcal{P}_{n_{j+1}}$ in $B_O(R_{n_j})$ and then using Boole's inequality, we obtain

$$\begin{aligned} p_j &\le \sum_{k=0}^{\infty} \frac{(n_{j+1}\theta_d R_{n_j}^d)^k \mathrm{e}^{-n_{j+1}\theta_d R_{n_j}^d}}{k!} \frac{k}{\theta_d R_{n_j}^d} \int_{B_O(R_{n_j})} \mathrm{e}^{-\tau n_j \|B_O(r_{n_{j+1}}) \cap B_x(r_{n_{j+1}})\|} \,\mathrm{d}x \\ &\le \sum_{k=0}^{\infty} \frac{(n_{j+1}\theta_d R_{n_j}^d)^k \mathrm{e}^{-n_{j+1}\theta_d R_{n_j}^d}}{k!} \frac{k}{\theta_d R_{n_j}^d} \int_{B_O(R_{n_j})} \mathrm{e}^{-\tau n_j \|B_O(r_{n_{j+1}}) \cap B_{R_{n_j} e_1}(r_{n_{j+1}})\|} \,\mathrm{d}x \\ &= n_{j+1}\theta_d R_{n_j}^d \mathrm{e}^{-\tau n_j \theta_d r_{n_{j+1}}^d \eta(r_{n_{j+1}}^d, R_{n_j}^d)}, \end{aligned} \tag{5.8}$$

where $\eta(\cdot, \cdot)$ is as defined in (3.4). Since

$$\frac{R_{n_j}}{r_{n_{j+1}}} = \left(\frac{A_0 \log n_j}{\theta_d n_j} \frac{\tau n_{j+1} \theta_d}{a \log n_{j+1}} \right)^{1/d} \to \left(\frac{A_0 \tau}{a} \right)^{1/d},$$

by the continuity of $\eta(\cdot, \cdot)$ (this follows from (3.5)), we have

$$\eta(r_{n_{j+1}}^d, R_{n_j}^d) \ge (1 - \varepsilon)\eta(a, A_0\tau) \tag{5.9}$$

for all sufficiently large j. For all sufficiently large j, we also have $(j/(j+1))^b \ge (1 - \varepsilon)$. Using (5.9) and simplifying by substituting for R_{n_j} and $r_{n_{j+1}}$ in (5.8), for all sufficiently large j, we have

$$\begin{aligned} p_j &\le \frac{(j+1)^b A_0 b \log j}{j^b} \exp\left(-\frac{j^b}{(j+1)^b} (1-\varepsilon)\eta(a, A_0\tau) ab \log(j+1) \right) \\ &\le \frac{A_0 b \log j}{(1-\varepsilon)} \exp(-(1-\varepsilon)^2 \eta(a, A_0\tau) ab \log(j+1)) \\ &= \frac{A_0 b \log j}{(1-\varepsilon)(j+1)^{(1-\varepsilon)^2 \eta(a, A_0\tau) ab}}. \end{aligned}$$

Hence,

$$n_{j+1} p_j \le \frac{A_0 b \log j}{(1-\varepsilon)(j+1)^{((1-\varepsilon)^2 \eta(a, A_0\tau) a - 1) b}}. \tag{5.10}$$

Using (5.5), we can choose b large enough so that $((1-\varepsilon)^2 \eta(a, A_0\tau) a - 1) b > 1$. It then follows from (5.10) that the first term on the right-hand side of (5.7) is summable in j. From Penrose (2003, Lemma 1.4), the second term on the right-hand side of (5.7) is also summable. Hence, by the Borel–Cantelli lemma, almost surely, only finitely many of the events

$$\bigcup_{n=n_j}^{n_{j+1}} B(n, \tau n, r_n, R_n)$$

occur, and, hence, only finitely many of the events $B(n, \tau n, r_n, R_n)$ occur. This implies that, almost surely, every vertex in $G_n(\tau n, r_n)$ is connected to every other vertex that is within a distance $R_n(A_0)$ from it for all large n. Since $A_0 > 1$, it follows from Theorem 5.1 that, almost surely, $G_n(\tau n, r_n)$ is connected eventually. This proves (5.4).

