Bull. Aust. Math. Soc. **108** (2023), 133–141
doi:10.1017/S0004972722001071

MULTIPLIERS ON THE SECOND DUAL OF ABSTRACT SEGAL ALGEBRAS

MEHDI NEMATI ✉ and ZHILA SOHAEI

(Received 9 August 2022; accepted 24 August 2022; first published online 6 October 2022)

Abstract

We characterise the existence of certain (weakly) compact multipliers of the second dual of symmetric abstract Segal algebras in both the group algebra $L^1(G)$ and the Fourier algebra $A(G)$ of a locally compact group G.

2020 *Mathematics subject classification*: primary 43A07; secondary 43A20, 46H10, 46H20.

Keywords and phrases: abstract Segal algebra, topologically invariant φ-mean, multiplier.

1. Introduction

Let G be a locally compact group. By a classical result of Sakai [14], G is compact if and only if the group algebra $L^1(G)$ has a nonzero (weakly) compact right multiplier. In [10], Lau showed that an analogous result is true on the dual side, that is, G is discrete if and only if its Fourier algebra $A(G)$ has a nonzero (weakly) compact multiplier. Along this line of research, Ghahramani and Lau proved that G is compact if and only if any symmetric Segal algebra $S^1(G)$ of $L^1(G)$ has a nonzero (weakly) compact right or left multiplier [6].

Moreover, it was shown in [4] that G is amenable if and only if $L^\infty(G)^* = L^1(G)^{**}$, the second dual of $L^1(G)$ equipped with the first Arens product, has a nonzero (weakly) compact right multiplier. Along the way, Ghahramani and Lau proved that G is compact if and only if $L^1(G)^{**}$ has a (weakly) compact left multiplier T with $\langle T(n), 1\rangle \neq 0$ for some $n \in L^1(G)^{**}$ [5]. Dually, G is discrete if and only if $A(G)^{**}$ has a (weakly) compact left multiplier T with $\langle T(n), 1\rangle \neq 0$ for some $n \in A(G)^{**}$.

It is thus natural to try to determine when the second dual of a symmetric abstract Segal algebra of $L^1(G)$ or $A(G)$ has a nonzero (weakly) compact left or right multiplier. We answer this question by proving that if $\mathcal{B}$ is a symmetric abstract Segal algebra of a Banach algebra $\mathcal{A}$ and φ is a nonzero character on $\mathcal{A}$, then the existence of a

The research of the first author was in part supported by a grant from IPM (No. 1401170411).

(weakly) compact left or right multiplier on $\mathcal{B}$ is equivalent to the existence of the same multiplier on $\mathcal{A}$.

For a symmetric Segal algebra $S^1(G)$ of the group algebra $L^1(G)$, we denote by K the set of all right multipliers T on $S^1(G)^{**}$ with rank one such that $\langle T(n), \varphi_1 \rangle = 1$ whenever $\langle n, \varphi_1 \rangle = 1$, where φ_1 is the nonzero character on $L^1(G)$ defined by $\varphi_1(f) = \int_G f(x)\,dx$ for all $f \in L^1(G)$. We prove that if G is amenable and noncompact and $d(G)$ is the smallest possible cardinality of a covering of G by compact sets, then $|K| \geq 2^{2^{d(G)}}$.

2. Preliminaries

We shall now fix some notation. We denote the closed linear span by $\overline{\langle \cdot \rangle}$. Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}^*$ is naturally a Banach $\mathcal{A}$-bimodule with the actions

$$\langle f \cdot a, b \rangle = \langle f, ab \rangle, \quad \langle a \cdot f, b \rangle = \langle f, ba \rangle,$$

for all $f \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$. It is known that there is a multiplication $\Box$ on the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$, extending the multiplication on $\mathcal{A}$. The first Arens product in $\mathcal{A}^{**}$ is given as follows. For $m, n \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$,

$$\langle m \Box n, f \rangle = \langle m, n \cdot f \rangle, \quad \langle n \cdot f, a \rangle = \langle n, f \cdot a \rangle.$$

