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A HYPONORMAL TOEPLITZ COMPLETION PROBLEM

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Abstract. In this paper we consider the following 'Toeplitz completion' problem: Complete the unspecified analytic Toeplitz entries of the partial block Toeplitz matrix

$$A := \begin{bmatrix} T_{\overline{\psi}_1} & ? \\ ? & T_{\overline{\psi}_2} \end{bmatrix}$$

to make A hyponormal, where $\psi_i \in H^\infty$ is a non-constant rational function for $i = 1, 2$.

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1. Introduction. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators acting on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if its self-commutator $[T^*, T] \equiv T^*T - TT^*$ is positive semi-definite and *subnormal* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subseteq \mathcal{H}$ and $T = N|_{\mathcal{H}}$. Let $L^2 \equiv L^2(\mathbb{T})$ be the set of square-integrable measurable functions on $\mathbb{T}$ and $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. If P and $P^\perp$ denote the orthogonal projections from L^2 onto H^2 and $(H^2)^\perp$, respectively, and J denotes the unitary operator on L^2 defined by $J(f)(z) = \overline{z}f(\overline{z})$, then for every bounded measurable function $\phi \in L^\infty$, the operators T_ϕ and H_ϕ on H^2 are defined by

$$T_\phi g := P(\phi g) \quad \text{and} \quad H_\phi g := JP^\perp(\phi g) \qquad (g \in H^2),$$

which are called the *Toeplitz operator* and the *Hankel operator*, respectively, *with symbol* ϕ. The following is a basic connection between Hankel and Toeplitz operators:

$$T_\phi^* = T_{\overline{\phi}},\ \ H_\phi^* = H_{\widetilde{\phi}},\ \ H_{\phi\psi} = T_{\widetilde{\phi}}^* H_\psi + H_\phi T_\psi \ (\phi, \psi \in L^\infty), \text{ where } \widetilde{h}(z) := \overline{h(\overline{z})}.$$

Given a partially specified operator matrix with some known entries, the problem of finding suitable operators to complete the given partial operator matrix so that the resulting matrix satisfies certain given properties is called a *completion problem*. Dilation problems are special cases of completion problems: in other words, the dilation of A is a completion of the partial operator matrix $\left[\begin{smallmatrix} A & ? \\ ? & ? \end{smallmatrix}\right]$. In this paper we consider the hyponormal completion problem for

$$\begin{bmatrix} T_{\overline{\psi}_1} & ? \\ ? & T_{\overline{\psi}_2} \end{bmatrix},$$

where $\psi_i \in H^\infty$ is a non-constant rational function for $i = 1, 2$.

A *partial block Toeplitz matrix* is simply an $n \times n$ matrix, some of whose entries are specified Toeplitz operators and whose remaining entries are unspecified. A *hyponormal completion* of a partial operator matrix is a particular specification of the unspecified entries resulting in a hyponormal operator. For example,

$$\begin{bmatrix} T_z & 1 - T_z T_{\overline{z}} \\ 0 & T_{\overline{z}} \end{bmatrix}$$

is a hyponormal (even unitary) completion of the 2×2 partial operator matrix $\left[\begin{smallmatrix} T_z & ? \\ ? & T_{\overline{z}} \end{smallmatrix}\right]$. A *hyponormal Toeplitz completion* of the partial block Toeplitz matrix is a hyponormal completion whose unspecified entries are Toeplitz operators. Then we may ask whether or not there is a hyponormal Toeplitz completion of $\left[\begin{smallmatrix} T_z & ? \\ ? & T_{\overline{z}} \end{smallmatrix}\right]$? In [**3**], it was shown that no hyponormal Toeplitz completion of $\left[\begin{smallmatrix} T_z & ? \\ ? & T_{\overline{z}} \end{smallmatrix}\right]$ can exist. Moreover, in [**3**], the following problem was considered and then answered: Complete the unspecified Toeplitz entries of the partial block Toeplitz matrix

$$A := \begin{bmatrix} T_{\overline{z}} & ? \\ ? & T_{\overline{z}} \end{bmatrix} \tag{1.1}$$

to make A subnormal. However, in (1.1), if the entry $T_{\overline{z}}$ is replaced by a general co-analytic Toeplitz operator $T_{\overline{\psi}}$ ($\psi \in H^\infty$), then the above problem seems to be quite difficult to answer. First of all, for such a case, we need to solve the hyponormal completion problem.