Towards a proof of Lemma 3.1, we first derive a vacancy estimate similar to Hall (1988, Theorem 3.11). For any locally finite point process $\mathcal{X} \subset U$, the coverage process is defined as

$$\mathcal{C}(\mathcal{X}, r) := \bigcup_{X_i \in \mathcal{X}} B_{X_i}(r), \tag{5.11}$$

and we abbreviate $\mathcal{C}(\mathcal{P}_n^{(1)}, r)$ by $\mathcal{C}(n, r)$. Recall that, for any $A \subset \mathbb{R}^d$, we write $\mathcal{X}(A)$ to be the number of points of $\mathcal{X}$ that lie in the set A.

Lemma 5.1. *For $d = 2$ and $0 < r < \frac{1}{2}$, define $V(r) := 1 - \|B_O(r) \cap \mathcal{C}(n, r)\|/\pi r^2$, the normalized vacancy in the r-ball. Then*

$$\mathrm{P}(V(r) > 0) \le (1 + n\pi r^2 + 4(n\pi r^2)^2) \exp(-n\pi r^2).$$

Proof. Write $\mathrm{P}(V(r) > 0) \le p_1 + p_2 + p_3$, where

$$p_1 = \mathrm{P}(\mathcal{P}_n^{(1)}(B_O(r)) = 0) = \exp(-n\pi r^2),$$
$$p_2 = \mathrm{P}(\mathcal{P}_n^{(1)}(B_O(r)) = 1) = n\pi r^2 \exp(-n\pi r^2),$$
$$p_3 = \mathrm{P}(\mathcal{P}_n^{(1)}(B_O(r)) > 1,\ V(r) > 0).$$

We will now upper bound p_3 to complete the proof. A *crossing* is defined as a point of intersection of the boundaries of two balls (all the balls mentioned in this proof are assumed to have a radius r) centered at the points of $\mathcal{P}_n^{(1)}$. A crossing is said to be *covered* if it lies in the interior of another ball centered at a point of $\mathcal{P}_n^{(1)}$, otherwise it is said to be *uncovered*. If there is more than one point of $\mathcal{P}_n^{(1)}$ in $B_O(r)$ then there exists at least one crossing in U. If $V(r) > 0$ and there exists more than one ball centered at a point of $\mathcal{P}_n^{(1)}$ in $B_O(r)$, then there exists at least one such ball with two uncovered crossings on its boundary. Denoting the number of uncovered crossings by M, we have

$$p_3 \le \mathrm{P}(M \ge 2) \le \frac{\mathrm{E}(M)}{2}.$$

Note that balls centered at distinct points can have at most two crossings and, almost surely, all the points of $\mathcal{P}_n^{(1)}$ are distinct. Thus, given a ball, the number of crossings on the boundary of the ball is twice the number of balls centered at a distance within $2r$. This number has expectation $2\int_0^{2r} 2n\pi x \,\mathrm{d}x = 8n\pi r^2$, where $2n\pi x \,\mathrm{d}x$ is the expected number of balls whose centers lie between x and $x + \mathrm{d}x$ of the center of the given ball. Thus,

$$\mathrm{E}(M) = \mathrm{E}(\mathcal{P}_n^{(1)}(B_O(r)))8n\pi r^2\, \mathrm{P}(\text{a crossing is uncovered}) = 8(n\pi r^2)^2 \exp(-n\pi r^2).$$

Proof of Lemma 3.1. We first prove the second part of the lemma which is easier.

