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AN EXISTENCE RESULT FOR STEPANOFF ALMOST-PERIODIC DIFFERENTIAL EQUATIONS

BY
S. ZAIDMAN

Introduction. In this short paper we present an existence (an unicity) result for a first order differential equation in Hilbert spaces with right-hand side almost-periodic in the sense of Stepanoff.

The result and the proof below should be compared with the first part in Taam's paper [1], where a very similar result is given.(*)

§1. Let H be a Hilbert space; B is a linear bounded operator in H such that

$$\|\exp(B\zeta)\| \leq \exp(a\zeta), \quad \text{for a certain } a > 0 \text{ and } \forall\zeta < 0. \tag{1.1}$$

Let us consider furthermore a *continuous* function $f(t), -\infty < t < +\infty \to H$, which is almost-periodic in the sense S^2: this means that for any $\varepsilon > 0$ there is a number $1(\varepsilon)$ such that any interval of length 1 of the real line contains at least one point ξ for which

$$\sup_{\alpha\in R^1}\left\{\int_\alpha^{\alpha+1}\|f(t+\xi)-f(t)\|^2\,dt\right\}^{1/2} < \varepsilon \tag{1.2}$$

We have then

THEOREM 1. *Under the above given hypothesis, there exists one and only one strongly continuously differentiable function $u(t)$, $-\infty < t < +\infty \to H$ verifying the differential equation*

$$u'(t) = Bu(t)+f(t) \tag{1.3}$$

and which is almost-periodic in the sense of Bochner.

Proof of uniqueness. The given equation has at most one bounded solution $-\infty < t < +\infty \to H$ as follows easily (see for a more general result our paper [2, Theorem 3]).

Proof of existence. Let us consider, for any $n = 1, 2, \ldots$ the function $v_n(t)$ which is defined by the integral:

$$v_n(t) = -\int_{-n}^{-n+1}\exp(B\zeta)f(t-\zeta)\,d\zeta. \tag{1.4}$$

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(*) He considers almost-everywhere solutions.

We have the estimate (easy to get applying the Cauchy–Schwarz inequality and (1.1))

$$\|v_n(t)\| \le \frac{1}{\sqrt{2a}}(e^{-2a(n-1)}-e^{-2an})^{1/2}\left(\int_{t+n-1}^{t+n}\|f(u)\|^2\,du\right)^{1/2}. \tag{1.5}$$

As is well known, any function almost-periodic S^2, $h(t)$, has the property

$$\sup_{-\infty<\alpha<\infty}\left(\int_\alpha^{\alpha+1}\|h(t)\|^2\,dt\right)^{1/2} = \|h\|_{S^2} < \infty. \tag{1.6}$$

It follows

$$\|v_n(t)\| \le \frac{1}{\sqrt{2a}}(e^{-2a(n-1)}-e^{-2an})^{1/2}\|f\|_{S^2}, \tag{1.7}$$

$$n = 1, 2, \ldots, \quad -\infty<t<+\infty.$$

Remark now that

$$(2a)^{-1/2}\sum_{n=1}^{\infty}\sqrt{e^{-2a(n-1)}-e^{-2an}} = (1-e^{-2a})^{1/2}(2a)^{-1/2}(1-e^{-a})^{-1}.$$

Hence, by the Weierstrass test, the series $\sum_{n=1}^{\infty} v_n(t)$ is uniformly convergent on $-\infty<t<+\infty$. Let $u(t)$ be the sum of the series

$$u(t) = \sum_{n=1}^{\infty} v_n(t). \tag{1.8}$$

Then $u(t)$ is strongly continuous, $-\infty<t<+\infty \to H$, and is uniformly bounded: precisely

$$\|u(t)\| \le \sum_{n=1}^{\infty}\|v_n(t)\| \le (1-e^{-2a})^{1/2}(2a)^{-1/2}(1-e^{-a})^{-1}\|f\|_{S^2}. \tag{1.9}$$

Furthermore, all $v_n(t)$ are (Bochner) almost-periodic. Let in fact ξ be an $S^2-\varepsilon$ almost-period for $f(t)$, i.e. (1.2) holds. We have then

$$v_n(t+\xi)-v_n(t) = -\int_{-n}^{-n+1}\exp(B\zeta)(f(t+\xi-\zeta)-f(t-\zeta))\,d\zeta$$

and estimating as above we have

$$\begin{aligned}\|v_n(t+\xi)-v_n(t)\| &\le (2a)^{-1/2}(e^{-2a(n-1)}-e^{-2an})^{1/2}\\ &\quad\times\left(\int_{-n}^{-n+1}\|f(t-\zeta+\xi)-f(t-\zeta)\|^2\,d\zeta\right)^{1/2}\\ &= (2a)^{-1/2}(e^{-2a(n-1)}-e^{-2an})^{1/2}\\ &\quad\times\left(\int_{t+n-1}^{t+n}\|f(u+\xi)-f(u)\|^2\,du\right)^{1/2} \le (2a)^{-1/2}2^{1/2}\varepsilon\\ &= \varepsilon a^{-1/2}, \quad t\in(-\infty, +\infty).\end{aligned} \tag{1.10}$$

