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INDICATOR SETS, REGULI, AND A NEW CLASS OF SPREADS

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1. Introduction. Let Σ be the projective 3-space over the field $GF(q)$ where $q = p^e$, p an odd prime. A *spread* W in Σ is a set of $q^2 + 1$ lines in Σ which are such that each point of Σ lies on exactly one line of W. Thus the lines of W are all mutually skew. The notion of a spread extends to higher dimensions and also applies for arbitrary fields [**1**; **3**; **6**, p. 29; **7**, p. 5]. Our concern, however, will be within the narrower but still extensive bounds indicated.

A particular type of spread, which is now usually identified as a *regular* spread, appears in the classical literature as an *elliptic linear congruence* [**9**, p. 315]. Contemporary interest in spreads arises from their intimate association with translation planes: every spread determines a translation plane, and every translation plane can be so determined [**6**, p. 133]. Here the term "spread" must have a wider definition which includes spreads in higher dimensional spaces. Spreads in the projective 3-space Σ over $GF(q)$ determine the class of finite translation planes of odd characteristic which are of dimension two over their kernels [**6**, p. 133].

It is possible to study spreads in Σ through the study of *indicator sets*, which are sets of q^2 points in the affine plane π over $GF(q^2)$, with a characteristic property to be presently explained (Lemma 2). The appealing feature of this point of view is that a knowledge of the geometry of points in the plane π can be invoked to gain new knowledge about spreads, which may be obscured in direct methods of attack in the 3-space Σ. The method has been used to advantage by Bruen [**4**] who, among other interesting results, used indicator sets to give an example of a spread in Σ which contains no regulus, and to provide a construction of some semifield planes.

This paper is in the same spirit as [**4**], but addresses itself to different problems. In Section 2 we introduce indicator sets in a manner which is equivalent to that of [**4**] but which is more adaptable to our methods. Section 3 establishes the connection between collineation groups in Σ and in π, which are the main tools for achieving our results. The results of Section 3 are immediately applied in Section 4 to describe in precise detail the manner in which reguli in Σ are indicated by points in π.

With this preparation we then introduce in Section 5 the concept of *kd*-spreads and *md*-spreads, generalizations of the subregular spreads of Bruck

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[**2**]. We produce in Section 6 a class of *md*-spreads which appears to be new. Aside from the intrinsic interest of these new spreads, we feel that their production vindicates the indicator set approach. It is possible to re-produce these spreads by more conventional methods in the geometry of lines in Σ, but they do not appear in the "natural" context of this approach. Also one is less likely to produce them by other methods in the generality in which they are produced by the indicator set approach, and the properties of particular cases are less apparent.

A final section explores the *md*-spreads produced in Section **6**, showing that two particular cases determine irregular nearfield planes, looking at a case that has special features of interest, and suggesting a method of further generalization.

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2. The indicator plane and indicator sets. Let F be the field $GF(q)$, where $q = p^e$, p any odd prime and e any positive integer. Let K be the quadratic extension of F. We shall denote the elements of F, with the exception of 0 and 1, by small Greek letters, and those of K by small Latin letters. Select a particular non-square ν in F (when $q \equiv -1 \pmod 4$) the natural choice is $\nu = -1$), so that the quadratic equation $t^2 - \nu = 0$ is irreducible over F. Then any element of K may be uniquely represented in the form $\alpha + t\beta$, where $\alpha, \beta \in F$ and $t^2 = \nu$.

$\Sigma = PG(3, q)$, the projective space of dimension 3 over F, and $\pi = AG(2, q^2)$, the affine plane over K, are both most frequently defined in terms of vector spaces in a well-known process [**6**, p. 27]. The points, lines, and planes of Σ are the $1-$, $2-$, and 3-dimensional subspaces of the four-dimensional vector space $V_4(q)$ over F, and incidence is defined by inclusion. The collineations of Σ are the semilinear transformations of $V_4(q)$, and the projective collineations are the linear transformations [**6**, p. 31]. π may be defined either through $V_2(q^2)$ or $V_3(q^2)$, but our purposes are better suited by the latter route. That is, we first define the projective plane $PG(2, q^2)$ from $V_3(q^2)$, and then remove a line l_∞ from $PG(2, q^2)$ to produce π. An *affinity* of π is a projective collineation of $PG(2, q^2)$ that fixes l_∞.

Let l be a line in the projective 3-space Σ and let $\mathscr{L}$ be the set of all lines in Σ which have no points in common with l. We establish a bijection between the lines of $\mathscr{L}$ and the points of the affine plane π. Expressing the points of Σ in terms of the homogeneous coordinates $(\sigma_1, \sigma_2, \tau_1, \tau_2)$, let l be the line $\sigma_1 = \sigma_2 = 0$. Then any line of $\mathscr{L}$ has equations $\tau_1 = \sigma_1\mu_1 + \sigma_2\mu_2$, $\tau_2 = \sigma_1\mu_3 + \sigma_2\mu_4$. In convenient matrix form this is

$$Y = XM,$$

where X, Y, M are the matrices $(\sigma_1\ \sigma_2)$, $(\tau_1\ \tau_2)$ and $\begin{bmatrix} \mu_1 & \mu_3 \\ \mu_2 & \mu_4 \end{bmatrix}$ respectively.

Thus each line of $\mathscr{L}$ is represented by a 2×2 matrix M over F. Conversely, each matrix M represents a distinct line of $\mathscr{L}$. We now match M with the point $(\mu_1 + t\mu_2, \mu_3 + t\mu_4)$ in π, and the bijection is complete.

Definition. π is called the *indicator plane* of Σ *relative to* l, or simply the *indicator plane* of $\mathscr{L}$.

Notation. Let the lines $a, b, \ldots$ of Σ correspond to the points $A, B, \ldots$ respectively of π under the above bijection.

LEMMA 1. *Two distinct lines a and b of $\mathscr{L}$ intersect if and only if the corresponding points A and B in π are joined by a line with slope from the set $\{\infty\} \cup F$.*

By a line with slope ∞, we mean a line with equation $x = a$ $(a \in K)$. The slope of the line joining two points (x_1, y_1) and (x_2, y_2), $x_2 \neq x_1$, is $(y_2 - y_1)/(x_2 - x_1)$.

Proof. Let a and b be represented by the matrices $M = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_2 & \mu_4 \end{bmatrix}$ and $N = \begin{bmatrix} \nu_1 & \nu_3 \\ \nu_2 & \nu_4 \end{bmatrix}$ respectively. These two lines intersect if and only if $(\sigma_1\sigma_2)M = (\sigma_1\sigma_2)N$ for some pair σ_1, σ_2 which are not both zero. The necessary and sufficient condition for this is that the matrix $M - N$ be singular, i.e. that

$$(\mu_1 - \nu_1)(\mu_4 - \nu_4) - (\mu_2 - \nu_2)(\mu_3 - \nu_3) = 0.$$

Thus either $\mu_1 - \nu_1 = \mu_2 - \nu_2 = 0$, or else $\mu_3 - \nu_3 = \lambda(\mu_1 - \nu_1)$, $\mu_4 - \nu_4 = \lambda(\mu_2 - \nu_2)$ for some $\lambda \in F$. In the former case, A and B both lie on the line $x = \mu_1 + t\mu_2$ of π, so that A has slope ∞, and the latter case, AB has slope

$$\frac{(\mu_3 + t\mu_4) - (\nu_3 + t\nu_4)}{(\mu_1 + t\mu_2) - (\nu_1 + t\nu_2)} = \frac{\lambda(\mu_1 - \nu_1) + \lambda t(\mu_2 - \nu_2)}{(\mu_1 - \nu_1) + t(\mu_2 - \nu_2)} = \lambda.$$

Definition. Two distinct points A and B in π are *compatible* if the slope of the line AB is in $K - F$. A and B are *incompatible* if they are not compatible.

Suppose now that W is a spread in Σ, i.e. W is a set of lines such that each point of Σ lies on exactly one line of W. Suppose further that the line $l : \sigma_1 = \sigma_2 = 0$ is a line of W. Then the remaining lines of W form a subset of the set $\mathscr{L}$ and correspond under the bijection to a subset $\mathscr{I} = \mathscr{I}(W)$ of points in π. $\mathscr{I}$ is called the *indicator set* of W, and we say that any subset of $\mathscr{I}$ *indicates* the corresponding subset $W - \{l\}$.

From Lemma 1 and the fact that any two lines of W are skew, we deduce the fundamental property of $\mathscr{I}$, namely

LEMMA 2. *The slope of the join of any two points of $\mathscr{I}$ is an element of $K - F$.*

In other words, any two points of $\mathscr{I}$ are compatible. Conversely, given any set $\mathscr{I}'$ of points in π any two of whose points are compatible, the image Z' of

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$\mathscr{I}'$ under the bijection is a set of mutually skew lines in $\mathscr{L}$, and, with l, is called a *partial spread* in Σ [**7**, p. 53]. If $W' = Z' \cup \{l\}$ contains all points of Σ, then W' is a spread and $\mathscr{I}'$ is the indicator set of W'.