2. Let $\widehat{W}_n(r)$ be the number of $\mathcal{P}_n^{(1)}$ nodes for which there is no other $\mathcal{P}_n^{(1)}$ node within distance r. Note that $\widehat{W}_n(2r) \le W_n(r)$. By this inequality and the Palm calculus, we obtain

$$\begin{aligned}
\mathrm{E}(W_n(r_n(\tau, \beta))) &\ge \mathrm{E}(\widehat{W}_n(2r_n(\tau, \beta))) \\
&= n \int_U \mathrm{P}(\mathcal{P}_n^{(1)}(B_x(2r_n(\tau, \beta))) = 0) \,\mathrm{d}x \\
&= n \exp(-2^d n\theta_d r_n^d(\tau, \beta)) \\
&= n \exp\left(-\frac{2^d}{\tau} \log\left(\frac{n}{\beta}\right)\right) \\
&\to \infty
\end{aligned}$$

as $n \to \infty$ since $\tau > 2^d$.

1. We prove the cases $d = 2$ and $d \geq 3$ separately. Let $d \geq 3$, and fix $\tau < 1$. Define $\widetilde{W}_n(\tau, r)$ to be the number of $\mathcal{P}_n^{(1)}$ nodes for which there is no $\mathcal{P}_{\tau n}^{(2)}$ nodes within distance r, and define $\overline{W}_n(\tau, r)$ be the number of $\mathcal{P}_{\tau n}^{(2)}$ nodes with only one $\mathcal{P}_n^{(1)}$ node within distance r. Note that

$$\widetilde{W}_n(\tau, r) \leq W_n(r) \leq \widetilde{W}_n(\tau, r) + \overline{W}_n(\tau, r). \tag{5.12}$$

By Palm calculus for Poisson point processes we have

$$\begin{aligned} \mathrm{E}(\widetilde{W}_n(\tau, r_n(\tau, \beta))) &= n \int_U \mathrm{P}(\mathcal{P}_{\tau n}^{(2)}(B_x(r_n(\tau, \beta))) = 0) \, \mathrm{d}x \\ &= n \exp(-\tau n \theta_d r_n^d(\tau, \beta)) \\ &= \beta, \end{aligned} \tag{5.13}$$

$$\begin{aligned} \mathrm{E}(\overline{W}_n(\tau, r_n(\tau, \beta))) &= \tau n \int_U \mathrm{P}(\mathcal{P}_n^{(1)}(B_x(r_n(\tau, \beta))) = 1) \, \mathrm{d}x \\ &= \tau n \exp(-n\theta_d r_n^d(\tau, \beta)) n\theta_d r_n^d(\tau, \beta) \\ &\to 0, \end{aligned} \tag{5.14}$$

since $\tau < 1$. It follows from (5.12), (5.13), and (5.14) that $\mathrm{E}(W_n(r_n(\tau, \beta))) \to \beta$ as $n \to \infty$ if $d \geq 3$ and $\tau < \tau_0 = 1$.

Now let $d = 2$, fix $\tau < \tau_0$, where τ_0 is as defined in (3.9), and let n be large enough so that $r_n(\tau, \beta) < \frac{1}{2}$. For any $X \in \mathcal{P}_n^{(1)}$, using (5.11), the degree of X in the graph $G_n(\tau n, r)$ can be written as

$$\deg_n(\tau n, X) := \sum_{X_j \in \mathcal{P}_n^{(1)}} \mathbf{1}\{\langle X_j, X \rangle \in E_n(\tau n, r)\} = \mathcal{P}_n^{(1)}(\mathcal{C}((\mathcal{P}_{\tau n}^{(2)} \cap B_X(r)), r) \setminus \{X\}).$$