The aim of this paper is to answer the following:

Problem 1. *Let $\psi_i \in H^\infty$ be a non-constant rational function for $i = 1, 2$. Complete the unspecified analytic Toeplitz entries of the partial block Toeplitz matrix*

$$A := \begin{bmatrix} T_{\overline{\psi}_1} & ? \\ ? & T_{\overline{\psi}_2} \end{bmatrix} \tag{1.2}$$

to make A hyponormal.

When we study hyponormality of the Toeplitz operator T_ϕ with symbol ϕ we may without loss of generality assume that $\phi(0) = 0$ because the hyponormality of an operator is invariant under translation by scalars.

In 1988, Cowen [**2**] has characterized the hyponormality of Toeplitz operators via a certain functional equation involving the operator's symbol ϕ.

Theorem A (Cowen's theorem) ([**2**, **8**]). *For each $\phi \in L^\infty$, T_ϕ is hyponormal if and only if there exists a function $k \in H^\infty$ such that $||k||_\infty \leq 1$ and $\phi - k\overline{\phi} \in H^\infty$.*

Recall that a function $\phi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are functions ψ_1, ψ_2 in $H^\infty(\mathbb{D})$ such that $\phi(z) = \psi_1(z)/\psi_2(z)$ for almost all $z \in \mathbb{T}$. Evidently, rational functions are of bounded type. It was known [**1**, Lemma 3]

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that if $\phi \in L^\infty$ then

$$\phi \text{ is of bounded type } \iff \ker H_\phi \neq \{0\} \iff \phi = \overline{\theta} b, \tag{1.3}$$

where θ is inner and $b \in H^\infty$. If $\phi \in L^\infty$, we write

$$\phi_+ \equiv P(\phi) \in H^2 \quad \text{and} \quad \phi_- \equiv \overline{P^\perp(\phi)} \in zH^2.$$

For an inner function θ, we write

$$\mathcal{H}(\theta) := H^2 \ominus \theta\, H^2\,.$$

If $\phi \in L^\infty$ is of bounded type then by (1.3) we can write

$$\phi_- = \theta \overline{a} \quad (\theta \text{ is inner and } a \in H^2), \tag{1.4}$$

where θ and a are coprime. We will refer the *coprime factorization* of ϕ_- for the representation (1.4). Note that if $f = \theta\overline{a} \in L^2$, then $f \in H^2$ if and only if $a \in \mathcal{H}(z\theta)$; in particular, if $f(0) = 0$ then $a \in \mathcal{H}(\theta)$. If ϕ_- is a rational function then in (1.4) θ can be chosen as a finite Blaschke product.

Let BMO denote the set of functions of bounded mean oscillation in L^1. It is well known that $L^\infty \subseteq BMO \subseteq L^2$. It is also known that if $f \in L^2$, then H_f is bounded on H^2 whenever $P^\perp f \in BMO$ (cf. [**9**]). If $\phi \in L^\infty$, then $\overline{\phi_-}, \overline{\phi_+} \in BMO$ so that $H_{\overline{\phi_-}}$ and $H_{\overline{\phi_+}}$ are well understood.

If both ϕ and $\overline{\phi}$ are of bounded type (e.g. ϕ is rational), then by the Beurling's theorem we can see that if T_ϕ is hyponormal then (also see [**6**, **7**])

$$\theta_+ H^2 = \ker H_{\overline{\phi_+}} \subset \ker H_{\overline{\phi_-}} = \theta_0 H^2, \tag{1.5}$$

which implies that θ_0 divides θ_+, i.e. $\theta_+ = \theta_0\theta_1$ for some inner function θ_1. Thus, if $\phi = \overline{\phi_-} + \phi_+ \in L^\infty$ such that ϕ and $\overline{\phi}$ are of bounded type such that T_ϕ is hyponormal then we can write

$$\phi_+ = \theta_0\theta_1\overline{a} \quad \text{and} \quad \phi_- = \theta_0\overline{b} \quad \text{(coprime factorizations)}\,,$$

where $a \in \mathcal{H}(z\theta_0\theta_1)$ and $b \in \mathcal{H}(\theta_0)$. If $g \in H^2$, the *reduced Cowen set* for g is defined by

$$G_{\overline{g}} := \{f \in H^2 : \overline{g} + f \in L^\infty \text{ and } T_{\overline{g}+f} \text{ is hyponormal}\}.$$