Since the spread W contains $q^2 + 1$ lines, $\mathscr{I}$ is a set of q^2 points of π, all mutually compatible.

Examples of indicator sets are any lines of π with slope in $K - F$. Less obvious examples are Baer subplanes of π [**6**, p. 118] all of whose lines have slope in $K - F$. Both of these examples indicate spreads which are called regular spreads, and which define Desarguesian planes.

3. Collineations in Σ and in π. Let Γ be any projective collineation of Σ that fixes the line $l : \sigma_1 = \sigma_2 = 0$. Then Γ permutes the lines $Y = XM$ of $\mathscr{L}$. In block matrix form Γ is therefore the mapping

$$(XY) \rightarrow (XY)\begin{bmatrix} E & T \\ O & A \end{bmatrix},$$

where E, A, O, T are 2×2 matrices over F; E and A are non-singular and O is the zero matrix. Because of homogeneity, the matrix $C = \begin{bmatrix} E & T \\ O & A \end{bmatrix}$ and the matrix λC $(\lambda \in F, \lambda \neq 0)$ yield the same collineation.

Since Γ permutes the lines of $\mathscr{L}$, it induces a permutation of the points of the indicator plane π. This permutation may or may not be an affinity in π; in its most general form, an affinity in π is the mapping $(x\, y\, 1) \rightarrow (x\, y\, 1)B$, where B is a 3×3 non-singular matrix over K:

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ h & k & 1 \end{bmatrix},$$

and $(x\, y\, 1)$ are the coordinates of the point (x, y) of π in homogeneous form. It is important for our purposes to identify those collineations Γ which induce affinities in π, thus establishing an isomorphism between the group of such collineations Γ and the corresponding group of affinities in π.

Accordingly, we note first that $C = C_1C_2$, where

$$C_1 = \begin{bmatrix} I & T \\ O & A \end{bmatrix}, \quad C_2 = \begin{bmatrix} E & O \\ O & I \end{bmatrix},$$

and I is the 2×2 identity matrix. Considering C_1 first, we note that

$$(X\; XM)C_1 = (X\; XT + XMA);$$

thus C_1 carries the line $Y = XM$ of $\mathscr{L}$ into the line $Y = X(MA + T)$. If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ and $T = \begin{bmatrix} \theta_1 & \eta_1 \\ \theta_2 & \eta_2 \end{bmatrix}$, let $h = \theta_1 + t\theta_2$, $k = \eta_1 + t\eta_2$; recalling that $x = \mu_1 + t\mu_2$ and $y = \mu_3 + t\mu_4$, where $M = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_2 & \mu_4 \end{bmatrix}$, we conclude that the collineation given by C_1 induces the affinity

$$(x, y, 1) \to (x, y, 1)\begin{bmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ h & k & 1 \end{bmatrix}$$

in π.

Turning now to the collineation of Σ with matrix C_2, we have

$$(X\,XM)C_2 = (XE\,XM).$$

Therefore C_2 carries $Y = XM$ into the line $Y = XM'$, where $M' = E^{-1}M$. If $E^{-1} = \begin{bmatrix} \epsilon_1 & \epsilon_3 \\ \epsilon_2 & \epsilon_4 \end{bmatrix}$, C_2 induces the permutation $(x, y) \to (x', y')$ in π where

$$\text{(3.1)} \quad \begin{aligned} x' &= (\epsilon_1\mu_1 + \epsilon_3\mu_2) + t(\epsilon_2\mu_1 + \epsilon_4\mu_2) \\ y' &= (\epsilon_1\mu_3 + \epsilon_3\mu_4) + t(\epsilon_2\mu_3 + \epsilon_4\mu_4) \end{aligned}$$

and $(x, y) = (\mu_1 + t\mu_2, \mu_3 + t\mu_4)$.

Now the point $(0, 0)$ of π is fixed; hence the above permutation is an affinity in π if and only if

$$\text{(3.2)} \quad \begin{aligned} x' &= xe_1 + ye_3 \\ y' &= xe_2 + ye_4 \end{aligned}$$

for some non-singular matrix $\begin{bmatrix} e_1 & e_3 \\ e_2 & e_4 \end{bmatrix}$ over K. Successively letting (x, y) be $(1, 0)$, $(0, 1)$, $(t, 0)$, and comparing (3.1) with (3.2), we obtain $e_1 = e_4 = \epsilon_1 + t\epsilon_2$, $e_2 = e_3 = 0$, $\epsilon_3 + t\epsilon_4 = \nu\epsilon_2 + t\epsilon_1$, where $\nu = t^2$. Therefore C_2 induces an affinity in π if and only if $E^{-1} = \begin{bmatrix} \epsilon_1 & \nu\epsilon_2 \\ \epsilon_2 & \epsilon_1 \end{bmatrix}$, and in this case the affinity has the form $(x, y) \to (ex, ey)$ where $e = \epsilon_1 + t\epsilon_2$.

Summarizing, we have

LEMMA 3. *The group* Γ *of projective collineations*

$$(XY) \to (XY)\begin{bmatrix} \varphi_1 & \nu\varphi_2 & \theta_1 & \eta_1 \\ \varphi_2 & \varphi_1 & \theta_2 & \eta_2 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \gamma & \delta \end{bmatrix}$$

in Σ *induces, and is isomorphic to, the group* Λ *of affinities*

$$(x\,y\,1) \to (x\,y\,1)\begin{bmatrix} e\alpha & e\beta & 0 \\ e\gamma & e\delta & 0 \\ h & k & 1 \end{bmatrix}$$

in π, *where* $h = \theta_1 + t\theta_2$, $k = \eta_1 + t\eta_2$, *and* $e = (\varphi_1 + t\varphi_2)^{-1}$.

Some familiarity with the action of the group Λ on the points and lines of π is necessary. In the remainder of this section we note the properties of Λ that are most important for our purposes.

First, we observe that Λ contains the following well-known affine collineations:

I. All translations in π : $(x, y) \to (x + h, y + k)$.

II. All central dilatations with any given centre, as for example the central dilatations $(x, y) \to (ex, ey)$ with centre $(0, 0)$. We note that the largest subgroup of dilatations with a given centre is of order $q^2 - 1$.

III. Conjugates in Λ of the General Linear Group $GL(2, q)$. In more geometrical terms, $GL(2, q)$ is the group of all affinities of the subplane $AG(2, q)$ of π which fix the point $(0, 0)$, and are simply the transformations $(x, y) \to (x, y)A$, where A is any non-singular 2×2 matrix over F.

LEMMA 4. (i) *Λ fixes the set of slopes $\{\infty\} \cup F$.*

(ii) *Λ is transitive on all ordered pairs of points P, Q such that the slope of the line PQ is in $K - F$.*

Proof. (i) It is clear from Lemma 3 that any element of Λ is a product of elements of types I, II and III listed above. Each of these types fixes the set of slopes $\{\infty\} \cup F$.

(ii) We note that Λ is transitive on the points of π, because Λ contains all translations. It suffices therefore to show that Λ_0, the stabilizer of $(0, 0)$ in Λ, is transitive on the lines $y = xm$ $(m \in K - F)$ and also transitive on the points $\neq (0, 0)$ on the line $y = xm$. To prove these we note that Λ_0 is the group of affinities

$$(x\, y\, 1) \to (x\, y\, 1)\begin{bmatrix} e\alpha & e\beta & 0 \\ e\gamma & e\delta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Under Λ_0, $(1, t, 0) \to (e(\alpha + t\gamma), e(\beta + t\delta), 0) = (\alpha + t\gamma, \beta + t\delta, 0)$. Now the only restriction on α, β, γ, δ is that $\alpha\delta - \beta\gamma \neq 0$. Therefore, setting $\alpha = 1$ and $\gamma = 0$, we see that there is an element of Λ_0 which carries slope t into any slope in $K - F$, i.e. Λ_0 is transitive on the slopes $K - F$. Finally, Λ_0 contains all dilatations $(x, y) \to (ex, ey)$; therefore there is an element of Λ_0 taking $(1, m)$ into (x, xm) for all $x \neq 0$, and Λ_0 is transitive on the points $\neq (0, 0)$ on $y = xm$.