Since

$$\{\mathcal{P}_n^{(1)}(\mathcal{C}((\mathcal{P}_{\tau n}^{(2)} \cap B_X(r)), r) \setminus \{X\}) = 0\} = \{\mathcal{P}_{\tau n}^{(2)}(B_X(r) \cap \mathcal{C}(\mathcal{P}_n^{(1)} \setminus \{X\}, r)) = 0\}, \tag{5.15}$$

we have

$$\begin{aligned} W_n(r) &= \sum_{X_i \in \mathcal{P}_n^{(1)}} \mathbf{1}\{\deg_n(\tau n, X_i) = 0\} \\ &= \sum_{X_i \in \mathcal{P}_n^{(1)}} \mathbf{1}\{\mathcal{P}_{\tau n}^{(2)}(B_{X_i}(r) \cap \mathcal{C}(\mathcal{P}_n^{(1)} \setminus \{X_i\}, r)) = 0\}. \end{aligned} \tag{5.16}$$

By Palm calculus for Poisson point processes (and the metric being toroidal) we have

$$\mathrm{E}(W_n(r)) = n \int_U \mathrm{E}(\mathbf{1}\{\deg_n(\tau n, x) = 0\}) \, \mathrm{d}x = n \, \mathrm{P}(\mathcal{P}_{\tau n}^{(2)}(B_O(r) \cap \mathcal{C}(n, r)) = 0), \tag{5.17}$$

where $\mathcal{C}(n, r) = \mathcal{C}(\mathcal{P}_n^{(1)}, r)$. For any bounded random closed set F, conditioning on F and then taking the expectation, we have

$$\mathrm{P}(\mathcal{P}_{\tau n}^{(2)}(F) = 0) = \mathrm{E}(\exp(-\tau n \|F\|)). \tag{5.18}$$

Thus, from (5.17) and (5.18) we obtain

$$\mathrm{E}(W_n(r)) = n \, \mathrm{E}(\exp(-\tau n \|B_O(r) \cap \mathcal{C}(n, r)\|)) = n \, \mathrm{E}(\exp(-\tau n \pi r^2 (1 - V(r)))), \tag{5.19}$$

where $V(r)$ is as defined in Lemma 5.1. Let $\eta(\tau) = \eta(1, \tau)$ be as defined in (3.4) and

$e_1 = (1, 0)$. Since $r_n(1, \beta)/r_n(\tau, \beta) = \tau^{1/2}$, by (3.5) we have

$$\frac{\|B_O(r_n(\tau, \beta)) \cap B_{r_n(1,\beta)e_1}(r_n(\tau, \beta))\|}{\pi r_n(\tau, \beta)^2} = \eta(\tau).$$

Given $\tau < \tau_0$, by continuity, we can choose an $\varepsilon \in (0, 1)$ such that

$$\eta_*(\tau, \varepsilon) = \frac{\|B_O(r_n(\tau, \beta)) \cap B_{r_n(1-\varepsilon,\beta)e_1}(r_n(\tau, \beta))\|}{\pi r_n(\tau, \beta)^2} \quad \text{satisfies} \quad \eta_*(\tau, \varepsilon) + \frac{1}{\tau} > 1. \tag{5.20}$$

Let $N_n = \mathcal{P}_n^{(1)}(B_O(r_n(1 - \varepsilon, \beta)))$. Thus, we have

$$\begin{aligned} \mathrm{E}(W_n(r_n(\tau, \beta))) = {} & n\,\mathrm{E}(\mathrm{e}^{-\tau n\pi r_n^2(\tau,\beta)(1-V(r_n(\tau,\beta)))}\,\mathbf{1}\{V(r_n(\tau, \beta)) = 0\}) \\ & + n\,\mathrm{E}(\mathrm{e}^{-\tau n\pi r_n^2(\tau,\beta)(1-V(r_n(\tau,\beta)))}\,\mathbf{1}\{V(r_n(\tau, \beta)) > 0,\ N_n = 0\}) \\ & + n\,\mathrm{E}(\mathrm{e}^{-\tau n\pi r_n^2(\tau,\beta)(1-V(r_n(\tau,\beta)))}\,\mathbf{1}\{V(r_n(\tau, \beta)) > 0,\ N_n > 0\}). \end{aligned} \tag{5.21}$$