We next introduce the notion of block Toeplitz operators. For a Hilbert space $\mathcal{X}$, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ the corresponding Hardy space. We observe that $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$. Let $M_{m\times n}$ denote the set of $m \times n$ complex matrices and write $M_n := M_{n\times n}$. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ $(= L^\infty(\mathbb{T}) \otimes M_n)$ then the block Toeplitz operator T_Φ and the block Hankel operator H_Φ on $H^2_{\mathbb{C}^n}$ are defined as

$$T_\Phi f = P_n(\Phi f) \quad \text{and} \quad H_\Phi f = JP_n^\perp(\Phi f) \quad (f \in H^2_{\mathbb{C}^n}),$$

where P_n and $P_n^\perp$ denote the orthogonal projections that map from $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$ and $\left(H^2_{\mathbb{C}^n}\right)^\perp$, respectively, and J denotes the unitary operator from $L^2_{\mathbb{C}^n}$ to $L^2_{\mathbb{C}^n}$ given by $J(g)(z) = \overline{z} I_n g(\overline{z})$ for $g \in L^2_{\mathbb{C}^n}$ (I_n :=the $n \times n$ identity matrix). In 2006, Gu et al.

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[**5**] considered the hyponormality of block Toeplitz operators and characterized the hyponormality of block Toeplitz operators in terms of their symbols.

Theorem B (Hyponormality of block Toeplitz operators) ([**5**]). *For each $\Phi \in L^\infty_{M_n}$, T_Φ is hyponormal if and only if Φ is normal and there exists $K \in H^\infty_{M_n}$ such that $||K||_\infty \leq 1$ and $\Phi - K\Phi^* \in H^\infty_{M_n}$.*

2. The main result.

For $\Phi \in L^\infty_{M_n}$, the *pseudo-self commutator* of T_Φ is defined by

$$[T^*_\Phi, T_\Phi]_p := H^*_{\Phi^*}H_{\Phi^*} - H^*_\Phi H_\Phi.$$

Then T_Φ is said to be *pseudo-hyponormal* if $[T^*_\Phi, T_\Phi]_p \geq 0$. Evidently, if $\Phi \in L^\infty_{M_n}$, then

$$[T_{\Phi^*}, T_\Phi] = [T_{\Phi^*}, T_\Phi]_p + T_{\Phi^*\Phi - \Phi\Phi^*}.$$

We thus have

$$T_\Phi \text{ is hyponormal} \iff T_\Phi \text{ is pseudo-hyponormal and } \Phi \text{ is normal} \tag{2.1}$$

and that if we write

$$\mathcal{E}(\Phi) := \left\{K \in H^\infty_{M_n} :\ ||K||_\infty \leq 1 \ \text{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n}\right\},$$

then (via [**5**, Theorem 3.3]) T_Φ is pseudo-hyponormal if and only if $\mathcal{E}(\Phi) \neq \emptyset$.

Our main theorem answers Problem 1.

THEOREM 2.1. *For $i = 1, 2$, let $\psi_i \in H^\infty$ be a non-constant rational function and consider*

$$\Phi \equiv \begin{bmatrix} \overline{\psi}_1 & \phi_1 \\ \phi_2 & \overline{\psi}_2 \end{bmatrix} \quad (\phi_1, \phi_2 \in H^\infty).$$

Then

$$\{(\phi_1, \phi_2) \in H^\infty \times H^\infty :\ T_\Phi \textit{ is hyponormal}\}$$

$$= \begin{cases} \left\{(\phi_1, \phi_2) \in G_{\overline{\psi}_1} \times G_{\overline{\psi}_2} :\ |\phi_1| = |\phi_2|\right\} & \textit{if } \psi_1 = \psi_2 \\ \emptyset & \textit{if } \psi_1 \neq \psi_2\,. \end{cases} \tag{2.2}$$

Proof. We first observe

$$[T^*_\Phi, T_\Phi]_p = \begin{bmatrix} [T^*_{\phi_1+\overline{\psi}_1}, T_{\phi_1+\overline{\psi}_1}] & 0 \\ 0 & [T^*_{\phi_2+\overline{\psi}_2}, T_{\phi_2+\overline{\psi}_2}] \end{bmatrix}, \tag{2.3}$$

which implies

$$T_\Phi \text{ is pseudo-hyponormal} \iff \text{each } \bar{\phi}_i \text{ is of bounded type and } (\phi_1, \phi_2) \in G_{\bar{\psi}_1} \times G_{\bar{\psi}_2}, \tag{2.4}$$

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where the second condition follows from [**1**, Lemma 6]. Suppose $\psi_i \in H^\infty$ is a non-constant rational function for $i = 1, 2$. We note that if

$$\Phi = \begin{bmatrix} \overline{\psi}_1 & \phi_1 \\ \phi_2 & \overline{\psi}_2 \end{bmatrix},$$

then T_Φ is hyponormal if and only if T_Φ is pseudo-hyponormal and Φ is normal. A straightforward calculation shows that Φ is normal if and only if

$$\begin{cases} |\phi_1| = |\phi_2| \\ \phi_1(\psi_1 - \psi_2) = \overline{\phi_2(\psi_1 - \psi_2)} \end{cases}. \tag{2.5}$$

