On Spheroidal Harmonics.

By Edward Blades.

(*Read 12th February 1915. Received 23rd February 1915*).

1. *Introductory.*

Ellipsoidal harmonics are defined to be those solutions of Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

(where x, y, z are rectangular coordinates) which are useful in problems relating to ellipsoids. If the equation

$$\frac{x^2}{t} + \frac{y^2}{t - b^2} + \frac{z^2}{t - c^2} = 1 \qquad (1)$$

represents a family of confocal quadrics, it is known that the ellipsoidal harmonics belonging to the family are products of the form

$$\left[\begin{matrix} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{matrix}\right] \left(\frac{x^2}{l_1} + \frac{y^2}{l_1 - b^2} + \frac{z^2}{l_1 - c^2} - 1\right)\left(\frac{x^2}{l_2} + \frac{y^2}{l_2 - b^2} + \frac{z^2}{l_2 - c^2} - 1\right) \ldots\ldots \qquad (2)$$

where $l_1, l_2 \ldots$ are constants: one term is to be picked out of the square brackets as a multiplier of the other factors. Now if we consider the case in which two of the principal axes of the ellipsoids are equal, the latter become spheroids. If then we put $b = 0$ in (1) the family of confocal spheroids has the equation

$$\frac{x^2 + y^2}{t} + \frac{z^2}{t - c^2} = 1.$$

and belonging to this family there will be spheroidal harmonics of the form given by (2) with b zero.

It is to be noted, however, that all the harmonics of the spheroid cannot be included in this formula. Certain of them are found to be anomalous. When n is even, for example, there are solutions of Laplace's equation of the form

$$V=\prod_{r=1}^{r=\frac{1}{2}n}\left(\frac{x^2}{l_r}+\frac{y^2}{l_r-b^2}+\frac{z^2}{l_r-c^2}-1\right) \qquad \text{(3)}$$

and, in particular, for $n=2$, l is a root of the equation

$$\frac{1}{l_1}+\frac{1}{l_1-b^2}+\frac{1}{l_1-c^2}=0,$$

and with b tending to zero, c remaining finite, one of the spheroidal harmonics becomes of the form

$$V=x^2-y^2,$$

the other value of l providing for the normal type of harmonic.

Similarly for n odd equal to 3 (say), l is a root of the equation

$$\frac{3}{l_1}+\frac{1}{l_1-b^2}+\frac{1}{l_1-c^2}=0$$

(b tending to zero): hence we obtain the spheroidal harmonic

$$V=\text{const}\times x(x^2-3y^2),$$

which again does not conform to the type got from

$$V=x\prod_{r=1}^{\frac{n-1}{2}}\left(\frac{x^2}{l_r}+\frac{y^2}{l_r-b^2}+\frac{z^2}{l_r-c^2}-1\right) \qquad \text{(3A)}$$

Thus we have the interesting fact that although (3) and (3A) are of the form

$$\left\{\begin{matrix} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{matrix}\right\}\left(\frac{x^2+y^2}{l_1}+\frac{z^2}{l_1-c^2}-1\right)\left(\frac{x^2+y^2}{l_2}+\frac{z^2}{l_2-c^2}-1\right) \qquad \text{(A)}$$

there are spheroidal harmonics got from them which are not of this form.

If we transform Laplace's equation to spheroidal coordinates (r, θ, ϕ) defined by the equations

$$\left.\begin{aligned} x&=c\sqrt{r^2+1}\,\sin\theta\cos\phi \\ y&=c\sqrt{r^2+1}\,\sin\theta\sin\phi \\ z&=c\,r\cos\theta \end{aligned}\right\} \qquad \text{(4)}$$

and try to find a solution of the form

$$V=R\,\Theta\,\Phi$$

where R, Θ, and Φ are respectively functions of r alone, θ alone, and ϕ alone, we find that the solution is

$$V=P_n^m(ri)\,P_n^m(\cos\theta)\cos m\phi$$

$$\text{or}\quad V=P_n^m(ri)\,P_n^m(\cos\theta)\sin m\phi \qquad \text{(5)}$$

where $P_n^m(z)$ is defined by the equation $P_n^m(z) = (1-z^2)^{\frac{m}{2}} \dfrac{d^m P_n(z)}{dz^m}$, $P_n(z)$ is the Legendre function of the first kind, and $i = \sqrt{-1}$.

2. *A general formula for spheroidal harmonics.*

The general solution of Laplace's equation is*

$$\int_0^{2\pi} f(x\cos t + y\sin t + iz,\ t)\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where f is any arbitrary function of the two arguments, $x\cos t + y\sin t + iz$ and t.