Consider the first term in (5.21). From Lemma 5.1 we obtain the bound,

$$\mathrm{P}(V(r_n(\tau, \beta)) > 0) \le D(1 + \log n + 4(\log n)^2)n^{-1/\tau} \tag{5.22}$$

for some constant D. Hence,

$$\begin{aligned} & n\,\mathrm{E}(\mathrm{e}^{-\tau n\pi r_n^2(\tau,\beta)(1-V(r_n(\tau,\beta)))}\,\mathbf{1}\{V(r_n(\tau, \beta)) = 0\}) \\ & \quad = n\exp(-\tau n\pi r_n(\tau, \beta)^2)\,\mathrm{P}(V(r_n(\tau, \beta)) = 0) \\ & \quad = \beta\,\mathrm{P}(V(r_n(\tau, \beta)) = 0) \\ & \quad \to \beta \quad \text{as } n \to \infty. \end{aligned}$$

The second term in (5.21) is bounded by

$$n\,\mathrm{P}(N_n = 0) = n\exp(-n\pi r_n(1 - \varepsilon, \beta)^2) = n^{1-1/(1-\varepsilon)}\beta^{1/(1-\varepsilon)} \to 0 \quad \text{as } n \to \infty. \tag{5.23}$$

We will now show that the third term in (5.21) converges to 0. On the event $\{N_n > 0\}$, we have

$$1 - V(r_n(\tau, \beta)) > \eta_*(\tau, \varepsilon). \tag{5.24}$$

Using (5.24) first and then (5.22), the third term in (5.21) can be bounded by

$$\begin{aligned} & n\mathrm{e}^{-\tau n\pi r_n(\tau,\beta)^2\eta_*(\tau,\varepsilon)}\,\mathrm{P}(V(r_n(\tau, \beta)) > 0,\ N_n > 0) \\ & \quad \le n^{1-\eta_*(\tau,\varepsilon)}\beta^{\eta_*(\tau,\varepsilon)}\,\mathrm{P}(V(r_n(\tau, \beta)) > 0) \\ & \quad \le Dn^{1-\eta_*(\tau,\varepsilon)-1/\tau}(1 + \log n + 4(\log n)^2)\beta^{\eta_*(\tau,\varepsilon)} \\ & \quad \to 0 \quad \text{as } n \to \infty \end{aligned} \tag{5.25}$$

by (5.20).

It follows from (5.21), (5.23), and (5.25) that $\mathrm{E}(W_n(r_n(\tau, \beta))) \to \beta$ as $n \to \infty$.

The *total variation distance* between two integer-valued random variables ψ and ζ is defined as

$$d_{\mathrm{TV}}(\psi, \zeta) = \sup_{A\subset\mathbb{Z}} |\mathrm{P}(\psi \in A) - \mathrm{P}(\zeta \in A)|.$$

The following estimate in the spirit of Theorem 6.7 of Penrose (2003) will be our main tool in proving Poisson convergence of $W_n(r_n(\tau, \beta))$. We denote the Palm version $\mathcal{P}_n^{(1)} \cup \{x\}$ of $\mathcal{P}_n^{(1)}$ by $\mathcal{P}_n^{(1,x)}$.

Lemma 5.2. *Let* $0 < r < 1$, *and let* $\mathcal{C}(\cdot,\cdot)$ *be the coverage process defined by (5.11). Define the integrals* $I_{in}(r)$, $i = 1, 2$ *and* $n \geq 1$, *by*

$$I_{1n}(r) := n^2 \int_U \mathrm{d}x \int_{B_x(5r)\cap U} \mathrm{d}y \, \mathrm{P}(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{\tau n}^{(2)} \cap B_x(r), r)) = 0) \times \mathrm{P}(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{\tau n}^{(2)} \cap B_y(r), r)) = 0), \tag{5.26}$$

$$I_{2n}(r) := n^2 \int_U \mathrm{d}x \int_{B_x(5r)\cap U} \mathrm{d}y \, \mathrm{P}(\mathcal{P}_n^{(1,x)}(\mathcal{C}(\mathcal{P}_{\tau n}^{(2)} \cap B_y(r), r)) = 0 = \mathcal{P}_n^{(1,y)}(\mathcal{C}(\mathcal{P}_{\tau n}^{(2)} \cap B_x(r), r))). \tag{5.27}$$