We now claim that $\psi_1 = \psi_2$. Assume to the contrary that $\psi_1 \neq \psi_2$. Since by (2.5), $\phi_1(\psi_1 - \psi_2) = c\,(c \in \mathbb{C})$, it follows from the F. and M. Riesz theorem that $c \neq 0$: indeed if $c = 0$ then $\phi_1 = 0$, and hence $T_{\overline{\psi}_1 + \phi_1} = T_{\overline{\psi}_1}$ is not hyponormal, which contradicts to the fact (2.4). Thus, ϕ_1 is invertible in H^∞, and hence ϕ_1 is an outer function (cf. [**4**]). Similarly, ϕ_2 is also an outer function. But since $|\phi_1| = |\phi_2|$, it follows that

$$\phi_2 = e^{i\xi}\phi_1 \quad \text{for some } \xi \in [0, 2\pi). \tag{2.6}$$

On the other hand, we note that if $f \in G_{\overline{g}}$ then $e^{i\mu} f \in G_{\overline{g}}$ for each $\mu \in [0, 2\pi)$. But since evidently $\phi_i \in G_{\overline{\psi}_i}$ $(i = 1, 2)$, it follows that $\phi_1 \in G_{\overline{\psi}_i}$ $(i = 1, 2)$. Thus, there exists a function $h_i \in \mathcal{E}(\overline{\psi}_i + \phi_1)$ for $i = 1, 2$, and hence $\frac{h_1 - h_2}{2} \in \mathcal{E}(\overline{\frac{\psi_1 - \psi_2}{2}} + \phi_1)$. Write

$$\psi := \frac{\psi_1 - \psi_2}{2}.$$

Then $T_{\overline{\psi} + \phi_1}$ is hyponormal. Since $\phi_1 \in G_{\overline{\psi}_1}$ and ψ_1 is non-constant, it follows that ϕ_1 is non-constant. But since by (2.5), $\phi_1 \psi = \frac{c}{2} \neq 0$, ψ is a non-constant rational function so that we may write

$$\psi = \zeta \frac{\prod_{j=1}^m (z - \beta_j)}{\prod_{i=1}^n (z - \alpha_i)} \quad (\alpha_i \neq \beta_j \text{ for any } i, j, \; \zeta \in \mathbb{C}, \; \zeta \neq 0).$$

Since $\phi_1 \psi = \frac{c}{2} \neq 0$ so that ψ is invertible in H^∞, we have $|\alpha_i| > 1$ and $|\beta_j| > 1$ for all i, j. Observe that

$$\begin{aligned} f \in \ker H_{\overline{\psi}} &\Longleftrightarrow \overline{\psi} f \in H^2 \\ &\Longleftrightarrow \frac{\prod_{j=1}^m (\overline{z} - \overline{\beta}_j)}{\prod_{i=1}^n (\overline{z} - \overline{\alpha}_i)} f \in H^2 \\ &\Longleftrightarrow z^{n-m} \frac{\prod_{j=1}^m (1 - \overline{\beta}_j z)}{\prod_{i=1}^n (1 - \overline{\alpha}_i z)} f \in H^2. \end{aligned}$$

If $n > m$, then

$$f \in \ker H_{\overline{\psi}} \Longleftrightarrow f\left(\frac{1}{\overline{\alpha}_i}\right) = 0.$$

In view of (1.4), if we write

$$\psi = \omega \overline{b} \quad \text{(coprime factorization)},$$

then a straightforward calculation shows that ω is a finite Blaschke product of the form

$$\omega := \prod_{i=1}^{n} \frac{z - \frac{1}{\overline{\alpha}_i}}{1 - \frac{1}{\alpha_i} z}$$

and

$$b := \overline{\zeta} \left(\prod_{i=1}^{n} \frac{\alpha_i}{\overline{\alpha}_i} \right) z^{n-m} \frac{\prod_{j=1}^{m}(1 - \overline{\beta}_j z)}{\prod_{i=1}^{n}(z - \alpha_i)},$$