Let us return to a closer consideration of the geometric nature of the subgroups of Λ which are of type III. Each subgroup of this type fixes some point of π, and in view of Lemma 4 any point of π is left fixed by at least one such subgroup. We lose no generality therefore in restricting our consideration to the group $GL(2, q)$ itself, which fixes $(0, 0)$. The elements of this group are represented in the usual fashion, namely by 2×2 non-singular matrices

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

over F, and the affinity is given by $(x\, y) \to (x\, y)A$. The centre of this group is the subgroup of dilatations with centre $(0, 0)$ in $AG(2, q)$: $(x, y) \to$

$(\lambda x, \lambda y)(\lambda \neq 0)$. The dilatations fix every line through $(0, 0)$ in $AG(2, q)$ (and indeed in π). The other affinities fix two, one, or no lines of $AG(2, q)$. We therefore have three types:

(a) A fixes exactly two lines in $AG(2, q)$. Then A is either a *strain*, (sometimes called an *affine homology*) pointwise fixing one line (its *axis*), and moving all points off its axis in the direction of the other fixed line, or else it is the product of a strain and a dilatation.

(b) A fixes exactly one line in $AG(2, q)$. Then A is either a *shear*, sometimes called an *affine elation*) pointwise fixing that line (its *axis*) and moving all other points in the direction of its axis, or else it is the product of a shear and a dilatation; cf. [**6**, p. 133].

(c) A fixes no lines in $AG(2, q)$. Then A must fix two slopes in the larger plane π, since K is the quadratic extension of F, and therefore the characteristic equation of A is solvable in K. Thus A fixes two slopes from the set $K - F$. We can be more specific:

LEMMA 5. *If A fixes the slope $m \in K$, A also fixes m^q. Conversely, if $m \in K - F$, there exist affinities of type* III(c) *which fix the directions m and m^q.*

Proof. Denote A by the matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. The line $y = xm$ of π is fixed (and therefore the slope m is fixed) if and only if

$$\gamma m^2 + (\alpha - \delta)m - \beta = 0. \tag{3.3}$$

This equation is invariant under the field automorphism $x \rightarrow x^q$, and therefore m^q also satisfies (3.3). To prove the last part of Lemma 5, we note first that the slopes t and $t^q = t\nu^{(q-1)/2} = -t$ are fixed by the affinities $A = \begin{bmatrix} \alpha & \nu\beta \\ \beta & \alpha \end{bmatrix}$, which are of type III(c) if $\beta \neq 0$. Next, the slope t is carried into the slope $m = \gamma + t\delta$ $(\delta \neq 0)$ by the affinity $T = \begin{bmatrix} 1 & \gamma \\ 0 & \delta \end{bmatrix}$. Therefore the affinities $T^{-1}AT$ are of type III(c) and fix m. By the first part of Lemma 5, they also fix m^q.

In virtue of Lemmas 4 and 5, we may analyse the affinities of type III(c) by examining those affinities of type III which fix t and $t^q = -t$. An easy calculation yields that these are precisely the affinities $A = \begin{bmatrix} \alpha & \nu\beta \\ \beta & \alpha \end{bmatrix}$, where $\alpha^2 - \nu\beta^2 \neq 0$. Now the determinant of A, $\alpha^2 - \nu\beta^2$, is either a square $(\neq 0)$ or a non-square, and A is the product $A'D$, where A' has determinant 1 or ν, and D is a dilatation. We are particularly concerned with the case that det $A = 1$.

Let us denote A by R in this case. Thus $R = \begin{bmatrix} \alpha & \nu\beta \\ \beta & \alpha \end{bmatrix}$ $(\alpha^2 - \nu\beta^2 = 1)$. The set of matrices of the above form is a cyclic group of order $q + 1$; we call these the *rotations* with *centre* (0, 0), by analogy with $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ in the real case.

A second important class is the *reflections* $\begin{bmatrix} \alpha & -\nu\beta \\ \beta & -\alpha \end{bmatrix}$ $(\alpha^2 - \nu\beta^2 = 1)$, the name arising again by analogy with the real case. The reflections are of period 2, and are special cases of type III(a), fixing the lines $y = x(-\alpha \pm 1)/\beta$ (unless $\beta = 0$, in which case $x = 0$ and $y = 0$ are fixed). Moreover, if $\beta \neq 0$, $y = x(1 - \alpha)/\beta$ is pointwise fixed, and if $\beta = 0$, $y = 0$ is pointwise fixed. Hence if P is any non-fixed point, and if P' is the image of P, then the line PP' has slope $(-1 - \alpha)/\beta$ if $\beta \neq 0$, and ∞ if $\beta = 0$. We therefore have

LEMMA 6. *Let P be any point which is not left invariant by a reflection R', and let $P \to P'$ under R'. Then P and P' are incompatible.*

The rotations $\begin{bmatrix} \alpha & \nu\beta \\ \beta & \alpha \end{bmatrix}$ and the reflections $\begin{bmatrix} \alpha & -\nu\beta \\ \beta & -\alpha \end{bmatrix}$ $(\alpha^2 - \nu\beta^2 = 1)$ fix any conic

$$C_r : -\nu x^2 + y^2 = r$$

in π [**9**, p. 245] which, again in analogy with real Euclidean geometry, we call a *circle* with centre (0, 0). The circles C_r $(r \in K)$ divide into three mutually exclusive classes, namely

(i) C_0, which is a degenerate circle consisting of the two lines $y = tx$ and $y = -tx$.

(ii) C_r (r a square $\neq 0$ in K). By easy calculation, all the lines $y = xm$ for which $m^2 - \nu$ is a square $\neq 0$ in K, meet each C_r, and these are the only lines through (0, 0) which meet C_r.

(iii) C_r (r a non-square in K). Again by easy calculation, all the lines $y = xm$ for which $m^2 - \nu$ is a non-square in K meet each C_r, and these are the only lines through (0, 0) which meet C_r.

Clearly the rotations and reflections which we have defined are relative to the two slopes t and $-t$, which are fixed by the rotations and interchanged by the reflections. Similarly, and in virtue of Lemmas 4 and 5, we have rotations and reflections relative to *any* pair of slopes m, m^q in $K - F$, and they have similar properties to the rotations and reflections which we have defined. Also we have circles relative to any pair m, m^q with similar properties to the circles C_r.

4. Reguli and their indication in π. After defining reguli in Σ, which are central in our subsequent work, we proceed to specify the exact manner in which reguli of the type that arise later are indicated in π. Theorems 1, 2, and 2′ which do this can be compared with Lemma 3.3 of [**4**, p. 526]. They embrace the results of this lemma, and go on to a more penetrating analysis of the indication of reguli.

The following definitions and preliminary results can be found in most good older books on three dimensional projective geometry, e.g. [**9**, Chapter XI].

Let a, b, c be any three skew lines in $\Sigma = PG(3, q)$. Through any point on a there passes exactly one line that meets b and c. Such a line is called a *transversal* of a, b, and c; there are exactly $q + 1$ transversals to any three skew lines, and they are all mutually skew. Now let a', b', c' be any three transversals to a, b and c. The set $\mathscr{R}$ of transversals to a', b' and c', which includes the lines a, b, c, is independent of the choice of a', b', c', and is called the *regulus* determined by a, b and c. It can be shown that the same regulus is determined by any three of its $q + 1$ lines. Moreover, the set of transversals to any three lines of a regulus $\mathscr{R}$ is independent of the particular choice of the three lines of $\mathscr{R}$, and forms a regulus $\mathscr{R}'$, called the *opposite* regulus to $\mathscr{R}$. Thus every line of $\mathscr{R}'$ meets every line of $\mathscr{R}$, and $\mathscr{R}$ and $\mathscr{R}'$ both cover the same $(q + 1)^2$ points of Σ, forming a *doubly-ruled quadric* $\mathscr{Q}$. The lines of $\mathscr{R}$ and the lines of $\mathscr{R}'$both lie in $\mathscr{Q}$ [**9**, p. 301; **2**, p. 435].

We are interested in the manner in which a regulus in Σ is indicated in π. As before, we consider the line $l : \sigma_1 = \sigma_2 = 0$, and regard $\pi : AG(2, q^2)$ as the indicator plane of the set of lines $\mathscr{L}$ which have no contact with l. Because we wish to deal later with spreads, our interest lies only with reguli $\mathscr{R}$ in Σ that either contain l as one line, or else have no contact with l. The two cases are dealt with in the next two theorems.