Then,

$$d_{\mathrm{TV}}(W_n(r), \mathrm{Po}(\mathrm{E}(W_n(r)))) \leq \min\left(3, \frac{1}{\mathrm{E}(W_n(r))}\right)(I_{1n}(r) + I_{2n}(r)). \tag{5.28}$$

Proof. The proof follows along the same lines as the proof of Theorem 6.7 (see Penrose (2003)). For every $m \in \mathbb{N}$, partition U into disjoint cubes of side length m^{-1} with corners at $m^{-1}\mathbb{Z}^d$. Let the cubes and their centers be denoted by $H_{m,1}, H_{m,2}, \ldots$ and $a_{m,1}, a_{m,2}, \ldots$, respectively. Define

$$I_m := \{i \in \mathbb{N} \colon H_{m,i} \subset [0,1]^d\} \quad \text{and} \quad E_m := \{\langle i, j\rangle \colon i, j \in I_m,\ 0 < \|a_{m,i} - a_{m,j}\| < 5r\}.$$

The graph $G_m = (I_m, E_m)$ forms a dependency graph (see Penrose (2003, Chapter 2)) for the random variables $\{\xi_{m,i}\}_{i\in I_m}$. The dependency neighborhood of a vertex i is $N_{m,i} = i \cup \{j \colon \langle i, j\rangle \in E_m\}$. Let

$$\xi_{m,i} := \mathbf{1}\{\{\mathcal{P}_n^{(1)}(H_{m,i}) = 1\} \cap \{\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{\tau n}^{(2)} \cap B_{a_{m,i}}(r), r) \cap H_{m,i}^c) = 0\}\}.$$

Here $\xi_{m,i} = 1$ provided there is exactly one point of $\mathcal{P}_n^{(1)}$ in the cube $H_{m,i}$ which is not connected to any other point of $\mathcal{P}_n^{(1)}$ that falls outside $H_{m,i}$ in the graph $G_n(\tau n, r)$. Let $W_m = \sum_{i\in I_m} \xi_{m,i}$. Then, almost surely,

$$W_n(r) = \lim_{m\to\infty} W_m.$$

Let $p_{m,i} = \mathrm{E}(\xi_{m,i})$ and $p_{m,i,j} = \mathrm{E}(\xi_{m,i}\xi_{m,j})$. The remaining part of the proof is based on the notion of dependency graphs and the Stein–Chen method. By Penrose (2003, Theorem 2.1) we have

$$d_{\mathrm{TV}}(W_m, \mathrm{Po}(\mathrm{E}(W_m))) \leq \min\left(3, \frac{1}{\mathrm{E}(W_m)}\right)(b_1(m) + b_2(m)), \tag{5.29}$$

where

$$b_1(m) = \sum_{i\in I_m} \sum_{j\in N_{m,i}} p_{m,i}p_{m,j} \quad \text{and} \quad b_2(m) = \sum_{i\in I_m} \sum_{j\in N_{m,i}/\{i\}} p_{m,i,j}.$$

The result follows if we show that the expressions on the left- and right-hand sides of (5.29) converge to the left- and right-hand expressions in (5.28), respectively.