If $\mathcal{A}$ is a Banach algebra, then a linear mapping $T : \mathcal{A} \to \mathcal{A}$ is a right (respectively left) multiplier if $T(ab) = aT(b)$ (respectively $T(ab) = T(a)b$) for all $a, b \in \mathcal{A}$. In particular, for each $a \in \mathcal{A}$, the multiplication operators $\lambda_a : \mathcal{A} \to \mathcal{A}$ and $\rho_a : \mathcal{A} \to \mathcal{A}$ defined by $\lambda_a(b) = ab$ and $\rho_a(b) = ba$ are respectively left and right multipliers on $\mathcal{A}$. We also denote by $\Delta(\mathcal{A})$ the set of all nonzero characters on $\mathcal{A}$.

We recall from Burnham [2] that a Banach algebra $\mathcal{B}$ is an *abstract Segal algebra* of $\mathcal{A}$ if:

(i) $\mathcal{B}$ is a dense left ideal in $\mathcal{A}$;
(ii) there exists $M > 0$ such that $\|b\|_{\mathcal{A}} \leq M\|b\|_{\mathcal{B}}$ for each $b \in \mathcal{B}$;
(iii) there exists $C > 0$ such that $\|ab\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$ for each $a, b \in \mathcal{B}$.

We further say that $\mathcal{B}$ is symmetric if it is also a two-sided dense ideal in $\mathcal{A}$ and for each $a, b \in \mathcal{B}$,

$$\|ba\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}.$$

In this case, by [2, Theorem 2.1], $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$ are homeomorphic.

Throughout this paper, we assume that G is a locally compact group with a fixed left Haar measure and let $L^1(G)$ be the group algebra of G. Then $L^1(G)$ is a Banach algebra with the convolution product defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\,dy \quad (f, g \in L^1(G)).$$

A linear subspace $S^1(G)$ of $L^1(G)$ is called a Segal algebra, if:

(i) $S^1(G)$ is dense in $L^1(G)$;
(ii) $S^1(G)$ is a Banach space under some norm $\|\cdot\|_S$ and $\|f\|_1 \leq \|f\|_S$ for all $f \in S^1(G)$;

https://doi.org/10.1017/S0004972722001071 Published online by Cambridge University Press

(iii) $S^1(G)$ is left translation invariant and the map $x \mapsto l_x f$ of G into $S^1(G)$ is continuous;
(iv) $\|l_x f\|_S = \|f\|_S$, for all $x \in G$ and $f \in S^1(G)$.

We note that every Segal algebra is an abstract Segal algebra of $L^1(G)$ by [13, Proposition 1]. A Segal algebra $S^1(G)$ is symmetric if it is right translation invariant, $\|r_x f\|_S = \|f\|_S$ and the map $x \mapsto r_x f$ from G into $S^1(G)$ is continuous for all $x \in G$ and $f \in S^1(G)$. Note that every symmetric Segal algebra is a two-sided ideal of $L^1(G)$ and has an approximate identity in which each term has norm one in $L^1(G)$ (see [13, Section 8, Proposition 1]).

3. Multipliers on the second dual

Let $\mathcal{B}$ be a symmetric abstract Segal algebra of a Banach algebra $\mathcal{A}$. We note that for every $f \in \mathcal{B}^*$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we can define $f \bullet b \in \mathcal{A}^*$ by

$$\langle f \bullet b, a\rangle = \langle f, ba\rangle.$$

Hence, for every $m \in \mathcal{A}^{**}$ and $f \in \mathcal{B}^*$, we may define the functional $m \bullet f \in \mathcal{B}^*$ by

$$\langle m \bullet f, b\rangle = \langle m, f \bullet b\rangle \quad (b \in \mathcal{B}).$$

Thus, for every $m \in \mathcal{A}^{**}$ and $n \in \mathcal{B}^{**}$, we can define the functional $n \odot m \in \mathcal{B}^{**}$ by

$$\langle n \odot m, f\rangle = \langle n, m \bullet f\rangle \quad (f \in \mathcal{B}^*).$$