The purpose of the present note is *to find what form the function f must have in order that the solution may be a spheroidal harmonic.*

Taking the Addition Theorem (Legendre—*Hist. Acad. Sc.*, Paris, 1789, éd. an II., p. 432) of Laplace's Coefficient

$$P_n\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi)'\} = P_n(\cos\theta)\,P_n(\cos\theta')$$
$$+ 2\sum_{m=1}^{n} \frac{(n-m)\,!}{(n+m)\,!} P_n^m(\cos\theta')\,P_n^m(\cos\theta)\cos m(\phi-\phi') \quad .. (7)$$

and multiplying each side by $\cos s\phi'$ and integrating between the limits 0 and 2π, we have

$$\int_0^{2\pi} P_n\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\}\cos s\phi'\,d\phi'$$
$$= \int_0^{2\pi} P_n(\cos\theta)\,P_n(\cos\theta')\cos s\phi'\,d\phi'$$
$$+ 2\sum_{m=1}^{n} \frac{(n-m)\,!}{(n+m)\,!} \int_0^{2\pi} P_n^m(\cos\theta')\,P_n^m(\cos\theta)\cos m(\phi-\phi')\cos s\phi'\,d\phi'$$
$$= \sum_{m=1}^{n} \frac{(n-m)\,!}{(n+m)\,!} \int_0^{2\pi} P_n^m(\cos\theta')\,P_n^m(\cos\theta)\,\{\cos(m\phi + \overline{s-m}\,\phi')$$
$$+ \cos(m\phi - \overline{s+m}\,\phi')\}\,d\phi'$$
$$= \sum_{m=1}^{n} \frac{(n-m)\,!}{(n+m)\,!} \int_0^{2\pi} P_n^m(\cos\theta')\,P_n^m(\cos\theta)\{\cos m\phi\cos\overline{s-m}\,\phi'$$
$$- \sin m\phi\sin\overline{s-m}\,\phi' + \cos(m\phi - \overline{s+m}\,\phi')\}\,d\phi'.$$

* Whittaker, *Mathematische Annalen* 57 (1903), pp. 333–355.

https://doi.org/10.1017/S0013091500002364 Published online by Cambridge University Press

Now each integral (except the first one for $m=s$) on the right hand side vanishes as m changes from 1 to n, and it becomes for $m=s$

$$\frac{(n-s)\,!}{(n+s)\,!}\,2\pi\,P_n^s(\cos\theta')\,P_n^s(\cos\theta)\cos s\phi,$$

and if for $\cos\theta'$ we write ri we get the first of the set of harmonics given in (5). Again, the left hand side

$$\int_0^{2\pi} P_n\{\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')\}\cos s\phi'\,d\phi'$$

can be written

$$\int_0^{2\pi} P_n\{\sqrt{r^2+1}\sin\theta\cos\phi\cos\phi'+\sqrt{r^2+1}\sin\theta\sin\phi\sin\phi'+ir\cos\theta\}$$
$$\times\cos s\phi'\,d\phi',$$

that is $\displaystyle\int_0^{2\pi} P_n\left(\frac{x\cos\phi'+y\sin\phi'+iz}{c}\right)\cos s\phi'\,d\phi'$ ....(8)

where x, y, and z are defined in (4), and this is the same form as the general solution given in (6).

If instead of multiplying both sides of the relation (7) by $\cos s\phi'$ we multiply by $\sin s\phi'$ and proceed as above, we obtain the corresponding result,

$$2\pi\frac{(n-s)\,!}{(n+s)\,!}\,P_n^s(ri)\,P_n^s(\cos\theta)\sin s\phi=\int_0^{2\pi}P_n\left(\frac{x\cos\phi'+y\sin\phi'+iz}{c}\right)$$
$$\times\sin s\phi'\,d\phi'\ldots\ldots\ldots\ldots(9)$$

which is again a case of (6).

It may be added that the integrals of (8) and (9) furnish respectively the $(n+1)$ and n harmonics in question, each of degree n, where s is an integer not greater than n.

Thus we obtain the general result that *all spheroidal harmonics may be represented in the form*

$$\int_0^{2\pi} P_n\left(\frac{x\cos t+y\sin t+iz}{c}\right)\begin{matrix}\sin\\ \cos\end{matrix}\,mt\,dt$$

where P_n *denotes the Legendre polynomial,* (x, y, z) *are rectangular coordinates, and* c *is the constant which defines the family of spheroids.*

https://doi.org/10.1017/S0013091500002364 Published online by Cambridge University Press