Let $w_m(x) = m^d p_{m,i}$ for $x \in H_{m,i}$. Then $\sum_{i\in I_m} p_{m,i} = \int_U w_m(x)\,\mathrm{d}x$. Clearly,

$$\lim_{m\to\infty} w_m(x) = n\,\mathrm{P}(\mathcal{P}_n^{(1,x)}(\mathcal{C}((\mathcal{P}_{\tau n}^{(2)} \cap B_x(r))/\{x\}, r)) = 0) = n\,\mathrm{P}(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{\tau n}^{(2)} \cap B_x(r), r)) = 0).$$

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Since $w_m(x) \le m^d \, \mathrm{P}(\mathcal{P}_n^{(1)}(H_{m,i}) = 1) \le n$,

$$\lim_{m\to\infty} \mathrm{E}(W_m) = n \int_U \mathrm{P}(\mathcal{P}_n^{(1)}(\mathcal{C}(\mathcal{P}_{\tau n}^{(2)} \cap B_x(r), r)) = 0) \, \mathrm{d}x = \mathrm{E}(W_n(r)),$$

where the first equality is due to the dominated convergence theorem and the second follows from (5.15)–(5.17). Similarly, by letting $u_m(x, y) = m^{2d} p_{m,i} p_{m,j} \mathbf{1}\{[j \in N_{m,i}]\}$ and $v_m(x, y) = m^{2d} p_{m,i,j} \mathbf{1}\{[j \in N_{m,i}/\{i\}]\}$ for $x \in H_{m,i}$ and $y \in H_{m,j}$, we can show that

$$b_1(m) = \int_U u_m(x, y) \, \mathrm{d}x \, \mathrm{d}y \to I_{1n}(r)$$

and

$$b_2(m) = \int_U v_m(x, y) \, \mathrm{d}x \, \mathrm{d}y \to I_{2n}(r).$$

Proof of Theorem 3.2. Equation (3.11) follows easily from (3.2) by noting that

$$\mathrm{P}(M_n \le r) = \mathrm{P}(W_n(r) = 0).$$

Hence, the proof is complete if we show (3.2), for which we will use Lemma 5.2. Let $I_{in}(r_n(\tau, \beta))$, $i = 1, 2$, be the integrals defined in (5.26) and (5.27) with r taken to be $r_n(\tau, \beta)$ satisfying (3.2). From Lemma 3.1, $\mathrm{E}(W_n(r_n(\tau, \beta))) \to \beta$ as $n \to \infty$. As convergence in *total variation distance* implies convergence in distribution, by Lemma 5.2 and the conclusion in the last statement, it suffices to show that $I_{in}(r_n(\tau, \beta)) \to 0$ as $n \to \infty$ for $i = 1, 2$.

Using (5.17) and Lemma 3.1, we obtain, for some finite positive constant C,

$$I_{1n}(r_n(\tau, \beta)) = \int_U \mathrm{d}x \int_{B_x(5r_n(\tau,\beta))\cap U} \mathrm{d}y (\mathrm{E}(W_n(r_n(\tau, \beta))))^2 \le C(5r_n(\tau, \beta))^d \to 0$$

as $n \to \infty$. We now compute the integrand in the inner integral in $I_{2n}(r)$. Let $\Gamma(x, r) = \|B_O(r) \cap B_x(r)\|$. For $x, y \in U$, using (5.18), we obtain

$$\begin{aligned}
&\mathrm{P}(\{\mathcal{P}_n^{(1,x)}(\mathcal{C}(\mathcal{P}_{\tau n}^{(2)} \cap B_y(r), r)) = 0\} \cap \{\mathcal{P}_n^{(1,y)}(\mathcal{C}(\mathcal{P}_{\tau n}^{(2)} \cap B_x(r), r)) = 0\}) \\
&\quad = \mathrm{P}(\mathcal{P}_{\tau n}^{(2)}(B_y(r) \cap (\mathcal{C}(n, r) \cup B_x(r))) = 0, \mathcal{P}_{\tau n}^{(2)}(B_x(r) \cap (\mathcal{C}(n, r) \cup B_y(r))) = 0) \\
&\quad \le \mathrm{P}(\mathcal{P}_{\tau n}^{(2)}(B_y(r) \cap \mathcal{C}(n, r)) = 0, \ \mathcal{P}_{\tau n}^{(2)}(B_x(r) \cap \mathcal{C}(n, r)) = 0) \\
&\quad = \mathrm{P}(\mathcal{P}_{\tau n}^{(2)}((B_y(r) \setminus B_x(r)) \cap \mathcal{C}(n, r)) = 0, \ \mathcal{P}_{\tau n}^{(2)}(B_x(r) \cap \mathcal{C}(n, r)) = 0) \\
&\quad = \mathrm{E}(\exp(-\tau n \|(B_y(r) \setminus B_x(r)) \cap \mathcal{C}(n, r)\|) \exp(-\tau n \|B_x(r) \cap \mathcal{C}(n, r)\|)). \qquad (5.30)
\end{aligned}$$