THEOREM 1. *If $\mathscr{R} \ni l$, $\mathscr{R} - \{l\}$ is indicated in π by a set of q collinear points $P_1, P_2, \ldots, P_q$ with the property that the subgroup of dilatations with centre $P_i (i = 1, \ldots, q)$ and order $q - 1$ is transitive on the remaining members of the set. Conversely, any such set in π indicates the lines $\neq l$ of a regulus containing l.*

Proof. Due to the extent of the group Γ of collineations in Σ (Lemmas 3 and 4), we lose no generality in assuming that $\mathscr{R} - \{l\}$ contains the line $a : \tau_1 = \tau_2 = 0$, which is indicated by the point $P_i(0, 0)$ in π. Let p_j be any other line of $\mathscr{R} - \{l\}$, and let $P_j(x, y)$ be the indicator point of p_j in π. By Lemma 1, the line P_iP_j of π has slope in $K - F$. Now the collineations

$$D = \left\{\begin{bmatrix} \lambda I & 0 \\ 0 & I \end{bmatrix}\right\} \quad \lambda \in F, \lambda \neq 0$$

fix both l and a pointwise, and therefore fix $\mathscr{R}'$ linewise. Hence D permutes the lines of $\mathscr{R} - \{l\} - \{a\}$, and D is easily seen to be transitive on this set. But D induces the dilatation $(x, y) \to (\lambda x, \lambda y)$ with centre $(0, 0)$, and therefore the indicator points of $\mathscr{R} - \{l\} - \{a\}$ in π are the images of P_j under these dilatations. Since these dilatations form the subgroup of order $q - 1$ of dilatations with centre $(0, 0)$, and since the choice of a as a line of $\mathscr{R} - \{l\}$ is arbitrary, the first part of Theorem 1 is established. As for the converse result, it follows by observing that the above argument is reversible in every step.

It is worth remarking that in the case that $\mathscr{R} \ni l$, which we have just considered, every line of $\mathscr{R}'$ meets l, and therefore no line of $\mathscr{R}'$ is indicated in π.

Notation. For economy, and with little risk of confusion, we let the same symbol denote both a subset of $\mathscr{L}$ in Σ and the corresponding set of indicator points in π. Thus for example, $\mathscr{R}$ may denote either a regulus of Σ having no contact with l or the set of $q + 1$ points of π which indicate this regulus.

THEOREM 2. *If $\mathscr{R}$ has no contact with l, then $\mathscr{R}$ is indicated by a set $\mathscr{R}$ of $q + 1$ points on a circle relative to fixed slopes m and m^q. This set is a single orbit under the group of rotations which fixes the circle. The opposite regulus $\mathscr{R}'$ is indicated by the images of $\mathscr{R}$ under any reflection fixing this circle.*

Proof. The reguli $\mathscr{R}$ and $\mathscr{R}'$ lie on a doubly-ruled quadric $\mathscr{Q}$, the equation of which is

$$(4.1)\qquad \sum_{i,j=1}^{4} \alpha_{ij}\zeta_i\zeta_j = 0$$

where $\alpha_{ij} = \alpha_{ji}$ for all $i, j = 1, 2, 3, 4$ [**5**, p. 259]. Since $\mathscr{R}$ has no contact with l, no point of l lies on $\mathscr{Q}$. Since the points of l are given by the coordinates $(0, 0, \tau_1, \tau_2)$ ($\tau_1, \tau_2 \in F$, τ_1, τ_2 not both $= 0$), substitution in (4.1) yields the necessary and sufficient condition that l does not meet $\mathscr{Q}$, namely

$$(4.2)\qquad \alpha_{34}^2 - \alpha_{33}\alpha_{44} \quad \text{is a non-square in } F.$$

In the matrix notation of Section 2, let $u : Y = XM$ be any line of $\mathscr{R}$ or of $\mathscr{R}'$. Thus u is made up of points (X, XM) where $X = (\sigma_1\,\sigma_2)$ and $X \neq (0\,0)$. Substitution in (4.1) yields the equation

$$(4.3)\qquad \varphi\sigma_1^2 + \theta\sigma_1\sigma_2 + \gamma\sigma_2^2 = 0$$

where φ, θ, and γ are (quadratic) functions of μ_1, μ_2, μ_3, and μ_4, the entries of M. Because (4.3) holds for *all* values of σ_1, σ_2 in F, $\varphi = \gamma = \theta = 0$. Successive steps of substitution of the particular points $(1, 0, \mu_1, \mu_3)$, $(0, 1, \mu_2, \mu_4)$ and $(1, 1, \mu_1 + \mu_2, \mu_3 + \mu_4)$ of u in (4.1) gives the explicit expressions for φ, γ, and θ, namely

$$(4.4)\qquad \begin{aligned} \varphi &= \alpha_{33}\mu_1^2 + 2\alpha_{34}\mu_1\mu_3 + \alpha_{44}\mu_3^2 + 2\alpha_{13}\mu_1 + 2\alpha_{14}\mu_3 + \alpha_{11} = 0 \\ \gamma &= \alpha_{33}\mu_2^2 + 2\alpha_{34}\mu_2\mu_4 + \alpha_{44}\mu_4^2 + 2\alpha_{23}\mu_2 + 2\alpha_{24}\mu_4 + \alpha_{22} = 0 \\ \theta &= 2\alpha_{33}\mu_1\mu_2 + 2\alpha_{34}(\mu_1\mu_4 + \mu_2\mu_3) + 2\alpha_{44}\mu_3\mu_4 + 2\alpha_{23}\mu_1 + 2\alpha_{13}\mu_2 \\ &\qquad + 2\alpha_{24}\mu_3 + 2\alpha_{14}\mu_4 + 2\alpha_{12} = 0 \end{aligned}$$

Consider the equation

$$\varphi + \nu\gamma + t\theta = 0.$$

Substituting the values of φ, γ, and θ given in (4.4), we obtain

$$(4.5)\qquad \alpha_{33}x^2 + 2\alpha_{34}xy + \alpha_{44}y^2 + 2gx + 2fy + c = 0,$$

where $x = \mu_1 + t\mu_2$, $y = \mu_3 + t\mu_4$, $g = \alpha_{13} + t\alpha_{23}$, $f = \alpha_{14} + t\alpha_{24}$, and $c = (\alpha_{11} + \nu\alpha_{22}) + 2t\alpha_{12}$. Equation (4.5) defines a conic $\mathscr{C}$ in π which contains the indicator points of $\mathscr{R}$ and $\mathscr{R}'$.

Solution of the equation $\alpha_{33}x^2 + 2\alpha_{34}xy + \alpha_{44}y^2 = 0$ establishes that $\mathscr{C}$ contains the infinite points $(1, m, 0)$ and $(1, m^q, 0)$ where, by (4.2), $m \in K - F$. By Lemma 4, the group Λ of affinities is transitive on the ordered pairs of slopes m, m^q $(m \in K - F)$. Consequently, we lose no generality here in assuming that $m = t$, and hence $m^q = t^q = -t$. Then $\alpha_{34} = 0$ and we may take $\alpha_{44} = 1$, so that $\alpha_{33} = -\nu$. Equation (4.5) then becomes

$$(4.6) \quad -\nu x^2 + y^2 + 2gx + 2fy + c = 0.$$

Re-arranged, this is

$$-\nu(x - g/\nu)^2 + (y + f)^2 = -g^2/\nu + f^2 - c.$$

Again by Lemma 4, we lose no generality in further assuming that $g = f = 0$, so that equation (4.6) becomes

$$(4.7) \quad -\nu x^2 + y^2 = r$$

$(r = -c)$, which is the equation of the circle C_r previously defined. Thus the indicator points $\mathscr{R}$ (and $\mathscr{R}'$) lie on the circle C_r, establishing part of Theorem 2. To complete the proof, let P_0 be any indicator point of $\mathscr{R}$, and consider the $q + 1$ reflections R' where $R' = \begin{bmatrix} \alpha & -\nu\beta \\ \beta & -\alpha \end{bmatrix}$ $(\alpha^2 - \nu\beta^2 = 1)$. Let $\{Q_0, Q_1, \ldots, Q_q\}$ be the images of P_0 under these reflections, and assume for the moment that $P_0 \neq Q_j$ for all $j = 0, 1, \ldots, q$. By Lemma 6, P_0 and Q_j are incompatible. Consider also the points $\{P_0, P_1, \ldots, P_q\}$, where P_i is the image of P_0 under a rotation $R = \begin{bmatrix} \alpha & \nu\beta \\ \beta & \alpha \end{bmatrix}$ $(\alpha^2 - \nu\beta^2 = 1)$. Since any rotation is the product $R'R_1'$ of two reflections, where the choice of the first reflection R' is arbitrary, the set $\{P_0, P_1, \ldots, P_q\}$ is the set of images of any point Q_j under the reflections. It follows that $\{Q_0, Q_1, \ldots, Q_q\}$ are the images of Q_0 under the rotations, and that P_i and Q_j $(i, j = 0, 1, \ldots, q)$ are incompatible. Moreover, the only points on C_r which are incompatible with P_i are the points $Q_0, Q_1, \ldots, Q_q$, since these are the only points of C_r which are on lines through P_i with slope in $\{\infty\} \cup F$. Therefore, since $P_0 \in \mathscr{R}$, and every point of $\mathscr{R}'$ is incompatible with P_0, $\{Q_0, Q_1, \ldots, Q_q\} = \mathscr{R}'$. Similarly, $\{P_0, P_1, \ldots, P_q\} = \mathscr{R}$.