For $f \in \mathcal{B}^*$ and $a \in \mathcal{A}$, we also can define $f \bullet a \in \mathcal{B}^*$ by

$$\langle f \bullet a, b\rangle = \langle f, ab\rangle.$$

Thus for $n \in \mathcal{B}^{**}$ and $f \in \mathcal{B}^*$, we may define the functional $n \bullet f \in \mathcal{A}^*$ by

$$\langle n \bullet f, a\rangle = \langle n, f \bullet a\rangle \quad (a \in \mathcal{A}).$$

Therefore, for $m \in \mathcal{A}^{**}$ and $n \in \mathcal{B}^{**}$, we can define the functional $m \odot n \in \mathcal{B}^{**}$ by

$$\langle m \odot n, f\rangle = \langle m, n \bullet f\rangle \quad (f \in \mathcal{B}^*).$$

Let $\iota : \mathcal{B} \to \mathcal{A}$ be the inclusion map. Then ι is an injective Banach $\mathcal{A}$-bimodule morphism. Consider the adjoints $\iota^* : \mathcal{A}^* \to \mathcal{B}^*$ and $\iota^{**} : \mathcal{B}^{**} \to \mathcal{A}^{**}$ of ι. Since ι has a dense range, ι^* is injective. It is not hard to see that ι^* is in fact the restriction map. The following lemma will prove useful.

LEMMA 3.1. *Let $\mathcal{B}$ be a symmetric abstract Segal algebra of $\mathcal{A}$. Then for every $m \in \mathcal{A}^{**}$ and $n, p \in \mathcal{B}^{**}$, the following statements hold:*

(i) $\|n \odot m\| \leq C\|n\|\,\|m\|$;
(ii) $\iota^{**}(n \odot m) = \iota^{**}(n)\Box m$;
(iii) $p \odot (m\Box\iota^{**}(n)) = (p \odot m)\Box n$;
(iv) $\|m \odot n\| \leq C\|n\|\,\|m\|$;

https://doi.org/10.1017/S0004972722001071 Published online by Cambridge University Press

(v) $\iota^{**}(m \odot n) = m\Box\iota^{**}(n)$;
(vi) $(\iota^{**}(n)\Box m) \odot p = n\Box(m \odot p)$.

PROOF. The proofs of (i), (ii), (iv) and (v) are straightforward.

(iii) For $f \in \mathcal{B}^*$, $b \in \mathcal{B}$ and $a \in \mathcal{A}$,

$$\begin{aligned}\langle \iota^{**}(n) \cdot (f \bullet b), a\rangle &= \langle \iota^{**}(n), f \bullet ba\rangle = \langle n, \iota^*(f \bullet ba)\rangle \\ &= \langle n, f \cdot ba\rangle = \langle n \cdot f, ba\rangle = \langle (n \cdot f) \bullet b, a\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle (m\Box\iota^{**}(n)) \bullet f), b\rangle &= \langle m\Box\iota^{**}(n), f \bullet b\rangle = \langle m, \iota^{**}(n) \cdot (f \bullet b)\rangle \\ &= \langle m, (n \cdot f) \bullet b\rangle = \langle m \bullet (n \cdot f), b\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\langle p \odot (m\Box\iota^{**}(n)), f\rangle &= \langle p, (m\Box\iota^{**}(n)) \bullet f\rangle = \langle p, m \bullet (n \cdot f)\rangle \\ &= \langle p \odot m, n \cdot f\rangle = \langle (p \odot m)\Box n, f\rangle.\end{aligned}$$

Hence, we obtain $p \odot (m\Box\iota^{**}(n)) = (p \odot m)\Box n$, as required.