We can and do choose an $\eta > 0$ so that, for any $r > 0$ and $|y - x| \le 5r$ (see Penrose (2003, Equation 8.21)), we have

$$\|B_x(r) \setminus B_y(r)\| \ge \eta r^{d-1} |y - x|.$$

Hence, if $|y - x| \le 5r$, the left-hand expression in (5.30) will be bounded above by

$$\mathrm{E}\bigg(\exp\bigg(-\tau n \eta r^{d-1} |y - x| \frac{\|(B_y(r) \setminus B_x(r)) \cap \mathcal{C}(n, r)\|}{\|B_y(r) \setminus B_x(r)\|}\bigg) \exp(-\tau n \|B_x(r) \cap \mathcal{C}(n, r)\|)\bigg).$$

Using the above bound, we obtain

$$
\begin{aligned}
&I_{2n}(r_n(\tau,\beta)) \\
&\quad \le \int_{B_O(5r_n^d(\tau,\beta))\cap U} n^2\,\mathrm{E}\bigg(\exp(-\tau n\|B_O(r_n(\tau,\beta))\cap \mathcal{C}(n,r_n(\tau,\beta))\|) \\
&\qquad \times \exp\bigg(-\tau n\eta r_n(\tau,\beta)^{d-1}|y|\frac{\|(B_y(r_n(\tau,\beta))\setminus B_O(r_n(\tau,\beta)))\cap \mathcal{C}(n,r_n(\tau,\beta))\|}{\|B_y(r_n(\tau,\beta))\setminus B_O(r_n(\tau,\beta))\|}\bigg)\bigg)\,\mathrm{d}y.
\end{aligned}
$$

Making the change of variable $w = nr_n(\tau,\beta)^{d-1}y$ and using (5.19), we obtain

$$
\begin{aligned}
&I_{2n}(r_n(\tau,\beta)) \\
&\quad \le \int_{B_x(5nr_n(\tau,\beta)^d)\cap U} (nr_n(\tau,\beta)^d)^{1-d}\,\mathrm{E}\bigg(n\exp(-\tau n\|B_O(r_n(\tau,\beta))\cap \mathcal{C}(n,r_n(\tau,\beta))\|) \\
&\qquad \times \exp\bigg(-c\eta|w|\frac{\|(B_{w(nr_n(\tau,\beta)^{d-1})^{-1}}(r_n(\tau,\beta))\setminus B_O(r_n(\tau,\beta)))\cap \mathcal{C}(n,r_n(\tau,\beta))\|}{\|B_{w(nr_n(\tau,\beta)^{d-1})^{-1}}(r_n(\tau,\beta))\setminus B_O(r_n(\tau,\beta))\|}\bigg)\bigg)\,\mathrm{d}w \\
&\quad \le (nr_n(\tau,\beta)^d)^{1-d}\,\mathrm{E}(W_n(r_n(\tau,\beta))) \\
&\quad \to 0
\end{aligned}
$$

as $n\to\infty$, since, by Lemma 3.1, $\mathrm{E}(W_n(r_n(\tau,\beta)))\to\beta$ and $nr_n(\tau,\beta)^d\to\infty$ as $n\to\infty$.

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