In reaching this conclusion, we assumed that $P_0 \neq Q_j$ for all $j = 0, 1, \ldots, q$. If this were not true, then $P_0 = Q_0$, say, and there are only q points on C_r, namely $Q_1, \ldots, Q_q$, which are incompatible with P_0, contradicting the fact that $\mathscr{R}'$ is on C_r and $|\mathscr{R}'| = q + 1$. Therefore the assumption is valid and the proof of Theorem 2 is complete.

In the proof of Theorem 2 we have just noted that no indicator point of $\mathscr{R}$ can be fixed by a reflection. In other words, an indicator point P of $\mathscr{R}$ cannot lie on the axis of a reflection. Therefore P cannot lie on any of the lines $x = 0$ or $y = \lambda x$ $(\lambda \in F)$. On the other hand, given any point P not on any of these lines, application of rotations and reflections as in the proof of Theorem 2 yields the converse of Theorem 2:

THEOREM 2′. *Let H be a group of order $q + 1$ consisting of rotations in π relative to fixed slopes m and m^q in $K - F$. Let $\mathscr{R}$ be an orbit of H consisting of mutually compatible points. Then $\mathscr{R}$ indicates a regulus $\mathscr{R}$ in Σ which has no contact with l.*

We conclude therefore that any regulus in Σ having no contact with l is indicated in π by an explicitly specified subset of points on a circle, which also contains the indicator points of the opposite regulus. Both the regulus and its opposite regulus therefore are related to a certain point, namely the centre of the circle on which they both lie, which we call also the *centre* of the regulus.

We noted in Section 3 that circles, as defined by rotations with a given centre and relative to given slopes m, m^q, divide into three mutually exclusive classes. One class consists of the single (degenerate) circle comprised of two lines with slopes m and m^q intersecting in the centre. By Lemmas 4 and 5, we lose no generality in restricting consideration to the case $m = t$, with the centre $(0, 0)$. Looking therefore at reguli indicated on the circle $-\nu x^2 + y^2 = 0$, and applying Theorems 2 and 2′, we see that $y = xt$ contains the set $\mathscr{R} = \{((\alpha + \beta t)x, (\nu\beta + \alpha t)x)\}$ $(\alpha^2 - \nu\beta^2 = 1)$ of points indicating a regulus $\mathscr{R}$ of Σ having no contact with l, and we note in passing that the opposite regulus, $\mathscr{R}'$, is indicated by the points $\mathscr{R}' : \{((\alpha - \beta t)x, (\nu\beta - \alpha t)x)\}$ on $y = -xt$.

Now $[(\alpha + \beta t)x]^{q+1} = (\alpha + \beta t)^{q+1}x^{q+1} = (\alpha^2 - \nu\beta^2)x^{q+1} = x^{q+1}$. The element x^{q+1} is a non-zero element γ of F, called the ***norm*** of x [**2**, p. 427, p. 508]. There are exactly $q + 1$ elements $z \in K$ such that $z^{q+1} = \gamma$; therefore $\mathscr{R}$ can be characterized as the set of points $\{(z, zt)\}$ on $y = xt$ such that $z^{q+1} = \gamma$. Looked at in a slightly different way, $\mathscr{R}$ is the set of points $\{(ux, uxt)\}$ where $x^{q+1} = \gamma$ and u is any element of K such that $u^{q+1} = 1$; in other words $\mathscr{R}$ is the set of images of the point (x, xt) under the group of dilatations $(x, y) \to (ux, uy)(u^{q+1} = 1)$. This point of view is helpful in that it is independent of the restriction of the slope pair m, m^q to t, $-t$, and is therefore descriptive of reguli on any line $y = xm$ $(m \in K - F)$ which have centre $(0, 0)$.

To generalize, again invoking Lemmas 4 and 5, the reguli with centre $(a, am + b)$ on $y = xm + b$ may be analysed by applying an appropriate element of Λ. The result is

THEOREM 3. *Let $\mathscr{R}$ be a regulus of Σ which has no contact with l, and which is indicated on a line $y = xm + b$. Then $\mathscr{R}$ is indicated by the orbit of a point under the group of dilatations whose centre is $(a, am + b)$ for some a, and whose order is $q + 1$. Also, $\mathscr{R}'$ is indicated by the orbit of a point on the line $y = xm^q + (m - m^q)a + b$, under the same dilatations.*

In the case of greatest interest to us, we can describe $\mathscr{R}$, the set of indicator points of a regulus with centre $(0, 0)$ on $y = xm$, as the set of points $\{(z, zm)\}$ where $z^{q+1} = \gamma$. In addition to calling γ the norm of z, we call it the norm of $\mathscr{R}$, and note that, for centre (0, 0), $\mathscr{R}$ can be characterized by its norm and the slope m of the line $y = xm$ on which it lies. There are therefore $q - 1$ different reguli so indicated on any line $y = xm$ ($m \in K - F$).

5. *kd*-spreads and *md*-spreads. We define a *kd-spread* to be a spread in $\Sigma = PG(3, q)$ that contains a set of k disjoint reguli. k, of course, is some non-negative integer. Since a spread in Σ contains $q^2 + 1$ lines, and since each regulus contains $q + 1$ lines, it follows that $k < q$. The maximal case $k = q - 1$ is particularly interesting. For ease of reference we shall call such a spread an *md-spread*, that is, an *md*-spread is a $(q - 1)d$-spread.

Examples of *kd*-spreads abound. Any subregular spread [**2**, p. 442] is a *kd*-spread; however, a *kd*-spread is not necessarily subregular as our examples in Section 7 will show. A regular spread (which can be defined as a spread which contains with any three of its lines the whole regulus determined by those three lines [**2**, p. 436]) contains a set $\{\mathscr{R}_i\}$ ($i = 1, \ldots, q - 1$) of disjoint reguli, called a *complete linear set* [**2**, p. 442]. Replacement of any s reguli $\mathscr{R}_i$ ($i = 1, \ldots, s$, $0 < s < q - 1$) by their opposite reguli $\mathscr{R}_i'$ produces a subregular spread *of index s*, which is the spread associated with an André plane of order q^2 with kernel of order q [**2**, p. 442; **7**, p. 33; **8**, p. 205].

The above remark concerning replacement bears emphasizing in the general case of *kd*-spreads. Any *kd*-spread $\mathscr{S}$ with disjoint reguli $\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_k$ gives rise to a set of *kd*-spreads which includes $\mathscr{S}$ and spreads obtained from $\mathscr{S}$ by replacing each regulus $\mathscr{R}_i$ in any subset of the disjoint reguli by its opposite regulus $\mathscr{R}_i'$. We may refer to any two spreads in such a class as being *related by replacement* [**6**, p. 225].

6. A new class of *md*-spreads. Let $\mathscr{S}$ be an *md*-spread in Σ. Then besides the $q - 1$ disjoint reguli, $\mathscr{S}$ contains two other lines, say l and a. If we let l be the line $\sigma_1 = \sigma_2 = 0$, as in previous sections, then $\mathscr{S}$ is indicated in the plane π by an indicator set $\mathscr{I}$ consisting of a single point (indicating a) and $q - 1$ reguli. The points of each regulus in $\mathscr{I}$ lie on a circle relative to a fixed slope pair m, m^q, and form an orbit under the rotation group fixing the circle (Theorem 2). Here, as suggested earlier, we are using the term "regulus" to denote the set of $q + 1$ points in π which indicate a regulus in Σ.

The indicator set $\mathscr{I}$ of $\mathscr{S}$ is equivalent under translations in the group Λ of affinities in π to a set in which the line a is indicated by the point (0, 0). We therefore assume without loss of generality that a is indicated by (0, 0). We may further assume if desirable that $\mathscr{I}$ contains another specified point, such as $(1, t)$, and lose no generality in doing so (Lemma 4), but for the present we shall leave this option open.

It would be very desirable to classify all *md*-spreads in $\Sigma = PG(3, q)$, and it may be that this can be done by considering their indicator sets. Our aim here is more modest, however. It is to use the indicator set approach to produce a particular *md*-spread $\mathscr{S}_m$ in Σ which belongs to a class of which only a few cases appear to have been previously known, and which has a number of interesting properties, The indicator set $\mathscr{I}_m$ of $\mathscr{S}_m$ will consist of $(0, 0)$ and $q - 1$ reguli, all with centre $(0, 0)$ and distributed on lines $y = xm$ for $(q + 3)/2$ distinct values of m.

From now on, whenever we refer to a regulus without specifying its centre or the circle on which it lies, it is to be understood that this regulus has centre $(0, 0)$ and lies on a line $y = xk$ $(k \in K - F)$.