Hence all $v_n(t)$ are almost-periodic Bochner, and an ε-almost-period for f in the S^2-sense is an $\varepsilon/\sqrt{a}$-almost-period for $v_n(t)$ in sense of Bochner. Consequently, the uniform sum $u(t)$ of the series $\sum_{n=1}^{\infty} v_n(t)$ is Bochner almost-periodic too.

We now make the obvious remark that all $v_n(t)$ are strongly continuously differentiable. If we put in (1.4), $t-\zeta=\sigma$, we get

$$v_n(t) = -\int_{t+n-1}^{t+n} \exp(B(t-\sigma))f(\sigma)\,d\sigma. \tag{1.11}$$

Computing the strong derivative we obtain

$$v_n'(t) = \exp(B(-n+1))f(t+n-1)-\exp(B(-n))f(t+n)+Bv_n(t). \tag{1.12}$$

Let us consider now the partial sums $u_N(t)=v_1(t)+\cdots+v_N(t)$; it is immediately seen that

$$u_N'(t) = \sum_{n=1}^{N} v_n'(t) = Bu_N(t)+f(t)-\exp(-NB)f(t+N). \tag{1.13}$$

If $f(t)$ would be bounded on $-\infty<t<+\infty$, using the estimate $\|\exp(-NB)\| \le \exp(-aN)$ which follows from (1.1), we could deduce as $N\to\infty$, that the right-hand side in (1.13) has uniform limit $Bu(t)+f(t)$. Then the strong derivative $u'(t)$ would exist, and would be equal to $Bu(t)+f(t)$. This happens for example when $f(t)$ is uniformly continuous on the real axis, because uniformly continuous S^2-almost-periodic functions are almost-periodic Bochner.

In our more general case, when $f(t)$ is continuous but is almost-periodic only in S^2-sense, we arrive at the same result but we shall use a slightly more involved way.

We obtain from (1.13) by integration between 0 and t the relation

$$u_N(t) = \int_0^t (Bu_N(\sigma)+f(\sigma))\,d\sigma-\int_0^t \exp(-NB)f(\sigma+N)\,d\sigma+u_N(0). \tag{1.14}$$

If we let $N\to\infty$, and keep t fixed, we get

$$u(t) = \int_0^t (Bu(\sigma)+f(\sigma))\,d\sigma-\lim_{N\to\infty}\int_0^t \exp(-NB)f(\sigma+N)\,d\sigma+u(0). \tag{1.15}$$

On the other part we have the estimate

$$\left\|\int_0^t \exp(-NB)f(\sigma+N)\,d\sigma\right\| \le \exp(-aN)\int_N^{t+N} \|f(\xi)\|\,d\xi;$$

now remark that

$$\int_N^{t+N} \|f(\xi)\|\,d\xi \le \sqrt{t}\left(\int_N^{t+N} \|f(\xi)\|^2\,d\xi\right)^{1/2}.$$

Moreover, we can write

$$\int_N^{t+N} \|f(\xi)\|^2\, d\xi \le \sum_{p=0}^{[t]} \int_{N+p}^{N+p+1} \|f(\xi)\|^2\, d\xi \le [t]\|f\|_{S^2}^2$$

where $[t]$ is the greatest integer $\le t$.

This implies

$$\lim_{N\to\infty} \exp(-aN) \int_N^{N+t} \|f(\xi)\|\, d\xi = 0$$

for any fixed t. Hence (1.15) gives

$$u(t) = u(0) + \int_0^t (Bu(\sigma) + f(\sigma))\, d\sigma. \tag{1.16}$$

As we know that both f and Bu are continuous it follows that $u(t)$ is strongly continuously differentiable and (1.3) is verified. This proves our theorem.

References

1. Choy-Tak Taam, *Stability, periodicity and almost-periodicity of solutions of non-linear differential equations in Banach spaces*, J. Math. Mech. (5) **15** (1966), 849–876.

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Université de Montréal,
Montréal, Québec

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