(vi) Let $f \in \mathcal{B}^*$, $b \in \mathcal{B}$ and $a \in \mathcal{A}$. Then

$$\begin{aligned}\langle (p \bullet f) \cdot b, a\rangle &= \langle p \bullet f, ba\rangle = \langle p, f \cdot ba\rangle \\ &= \langle p, (f \cdot b) \bullet a\rangle = \langle p \bullet (f \cdot b), a\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle m \cdot (p \bullet f), b\rangle &= \langle m, (p \bullet f) \cdot b\rangle = \langle m, p \bullet (f \cdot b)\rangle \\ &= \langle m \odot p, f \cdot b\rangle = \langle (m \odot p) \cdot f, b\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\langle (\iota^{**}(n)\Box m) \odot p, f\rangle &= \langle \iota^{**}(n)\Box m, p \bullet f\rangle = \langle \iota^{**}(n), m \cdot (p \bullet f)\rangle \\ &= \langle n, \iota^*(m \cdot (p \bullet f))\rangle = \langle n, m \cdot (p \bullet f)|_{\mathcal{B}}\rangle \\ &= \langle n, (m \odot p) \cdot f\rangle = \langle n\Box(m \odot p), f\rangle.\end{aligned}$$

Hence, $(\iota^{**}(n)\Box m) \odot p = n\Box(m \odot p)$ and the proof is complete. □

THEOREM 3.2. *Let $\mathcal{B}$ be a symmetric abstract Segal algebra of $\mathcal{A}$ and let $\varphi \in \Delta(\mathcal{A})$. Then the following statements are equivalent:*

(i) *there is a compact (weakly compact) left (right) multiplier T of $\mathcal{B}^{**}$ such that $\langle T(n), \varphi\rangle \neq 0$ for some $n \in \mathcal{B}^{**}$;*
(ii) *there is a compact (weakly compact) left (right) multiplier T of $\mathcal{A}^{**}$ such that $\langle T(m), \varphi\rangle \neq 0$ for some $m \in \mathcal{A}^{**}$.*

PROOF. Suppose that T is a compact (weakly compact) left multiplier of $\mathcal{B}^{**}$ with $\langle T(n), \varphi\rangle \neq 0$ for some $n \in \mathcal{B}^{**}$. Putting $p = T(n)$ makes $\lambda_p = T \circ \lambda_n$ a compact (weakly compact) left multiplier of $\mathcal{B}^{**}$. Now for each $n \in \mathcal{B}^{**}$, consider the continuous linear map $l_n : \mathcal{A}^{**} \to \mathcal{B}^{**}$ defined by $l_n(m) = n \odot m$ for all $m \in \mathcal{A}^{**}$. Since

https://doi.org/10.1017/S0004972722001071 Published online by Cambridge University Press

$\iota^{**} \circ \lambda_p = \lambda_{\iota^{**}(p)} \circ \iota^{**}$, by using Lemma 3.1(ii), $\lambda_{\iota^{**}(p^2)} = \lambda_{\iota^{**}(p)} \circ \iota^{**} \circ l_p = \iota^{**} \circ \lambda_p \circ l_p$ is a compact (weakly compact) left multiplier of $\mathcal{A}^{**}$ such that

$$\langle \lambda_{\iota^{**}(p^2)}(\iota^{**}(p)), \varphi \rangle = \langle \iota^{**}(p^3), \varphi \rangle = \langle p^3, \varphi \rangle = \langle p, \varphi \rangle^3 \neq 0.$$

Conversely, suppose that T is a compact (weakly compact) left multiplier of $\mathcal{A}^{**}$ such that $\langle T(m), \varphi \rangle \neq 0$ for some $m \in \mathcal{A}^{**}$. Then λ_p is a compact (weakly compact) left multiplier on $\mathcal{A}^{**}$, where $p = T(m)$. Choose $n_0 \in \mathcal{B}$ with $n_0(\varphi) = 1$. Using Lemma 3.1(iii), $n_0 \odot (p \square \iota^{**}(n)) = (n_0 \odot p) \square n$ for all $n \in \mathcal{B}^{**}$. Then the map $\lambda_{n_0 \odot p} = l_{n_0} \circ \lambda_p \circ \iota^{**}$ is a compact (weakly compact) left multiplier of $\mathcal{B}^{**}$ such that

$$\langle \lambda_{n_0 \odot p}(n_0), \varphi \rangle = \langle p, \varphi \rangle \neq 0,$$

as required. The result for a right multiplier T can be proved similarly. □

From [4, Theorem 2.1] and the above theorem, we obtain the following corollary.