LEMMA. 7. *Let $\mathscr{R}$ be a regulus* $\{(x, xk)|x^{q+1} = \delta\}$. *Then $\mathscr{R}'$, the opposite regulus to $\mathscr{R}$, is the regulus* $\{(x, xk^q)|x^{q+1} = \delta\}$.

Proof. Consider the regulus $\{(z, zk^q)|z^{q+1} = \delta\}$. If $z = x$, then the line $(x, xk)(x, xk^q)$ has slope ∞. If $z \neq x$, then the line $(x, xk)(z, zk^q)$ has slope $(zk^q - xk)/(z - x) = [(zk^q - xk)/(z - x)]^q$, which is therefore in F. Thus every point of this regulus is incompatible with every point of $\mathscr{R}$; the regulus is therefore $\mathscr{R}'$.

LEMMA 8. *Let $\mathscr{R}$ and $\bar{\mathscr{R}}$ be any two reguli on different lines $y = xk$, $y = x\bar{k}$. Denote the points of $\mathscr{R}$ by (ux_0, ux_0k), where $u^{q+1} = 1$, and those of $\bar{\mathscr{R}}$ by $(u\bar{x}_0, u\bar{x}_0\bar{k})$. If the line $(x_0, x_0k)(\bar{x}_0, \bar{x}_0\bar{k})$ has slope $s \in \{\infty\} \cup K$, then for each value of u, the line $(ux_0, ux_0k)(u\bar{x}_0, u\bar{x}_0\bar{k})$ has slope s.*

Proof. The dilatation $(x, y) \rightarrow (ux, uy)$, which preserves slopes, takes line $(x_0, x_0k)(\bar{x}_0, \bar{x}_0\bar{k})$ into line $(ux_0, ux_0k)(u\bar{x}_0, u\bar{x}_0\bar{k})$.

If follows from Lemma 8 that there are exactly $q + 1$ slopes associated with any two reguli $\mathscr{R}$ and $\bar{\mathscr{R}}$ on different lines, namely the slopes of the lines joining any one point of $\mathscr{R}$ to each point of $\bar{\mathscr{R}}$. We say that $\mathscr{R}$ and $\bar{\mathscr{R}}$ are *connected* by any one of these slopes, and that $\mathscr{R}$ and $\bar{\mathscr{R}}$ are *compatible* if all of their connecting slopes are in $K - F$; otherwise $\mathscr{R}$ and $\bar{\mathscr{R}}$ are *incompatible.* Note that $\mathscr{R}$ and $\bar{\mathscr{R}}$ are compatible if and only if every point of $\mathscr{R}$ is compatible with every point of $\bar{\mathscr{R}}$. Thus the earlier definition of compatibility of points agrees with the definition of compatibility of reguli. Two reguli on the same line are compatible if and only if the line has slope in $K - F$. The disjoint reguli in the indicator set of any md-spread must of course be mutually compatible.

Notation. Let $\mathscr{R}_\gamma$ denote the regulus on $y = xt$ with norm γ, and let $\mathscr{R}$ denote the regulus on $y = xm(m \neq t, -t)$ with norm σ. Thus $\mathscr{R}_\gamma$ is the set of points $\{(x, xt)|x^{q+1} = \gamma\}$ and $\mathscr{R}$ is the set $\{(x, xm)|x^{q+1} = \sigma\}$.

LEMMA 9. *$\mathscr{R}$ is incompatible with $\mathscr{R}_\sigma$ and with $\mathscr{R}_\gamma$, where*

$$\gamma = [\lambda^2 - (m + m^q)\lambda + m^{q+1}]\sigma/(\lambda^2 - \nu) \quad (\lambda \in F).$$

$\mathscr{R}$ and $\mathscr{R}_\gamma$ are connected by the slope λ.

Proof. Since $\mathscr{R}$ has norm σ, $\mathscr{R}$ and $\mathscr{R}_\sigma$ are incompatible, being connected by the slope ∞. If $\mathscr{R}$ and $\mathscr{R}_\gamma$ $(\gamma \neq \sigma)$ are incompatible, there is a slope $\lambda \in F$

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connecting $\mathscr{R}$ and $\mathscr{R}_\gamma$. Thus, given any point (x, xm) $(x^{q+1} = \sigma)$ in $\mathscr{R}$, there is a point (y, yt) $(y^{q+1} = \gamma)$ in $\mathscr{R}_\gamma$ such that $\lambda = (yt - xm)/(y - x)$. Hence $y = (\lambda - m)x/(\lambda - t)$. Therefore

$$\begin{aligned}\gamma = y^{q+1} = y^q y &= [(\lambda - m)^q x^q/(\lambda - t)^q][(\lambda - m)x/(\lambda - t)]\\ &= (\lambda - m^q)(\lambda - m)x^{q+1}/(\lambda - t^q)(\lambda - t)\\ &= [\lambda^2 - (m + m^q)\lambda + m^{q+1}]\sigma/(\lambda^2 - \nu).\end{aligned}$$

LEMMA 10. *If $\mathscr{R}$ and $\mathscr{R}_\gamma$ are incompatible, then they are connected by at most two slopes from $\{\infty\} \cup F$.*

Proof. If $\mathscr{R}$ and $\mathscr{R}_\gamma$ are connected by three or more slopes from $\{\infty\} \cup F$, then any point of $\mathscr{R}$ is incompatible with at least three points of $\mathscr{R}_\gamma$. In terms of the reguli of Σ which $\mathscr{R}$ and $\mathscr{R}_\gamma$ indicate, this means that each line of $\mathscr{R}$ is a transversal of $\mathscr{R}_\gamma$, i.e., $\mathscr{R}_\gamma = \mathscr{R}'$, the opposite regulus to $\mathscr{R}$, which in view of Lemma 7 is a contradiction.

LEMMA 11. (i). *$\mathscr{R}$ and $\mathscr{R}_\sigma$ are connected by a slope $\lambda \in F$ as well as slope ∞ if and only if $m + m^q \neq 0$. In this case, the slope λ is $(m^{q+1} + \nu)/(m + m^q)$.*

(ii). *The slopes λ and $\mu \neq \lambda$ in F both connect $\mathscr{R}$ with the same incompatible regulus $\mathscr{R}_\gamma$ if and only if*

$$(5.1)\qquad (m + m^q)\lambda\mu - (m^{q+1} + \nu)(\lambda + \mu) + \nu(m + m^q) = 0.$$

Proof. (i) If slope $\lambda \in F$ connects $\mathscr{R}$ and $\mathscr{R}_\sigma$, then, by Lemma 9,

$$\begin{aligned}\sigma &= [\lambda^2 - (m + m^q)\lambda + m^{q+1}]\sigma/(\lambda^2 - \nu),\\ \lambda^2 - \nu &= \lambda^2 - (m + m^q)\lambda + m^{q+1},\\ (m + m^q)\lambda &= m^{q+1} + \nu.\end{aligned}$$

Now $m + m^q = m^{q+1} + \nu = 0$ implies that $m = t$ or $-t$, which is false. hence λ exists if and only if $m + m^q \neq 0$.

(ii). By Lemma 9, slopes λ and μ connect $\mathscr{R}$ with $\mathscr{R}_\gamma$ if and only if

$$\begin{aligned}&[\lambda^2 - (m + m^q)\lambda + m^{q+1}]\sigma/(\lambda^2 - \nu) = \gamma\\ &\quad = [\mu^2 - (m + m^q)\mu + m^{q+1}]\sigma/(\mu^2 - \nu),\\ &(\mu^2 - \nu)[\lambda^2 - (m + m^q)\lambda + m^{q+1}] = (\lambda^2 - \nu)[\mu^2 - (m + m^q)\mu + m^{q+1}],\\ &\qquad [(m + m^q)\lambda\mu - (m^{q+1} + \nu)(\lambda + \mu) + \nu(m + m^q)](\lambda - \mu) = 0,\end{aligned}$$

from which (5.1) follows.