where ω and b are coprime because $b(\frac{1}{\overline{\alpha}_i}) \neq 0$ for each $i = 1, \ldots, n$ by our assumption $\alpha_i \neq \beta_j$ for each i, j. We thus have $\mathcal{Z}(\omega) = \{\frac{1}{\overline{\alpha}_i} : \ i = 1, \ldots, n\}$, where $\mathcal{Z}(\omega)$ denotes the set of zeros of ω. Note that

$$\phi_1 = \frac{c}{2\zeta} \frac{\prod_{i=1}^{n}(z - \alpha_i)}{\prod_{j=1}^{m}(z - \beta_j)},$$

and that

$$\begin{aligned} f \in \ker H_{\overline{\phi}_1} &\iff \overline{\phi}_1 f \in H^2 \\ &\iff \frac{\prod_{i=1}^{n}(\overline{z} - \overline{\alpha}_i)}{\prod_{j=1}^{m}(\overline{z} - \overline{\beta}_j)} f \in H^2 \\ &\iff \overline{z}^{n-m} \frac{\prod_{i=1}^{n}(1 - \overline{\alpha}_i z)}{\prod_{j=1}^{m}(1 - \overline{\beta}_j z)} f \in H^2. \end{aligned}$$

But since $n \geq m$, it follows that $f \in \ker H_{\overline{\phi}_1}$ if and only if

$$f = z^{n-m} f_1 \ \text{ and } \ f_1\left(\frac{1}{\overline{\beta}_j}\right) = 0.$$

Thus, if we write $\phi_1 = \theta_1 \overline{a}_1$ (coprime factorization), then the same argument shows that $\mathcal{Z}(\theta_1) = \{0, \frac{1}{\overline{\beta}_j} : \ j = 1, \ldots, m\}$. But since $T_{\overline{\psi} + \phi_1}$ is hyponormal, it follows from (1.5) that $\mathcal{Z}(\omega) \subseteq \mathcal{Z}(\theta_1)$, and hence $\alpha_i = \beta_j$ for some i, j, a contradiction.

If $n = m$, then the same argument shows that

$$\mathcal{Z}(\omega) = \left\{ \frac{1}{\overline{\alpha}_i} : \ i = 1, \ldots, n \right\} \subseteq \mathcal{Z}(\theta_1) = \left\{ \frac{1}{\overline{\beta}_j} : \ j = 1, \cdots, m \right\},$$

a contradiction.

If $n < m$, then the same argument shows that

$$\mathcal{Z}(\omega) = \left\{ 0, \frac{1}{\overline{\alpha}_i} : \ i = 1, \cdots, n \right\} \subseteq \mathcal{Z}(\theta_1) = \left\{ \frac{1}{\overline{\beta}_j} : \ j = 1, \cdots, m \right\},$$

a contradiction.

Consequently, if T_Φ is hyponormal then $\psi_1 = \psi_2$. Thus, (2.2) follows at once from (2.4) and (2.5). This completes the proof. $\square$

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REMARK 2.2. (a) We need not expect that if

$$\Phi = \begin{bmatrix} \overline{\psi} & \phi_1 \\ \phi_2 & \overline{\psi} \end{bmatrix} \quad (\psi \in H^\infty \text{ is such that } \overline{\psi} \text{ is of bounded type})$$

is such that T_Φ is hyponormal then ϕ_1 and ϕ_2 are analytic. Indeed, if

$$\Phi \equiv \begin{bmatrix} \overline{z} & \overline{z} + 2z \\ \overline{z} + 2z & \overline{z} \end{bmatrix},$$

then Φ is normal and if we put $K := \frac{1}{2}\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, then $\Phi - K\Phi^* \in H^2_{M_2}$ and $||K||_\infty = 1$ so that by Theorem B, T_Φ is hyponormal. However, we have not been able to characterize all hyponormal Toeplitz completion of

$$\begin{bmatrix} T_{\overline{z}} & ? \\ ? & T_{\overline{z}} \end{bmatrix}.$$

(b) We also need not expect that if

$$\begin{bmatrix} T_{\overline{z}} & T_{\phi_1} \\ T_{\phi_2} & T_{\overline{z}} \end{bmatrix}$$

is hyponormal then ϕ_1 and ϕ_2 are trigonometric polynomials. Indeed, if

$$\Phi \equiv \begin{bmatrix} \overline{z} & \overline{z} + 2zb \\ \overline{z} + 2zb & \overline{z} \end{bmatrix} \quad \text{with } b(z) := \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z},$$

then a straightforward calculation shows that Φ is normal and if we put $K := \frac{1}{2}b\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, then $\Phi - K\Phi^* \in H^2_{M_2}$ and $||K||_\infty = 1$ so that by Theorem B, T_Φ is hyponormal.

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