COROLLARY 3.3. *Let $S(G)$ be a symmetric abstract Segal algebra of $L^1(G)$. Then G is amenable if and only if there is a compact (weakly compact) right multiplier T of $S(G)^{**}$ such that $\langle T(n), \varphi_1 \rangle \neq 0$ for some $n \in L^1(G)^{**}$.*

From [5, Theorem 4.1] and Theorem 3.2, we also obtain the following result.

COROLLARY 3.4. *Let $S(G)$ be a symmetric abstract Segal algebra of $L^1(G)$. Then G is compact if and only if there is a compact (weakly compact) left multiplier T of $S(G)^{**}$ such that $\langle T(n), \varphi_1 \rangle \neq 0$ for some $n \in S(G)^{**}$.*

To state the next corollary, let $A(G)$ be the Fourier algebra of a locally compact group G as defined in [3]. Combining Theorem 3.2 with [5, Theorem 4.3], we obtain the following characterisation of discrete groups.

COROLLARY 3.5. *Let $SA(G)$ be an abstract Segal algebra of the Fourier algebra $A(G)$. Then G is discrete if and only if there is a compact (weakly compact) left multiplier T of $SA(G)^{**}$ such that $\langle T(n), 1 \rangle \neq 0$ for some $n \in SA(G)^{**}$.*

4. Multipliers with rank one

Let $\mathcal{A}$ be a Banach algebra and let $\varphi \in \Delta(\mathcal{A})$. Following [8], we call an element $m \in \mathcal{A}^{**}$ a topologically left invariant φ-mean if $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a) m(f)$ for every $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$, or equivalently $a \square m = \varphi(a) m$. We denote the set of all topologically left invariant φ-means on $\mathcal{A}^*$ by $TLI_\varphi(\mathcal{A}^{**})$. We also put $I_\varphi := \{a \in \mathcal{A} : \varphi(a) = 0\}$ which is a co-dimension one closed ideal in $\mathcal{A}$. Recall that a locally compact group G is called amenable if there exists a *topologically left invariant mean* m on $L^\infty(G)$, that is, a bounded linear functional with $\|m\| = m(1) = 1$ such that $m(f \cdot a) = a(1) m(f)$ for all $f \in L^\infty(G)$ and $a \in L^1(\mathbb{G})$. Topologically right invariant means and (two-sided) invariant means on $L^\infty(G)$ are defined similarly. It is known that the existence of a topologically right invariant mean and the existence of a topologically invariant mean are both equivalent to G being amenable.

https://doi.org/10.1017/S0004972722001071 Published online by Cambridge University Press

A standard argument, used in the proof of [11, Theorem 4.1] on F-algebras, a class of Banach algebras including group algebras, shows that amenability of G is equivalent to the existence of a topologically left invariant φ_1-mean on $L^\infty(G)$ (see also [7, Remark 1.3]).

THEOREM 4.1. *Let $S(G)$ be an abstract Segal algebra of $L^1(G)$. Then G is amenable if and only if there is a nonzero idempotent $m \in S(G)^{**}$ such that ρ_m has rank one.*

PROOF. Suppose that G is amenable. Then by [1, Corollary 3.4], there is a topologically left invariant φ_1-mean m on $S(G)^*$. It is clear that m is a nonzero idempotent and the map ρ_m on $S(G)^{**}$, defined by $\rho_m(n) = n\Box m = \langle n, \varphi_1\rangle m$ for all $n \in S(G)^{**}$, has rank one.