Definition. Two incompatible reguli are *doubly connected* if they are connected by two distinct slopes from $\{\infty\} \cup F$. If they are connected by only one slope from $\{\infty\} \cup F$, they are *singly connected.*

LEMMA 12. *Suppose $\mathscr{R}$ and $\mathscr{R}_\gamma$ are incompatible and singly connected. Then $\Delta = (m^{q+1} + \nu)^2 - \nu(m + m^q)^2$ is a square in F. If Δ is a square in F, then there are exactly two reguli $\mathscr{R}_\gamma$ on $y = xt$ which are singly connected with $\mathscr{R}$.*

Proof. If $\mathscr{R}$ and $\mathscr{R}_\sigma$ are singly connected, then, by Lemma 11 (i), $m + m^q = 0$, so $\Delta = (m^{q+1} + \nu)^2$ is a square in F. If $\mathscr{R}$ and $\mathscr{R}_\gamma$ $(\gamma \neq \sigma)$ are singly connected, let $\lambda \in F$ be the slope connecting $\mathscr{R}$ and $\mathscr{R}_\gamma$. By Lemma 11 (ii), the equation (5.1) has no solution μ distinct from λ. Re-arranged, (5.1) is

$$(5.2) \quad [(m + m^q)\lambda - (m^{q+1} + \nu)]\mu = (m^{q+1} + \nu)\lambda - \nu(m + m^q).$$

Now $m + m^q = m^{q+1} + \nu = 0$ implies that $m = t$ or $-t$, contrary to the choice of $\mathscr{R}$. Hence if $(m + m^q)\lambda - (m^{q+1} + \nu) = 0$ in (5.2), then also $(m^{q+1} + \nu)\lambda - \nu(m + m^q) = 0$, implying $(m + m^q)(\lambda^2 - \nu) = 0$. Since ν is a non-square in F, we must have $m + m^q = 0$. But then $m^{q+1} + \nu = 0$ also, which possibility we have just eliminated. Therefore $(m + m^q)\lambda - (m^{q+1} + \nu) \neq 0$, and so $\mu = [(m^{q+1} + \nu)\lambda - \nu(m + m^q)]/[(m + m^q)\lambda - (m^{q+1} + \nu)]$ is a solution of (5.1). We require $\mu = \lambda$. Thus (5.1) becomes a quadratic equation in λ which is reducible in F. The necessary and sufficient condition for this is that the discriminant, $4(m^{q+1} + \nu)^2 - 4\nu(m + m^q)^2 = 4\Delta$ be a square in F.

Suppose now that Δ is a square in F. If $m + m^q = 0$, then by Lemma 11 (i), $\mathscr{R}$ and $\mathscr{R}_\sigma$ are singly connected, namely by the slope ∞. Equation (5.1) reduces to

$$(m^{q+1} + \nu)(\lambda + \mu) = 0.$$

Since $m + m^q$ and $m^{q+1} + \nu$ are not simultaneously $= 0$, $\mu = -\lambda$, yielding $\mu \neq \lambda$ except when $\lambda = 0$. Therefore in the case that $m + m^q = 0$, any regulus $\mathscr{R}_\gamma$ which is connected with $\mathscr{R}$ by slope λ is also connected with $\mathscr{R}$ by slope $-\lambda$, and Lemma 12 follows in this case. On the other hand, if $m + m^q \neq 0$, $\mathscr{R}$ and $\mathscr{R}_\sigma$ are doubly connected, by Lemma 11 (i). By Lemma 11 (ii), $\mathscr{R}$ and $\mathscr{R}_\gamma$ $(\gamma \neq \sigma)$ are connected by slopes λ and $\mu \neq \lambda$ in F if and only if equation (5.1) holds. Now equation (5.1) has two distinct solutions λ and $\mu \neq \lambda$ except when

$$(5.3) \quad (m + m^q)\lambda^2 - 2(m^{q+1} + \nu)\lambda + \nu(m + m^q) = 0.$$

Since $m + m^q \neq 0$ and the discriminant of (5.3), 4Δ, is a square $\neq 0$ in F, (5.3) has two distinct solutions λ_1, λ_2. Hence there are two reguli, $\mathscr{R}_{\gamma_i}$ $(i = 1, 2)$ which are connected with $\mathscr{R}$ by exactly one slope, namely λ_i, and the proof of Lemma 12 is complete.

Notation. $\mathscr{T} = \mathscr{T}(\mathscr{R})$ denotes the set of reguli $\mathscr{R}_\gamma$ on $y = xt$ that are incompatible with $\mathscr{R}$.

Since $|\{\infty\} \cup F| = q + 1$, Lemma 12 has the following immediate corollary:

COROLLARY 1.

$$|\mathscr{T}| = \begin{cases} (q+1)/2 & \text{if } \Delta \text{ is a non-square in } F, \\ (q+3)/2 & \text{if } \Delta \text{ is a square in } F. \end{cases}$$

LEMMA 13. $\Delta = (m^2 - \nu)^{q+1}$, *and* Δ *is a square in* F *if and only if* $m^2 - \nu$ *is a square in* K.

Proof.

$$\begin{aligned}(m^2 - \nu)^{q+1} &= (m^2 - \nu)(m^2 - \nu)^q = (m^2 - t^2)(m^{2q} - t^2)\\ &= (m - t)(m^q - t)(m + t)(m^q + t)\\ &= [m^{q+1} + \nu - t(m + m^q)][m^{q+1} + \nu + t(m + m^q)]\\ &= (m^{q+1} + \nu)^2 - \nu(m + m^q)^2 = \Delta.\end{aligned}$$

Now $m^2 - \nu = e^i$, where e is a primitive root in K, and therefore $\Delta = e^{i(q+1)}$. Δ is a square in F if and only if i is even, and i is even if and only if $m^2 - \nu$ is a square in K.

Let H be the cyclic subgroup of order $q + 1$ of the affinities of π which are the rotations $\begin{bmatrix}\alpha & \nu\beta\\ \beta & \alpha\end{bmatrix}$, and let R be a generator of that group. Let D be the (cyclic) group or order $q + 1$ of dilatations $(x, y) \to (ux, uy)$ where $u^{q+1} = 1$. $H \cap D$ consists of the identity and the half-turn $(x, y) \to (-x, -y)$, and therefore $G = HD$ is an Abelian group of order $(q + 1)^2/2$.

LEMMA 14. *G fixes each regulus $\mathscr{R}_\gamma$ on $y = xt$ and moves the regulus $\mathscr{R}$ in an orbit of length $(q + 1)/2$.*

Proof. $G = HD$. It follows from Theorem 2 that H fixes $\mathscr{R}_\gamma$, and from Theorem 3 that D also fixes $\mathscr{R}_\gamma$. That D fixes $\mathscr{R}$ is also a consequence of Theorem 3. Now the rotation R moves the line $y = xm$ ($m \neq t$ or $-t$) in an orbit of length $(q + 1)/2$, the half-turn $R^{(q+1)/2}$ fixing each line of the orbit. Also, $R^{(q+1)/2} \in D$, and therefore it fixes the regulus $\mathscr{R}$.

We now wish to examine the compatibility properties of the reguli in the orbit of reguli under H to which $\mathscr{R}$ belongs. To that end, we denote by $\mathscr{R}^i$ the regulus which is the image of $\mathscr{R}$ under the rotation R^i ($i = 1, \ldots, (q + 1)/2$). By Lemma 14, $\mathscr{R}^{(q+1)/2} = \mathscr{R}$.

LEMMA 15. *Let $\mathscr{T}'$ be the set of reguli $\mathscr{R}_\delta$ with which $\mathscr{R}$ is compatible. Then each $\mathscr{R}^i$ is also compatible with the reguli of $\mathscr{T}'$.*

Proof. $\mathscr{T}'$ is of course the complement of $\mathscr{T} = \mathscr{T}(\mathscr{R})$ on the line $y = xt$. Each regulus $\mathscr{R}_\gamma$ of $\mathscr{T}$ is connected to $\mathscr{R}$ by at least one slope from $\{\infty\} \cup F$. Now R^i preserves $\mathscr{R}_\gamma$, and also preserves the set of slopes $\{\infty\} \cup F$ (Lemma 4). Therefore $\mathscr{R}_\gamma$ is incompatible with $\mathscr{R}^i$ if and only if it is incompatible with $\mathscr{R}$. Hence $\mathscr{T}(\mathscr{R}^i) = \mathscr{T}(\mathscr{R})$, and the result follows.

LEMMA 16. *If $m^2 - \nu$ is a non-square in K, then the reguli*

$$\{\mathscr{R}^i | i = 1, \ldots, (q + 1)/2\}$$

are mutually compatible.

Proof. Clearly it is sufficient to show that $\mathscr{R}$ is compatible with each $\mathscr{R}^i$ ($i \neq (q + 1)/2$). Suppose therefore that $\mathscr{R}$ and $\mathscr{R}^i$ ($i = 1, \ldots, (q + 1)/2$)

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are incompatible. Then $\mathscr{R}$ and $\mathscr{R}^i$ are connected by a slope μ from $\{\infty\} \cup F$. The rotation R^i fixes no slope from $\{\infty\} \cup F$; hence there is a slope λ in $\{\infty\} \cup F$ $\lambda \neq \mu$, such that R^i takes λ into μ. Now let $\mathscr{R}_\gamma$ be the regulus on $y = xt$ which is connected with $\mathscr{R}$ by slope λ. Since R^i fixes $\mathscr{R}_\gamma$ (Lemma 14) and carries $\mathscr{R}$ into $\mathscr{R}^i$, $\mathscr{R}^i$ and $\mathscr{R}_\gamma$ are connected by the slope μ. But $\mathscr{R}$ and $\mathscr{R}^i$ are connected by the slope μ, and so $\mathscr{R}$ and $\mathscr{R}_\gamma$ are connected by the slope μ. Therefore $\mathscr{R}$ and $\mathscr{R}_\gamma$ are doubly connected, by the two slopes λ and $\mu \neq \lambda$, where the rotation R^i carries λ into μ.