Conversely, let $m \in S(G)^{**}$ be a nonzero idempotent such that ρ_m on $S(G)^{**}$ has rank one. Then there is a functional $\varphi \in S(G)^{***}$ such that $n\Box m = \varphi(n)m$ for all $n \in S(G)^{**}$. Since m is a nonzero idempotent, we obtain $\varphi(m) = 1$. Moreover,

$$\begin{aligned}\varphi(a * b)m &= (a * b)\Box m = a\Box(b\Box m)\\ &= a\Box(\varphi(b)m) = \varphi(b)a\Box m\\ &= \varphi(b)\varphi(a)m,\end{aligned}$$

for all $a, b \in S(G)$. This implies that $\varphi(a * b) = \varphi(a)\varphi(b)$ for all $a, b \in S(G)$. Since the map $n \mapsto n\Box m$ on $S(G)^{**}$ is weak*-weak* continuous and $\varphi(m) = 1$, it follows that $\varphi \in \Delta(S(G)) = \Delta(L^1(G))$. This shows that m is a topologically left invariant φ-mean on $S(G)^*$. Hence, G is amenable by [1, Corollary 3.4]. □

LEMMA 4.2. *Let $S^1(G)$ be a symmetric Segal algebra of $L^1(G)$ and let $\varphi \in \Delta(L^1(G))$. Then there is a one–one correspondence between the set of topologically left invariant φ-means on $S^1(G)^*$ and on $L^\infty(G)$.*

PROOF. Let $\iota : S^1(G) \to L^1(G)$ be the inclusion map. Consider the map $\iota^{**} : TLI_\varphi(S^1(G)^{**}) \to L^\infty(G)^*$. Let $n \in TLI_\varphi(S^1(G)^{**})$ and $m = \iota^{**}(n)$. It is clear that $m(\varphi) = 1$. Moreover, for every $a \in L^1(G)$, there is a sequence (a_i) in $S^1(G)$ such that $\|a_i - a\|_1 \to 0$. Since $\Delta(S^1(G)) = \Delta(L^1(G))$, we have

$$\begin{aligned}a\Box m &= \lim_i (a_i\Box\iota^{**}(n)) = \lim_i \iota^{**}(a_i\Box n)\\ &= \lim_i \varphi(a_i)\iota^{**}(n) = \varphi(a)\iota^{**}(n) = \varphi(a)m.\end{aligned}$$

Therefore, $\iota^{**}(TLI_\varphi(S^1(G)^{**})) \subseteq TLI_\varphi(L^\infty(G)^*)$. We next show that

$$\iota^{**} : TLI_\varphi(S^1(G)^{**}) \to TLI_\varphi(L^\infty(G)^*)$$

is injective. In fact, suppose that $m, n \in TLI_\varphi(S^1(G)^{**})$ with $m \neq n$. Then there exists $f \in S^1(G)^*$ such that $m(f) \neq n(f)$. Let $b_0 \in S^1(G)$ be such that $\varphi(b_0) = 1$. Then $m(f \cdot b_0) = m(f) \neq n(f) = n(f \cdot b_0)$. It follows that

$$\langle\iota^{**}(m), f \bullet b_0\rangle = \langle m, f \cdot b_0\rangle \neq \langle n, f \cdot b_0\rangle = \langle\iota^{**}(n), f \bullet b_0\rangle.$$

https://doi.org/10.1017/S0004972722001071 Published online by Cambridge University Press

Therefore, $\iota^{**}(m) \neq \iota^{**}(n)$. We now prove that ι^{**} is surjective. Suppose that $m \in TLI_\varphi(L^\infty(G)^*)$. Then for each $f \in S^1(G)^*$ and $a, b \in S^1(G)$, we have

$$\langle m \bullet f, a * b\rangle = \langle m, f \bullet a * b\rangle = \langle m, (f \bullet a) \cdot b\rangle = \varphi(b)\langle m \bullet f, a\rangle.$$

Thus, for $b \in I_\varphi$, we have

$$\langle m \bullet f, a * b\rangle = 0.$$

Since $S^1(G)$ has an approximate identity (not necessarily bounded), it follows that $\overline{\langle S^1(G) * I_\varphi\rangle} = I_\varphi$. Thus $(m \bullet f)|_{I_\varphi} = 0$. As $a * b - b * a \in I_\varphi$, we obtain

$$\langle m, f \bullet (a * b)\rangle = \langle m, f \bullet (b * a)\rangle.$$

Let $\varphi(b_0) = 1$ for some $b_0 \in S^1(G)$ and consider the functional $\tilde{m} \in S^1(G)^{**}$ defined by

$$\tilde{m}(f) = \langle m, f \bullet b_0\rangle, \quad f \in S^1(G)^*.$$

Then for each $b \in S^1(G)$ and $f \in S^1(G)^*$, we have

$$\begin{aligned}\tilde{m}(f \cdot b) &= \langle m, (f \cdot b) \bullet b_0\rangle = \langle m, f \bullet (b * b_0)\rangle \\ &= \langle m, f \bullet (b_0 * b)\rangle = \langle m, (f \bullet b_0) \cdot b)\rangle \\ &= \varphi(b)\langle m, f \bullet b_0\rangle = \varphi(b)\tilde{m}(f).\end{aligned}$$

Furthermore, it is obvious that $\tilde{m}(\varphi) = 1$. Hence, $\tilde{m} \in TLI_\varphi(S^1(G)^{**})$. We have to show that $\iota^{**}(\tilde{m}) = m$. In fact, for every $g \in L^\infty(G)$, we have

$$\langle \iota^{**}(\tilde{m}), g\rangle = \langle \tilde{m}, \iota^*(g)\rangle = \langle m, \iota^*(g) \bullet b_0\rangle - \langle m, g \cdot b_0\rangle = \langle m, g\rangle,$$

and the proof is complete. □

Before giving the next result, recall that the compactness of G is equivalent to the existence of a topologically invariant mean in $L^1(G)$. The following theorem is inspired by [4, Theorem 2.15].

THEOREM 4.3. *Let $S^1(G)$ be a symmetric Segal algebra of $L^1(G)$ and K be the set of all right multipliers T of $S^1(G)^{**}$ with rank one such that $\langle T(n), \varphi_1\rangle = 1$ whenever $\langle n, \varphi_1\rangle = 1$ for $n \in S^1(G)^{**}$. Then the following statements hold:*

(i) *$K \neq \emptyset$ if and only if G is amenable;*
(ii) *$|K| = 1$ if and only if G is compact;*
(iii) *if G is amenable and noncompact and $d(G)$ is the smallest possible cardinality of a covering of G by compact sets, then $|K| \geq 2^{2^{d(G)}}$.*

PROOF. (i) Suppose that G is amenable. Then by [1, Corollary 3.4], there is a topologically left invariant φ_1-mean m on $S^1(G)^*$. Since $\rho_m(n) = n \square m = \langle n, \varphi_1\rangle m$ for all $n \in S^1(G)^{**}$, it follows that ρ_m belongs to K.

Conversely, suppose that $T \in K$ and $\langle n, \varphi_1\rangle = 1$ for some $n \in S^1(G)^{**}$. Putting $m = T(n)$, we have $\langle m, \varphi_1\rangle = 1$. By the same argument as that used in the proof of Theorem 4.1, it is easy to show that m is a topologically left invariant φ_1-mean on $S^1(G)^*$. Thus, G is amenable by [1, Corollary 3.4].

https://doi.org/10.1017/S0004972722001071 Published online by Cambridge University Press

(ii) Let $T \in K$ and $n \in TLI_{\varphi_1}(S^1(G)^{**})$. Putting $m = T(n)$, by (i), m is a topologically left invariant φ_1-mean on $S^1(G)^*$. In particular, for each $p \in S(G)^{**}$ with $\langle p, \varphi_1 \rangle = 1$, we obtain $p \Box m = m$. Thus,

$$\rho_m(p) = p \Box m = m = T(p).$$

By linearity, we conclude that $\rho_m = T$ and so there is a one–one correspondence between K and $TLI_{\varphi_1}(S^1(G)^{**})$. By Lemma 4.2, $|K| = |TLI_{\varphi_1}(L^\infty(G)^*)|$.

Now suppose that G is compact. Then there is a topologically invariant mean m in $L^1(G)$. Thus, for each $n \in TLI_{\varphi_1}(L^\infty(G)^*)$, we have

$$m = n(\varphi_1)m = m \Box n = m(\varphi_1)n = n.$$

This shows that $|K| = |TLI_{\varphi_1}(L^\infty(G)^*)| = 1$.

Conversely, suppose that $|K| = 1$. Then $|TLI_{\varphi_1}(L^\infty(G)^*)| = 1$. Therefore, there is a unique topologically left invariant φ_1-mean m on $L^\infty(G)$. It follows that m belongs to $L^1(G)$ (see [9]), whence G is compact.

(iii) Suppose that G is noncompact. Then by [12, Theorem 1], the cardinality of $TLI_{\varphi_1}(L^\infty(G)^*)$ is at least $2^{2^{d(G)}}$. Therefore, $|K| = |TLI_{\varphi_1}(L^\infty(G)^*)| \geq 2^{2^{d(G)}}$. □

Acknowledgement

The authors would like to sincerely thank the referee for a careful reading of the paper.

References

[1] M. Alaghmandan, R. Nasr-Isfahani and M. Nemati, 'Character amenability and contractibility of abstract Segal algebras', *Bull. Aust. Math. Soc.* **82** (2010), 274–281.
[2] J. T. Burnham, 'Closed ideals in subalgebras of Banach algebras. I', *Proc. Amer. Math. Soc.* **32** (1972), 551–555.
[3] P. Eymard, 'L'algèbre de Fourier d'un groupe localement compact', *Bull. Soc. Math. France* **92** (1964), 181–236.
[4] F. Ghahramani and A. T.-M. Lau, 'Multipliers and ideals in second conjugate algebras related to locally compact groups', *J. Funct. Anal.* **132** (1995), 170–191.
[5] F. Ghahramani and A. T.-M. Lau, 'Multipliers and modulus on Banach algebras related to locally compact groups', *J. Funct. Anal.* **150** (1997), 478–497.
[6] F. Ghahramani and A. T.-M. Lau, 'Approximate weak amenability, derivations and Arens regularity of Segal algebras', *Studia Math.* **169** (2005), 189–205.
[7] E. Kaniuth, A. T.-M. Lau and J. Pym, 'On φ-amenability of Banach algebras', *Math. Proc. Cambridge Philos. Soc.* **144** (2008), 85–96.
[8] E. Kaniuth, A. T.-M. Lau and J. Pym, 'On character amenability of Banach algebras', *J. Math. Anal. Appl.* **344** (2008), 942–955.
[9] M. Klawe, 'On the dimension of left invariant means and left thick subsets', *Trans. Amer. Math. Soc.* **231** (1977), 507–518.
[10] A. T.-M. Lau, 'Uniformly continuous functionals on the Fourier algebra of any locally compact group', *Trans. Amer. Math. Soc.* **251** (1979), 39–59.
[11] A. T.-M. Lau, 'Analysis on a class of Banach algebras with application to harmonic analysis on locally compact groups and semigroups', *Fund. Math.* **118** (1983), 161–175.

[12] A. T.-M. Lau and A. L. T. Paterson, ‘The exact cardinality of the set of topological left invariant means on an amenable locally compact group’, *Proc. Amer. Math. Soc.* **98** (1986), 75–80.
[13] H. Reiter, *L^1-algebras and Segal Algebras*, Lecture Notes in Mathematics, 231 (Springer, Berlin, 1971).
[14] S. Sakai, ‘Weakly compact operators on operator algebras’, *Pacific J. Math.* **14** (1964), 659–664.

MEHDI NEMATI, Department of Mathematical Sciences,
Isfahan University of Technology, Isfahan 84156-83111, Iran
and
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran
e-mail: m.nemati@iut.ac.ir

ZHILA SOHAEI, Department of Mathematical Sciences,
Isfahan University of Technology, Isfahan 84156-83111, Iran
e-mail: j.sohaei@math.iut.ac.ir

https://doi.org/10.1017/S0004972722001071 Published online by Cambridge University Press