In the remainder of the proof we shall ignore the special cases in which either λ or μ is the slope ∞. These are easily handled as special adaptations of the general case, in which λ, μ are both in F. Letting R^i be the rotation $\begin{bmatrix} \alpha & \nu\beta \\ \beta & \alpha \end{bmatrix}(\alpha^2 - \nu\beta^2 = 1)$, and applying R^i to λ, we have that $\mu = (\nu\beta + \alpha\lambda)/(\alpha + \beta\lambda)$. By Lemma 11 (ii), λ and μ satisfy equation (5.1). Substituting the value $\mu = (\nu\beta + \alpha\lambda)/(\alpha + \beta\lambda)$ into (5.1) and collecting coefficients, we obtain the quadratic equation

$$(5.4) \quad [\alpha(m + m^q) - \beta(m^{q+1} + \nu)]\lambda^2 + 2[\nu\beta(m + m^q) - \alpha(m^{q+1} + \nu)]\lambda \\ + \nu[\alpha(m + m^q) - \beta(m^{q+1} + \nu)] = 0$$

in λ. Since λ exists in F, the discriminant of equation (5.4) must be a square in F. This discriminant is

$$4\{[\nu\beta(m + m^q) - \alpha(m^{q+1} + \nu)]^2 - \nu[\alpha(m + m^q) - \beta(^{q+1} + \nu)]^2\} \\ = 4[(\nu^2\beta^2 - \nu\alpha^2)(m + m^q)^2 + (\alpha^2 - \nu\beta^2)(m^{q+1} + \nu)^2] = 4\Delta,$$

since $\alpha^2 - \nu\beta^2 = 1$. Hence Δ must be a square in F, and therefore by Lemma 13, $m^2 - \nu$ is a square in K. Thus if $m^2 - \nu$ is a non-square in K, $\mathscr{R}$ and $\mathscr{R}^i$ are compatible.

THEOREM 4. *Let m be any element of $K - F$ different from t or $-t$ and such that $m^2 - \nu$ is a non-square* (in K). *Let $\mathscr{R}$ be the regulus $\{(x, xm)|x^{q+1} = \sigma\}$, and let $\{\mathscr{R}^i\}$ be the reguli in the orbit of $\mathscr{R}$ determined by the rotation group H. Let $\{\mathscr{R}_\delta\}$ be the set of reguli on $y = xt$ which are compatible with $\mathscr{R}$. Then the set of points*

$$\mathscr{I}_m = \{(0, 0) \cup \{\mathscr{R}_\delta\} \cup \{\mathscr{R}^i\}\}$$

is the indicator set of an md-spread $\mathscr{S}_m$.

Proof. By Lemma 14, the orbit of $\mathscr{R}$ has length $(q + 1)/2$. By Corollary 1, there are $q - 1 - (q + 1)/2 = (q - 3)/2$ reguli in the set $\{\mathscr{R}_\delta\}$. Hence $\mathscr{I}_m$ contains $q - 1$ reguli in addition to the point $(0, 0)$. By Lemmas 15 and 16, these $q - 1$ reguli are mutually compatible. Finally, $(0, 0)$ is compatible with each point of each regulus, since the reguli all lie on lines $y = xk$ with slope $k \in K - F$.

Thus $\mathscr{I}_m$ is the indicator set of an md-spread $\mathscr{S}_m$.

Other new examples of md-spreads can be obtained from $\mathscr{S}_m$ by replacing any reguli in $\{\mathscr{R}_\delta\} \cup \{\mathscr{R}^i\}$ by their opposite reguli, although of course these spreads are all related by replacement. Lemma 7 gives an explicit expression for the opposite reguli used in the replacement.

Note also that the group $G = HD$ is (sharply) transitive on the $(q + 1)^2/2$ indicator points of $\mathscr{I}_m$ not on $y = xt$. In particular, the subgroup H partitions this subset of indicator points into $(q + 1)/2$ orbits of $q + 1$ each; by Theorem 2′ each orbit indicates a regulus. Therefore the $(q + 1)^2/2$ lines of the set $\{\mathscr{R}^i | i = 1, \ldots, (q + 1)/2\}$, in addition to forming reguli $\mathscr{R}^i$ (which are orbits under D) may also be partitioned into reguli $\overline{\mathscr{R}}^j$, say, $(j = 1, \ldots, (q+1)/2)$ with centre $(0, 0)$, which are orbits under H. Thus $\mathscr{I}_m$ may be thought of as the set $\{(0, 0) \cup \{\mathscr{R}_\delta\} \cup \{\overline{\mathscr{R}}^j\}\}$, which again indicates an md-spread. Thus there are two distinct sets of $q - 1$ disjoint reguli in $\mathscr{S}_m$, and in that sense, $\mathscr{S}_m$ is an md-spread in two distinct ways. The reguli $\overline{\mathscr{R}}^j$, when thought of as indicator points, do not of course lie on lines of π. Rather $\overline{\mathscr{R}}^j$ is on the circle $C_{r_j} : -\nu x^2 + y^2 = r_j$, where $r_j = (m^2 - \nu)z_j^2$ $(z_j^{q+1} = \sigma)$. By Theorem 2, $\overline{\mathscr{R}}^{j\prime}$, the opposite regulus to $\overline{\mathscr{R}}^j$, also lies on C_{r_j}.

7. Properties of $\mathscr{S}_m$. We can now easily describe $\mathscr{S}_m$ in terms of its spread set $S = \{M\}$, where, as in Section 2, $\{Y = XM\}$ is the set of lines in Σ which are distinct from l [**6**, p. 220]. The indicator points of $\mathscr{I}_m$ in π are the points $\{(1\ t)M\}$, and are given in Theorem 4 as the set $\{(0, 0) \cup \{\mathscr{R}_\delta\} \cup \{\mathscr{R}^i\}\}$, where δ ranges over the $(q - 3)/2$ non-zero elements of F which are not in the set $T(\sigma) = \{[\lambda^2 - (m + m^q)\lambda + m^{q+1}]\sigma/(\lambda^2 - \nu)\}$ (Lemma 9). Hence $S \ni 0$, the zero matrix, and if we take up the option of including $(1, t)$ in $\mathscr{I}_m$, then $S \ni I$, the identity matrix (the inclusion of $(1, t)$ in $\mathscr{I}_m$ imposes the condition on σ that $1 \notin T(\sigma)$.). $\mathscr{R}_\delta = \{(x, xt) | x^{q+1} = \delta;\ \delta \notin T(\sigma),\ \delta \neq 0\}$; the corresponding elements of S are $\left\{\begin{bmatrix} \delta_1 & \nu\delta_2 \\ \delta_2 & \delta_1 \end{bmatrix} \middle| \delta_1^2 - \nu\delta_2^2 = \delta\right\}$. $\mathscr{R} = \mathscr{R}^0 = \{(x, xm) | x^{q+1} = \sigma\}$; the corresponding elements of S are

$$\left\{\begin{bmatrix} \sigma_1 & \nu\sigma_2 \\ \sigma_2 & \sigma_1 \end{bmatrix}\begin{bmatrix} 1 & \mu_1 \\ 0 & \mu_2 \end{bmatrix} \middle| \sigma_1^2 - \nu\sigma_2^2 = \sigma;\ m = \mu_1 + t\mu_2\right\}.$$

Finally, $\mathscr{R} \xrightarrow{R^i} \mathscr{R}^i$, where $R^i = \begin{bmatrix} \alpha & \nu\beta \\ \beta & \alpha \end{bmatrix}$ $(\alpha^2 - \nu\beta^2 = 1)$, and therefore the elements of S corresponding to $\mathscr{R}^i$ are $\left\{\begin{bmatrix} \sigma_1 & \nu\sigma_2 \\ \sigma_2 & \sigma_1 \end{bmatrix}\begin{bmatrix} 1 & \mu_1 \\ 0 & \mu_2 \end{bmatrix}\begin{bmatrix} \alpha & \nu\beta \\ \beta & \alpha \end{bmatrix}\right\}$.

Summarizing:

The spread set S of $\mathscr{S}_m$ is

$$\left\{0 \cup \left\{\begin{bmatrix} \delta_1 & \nu\delta_2 \\ \delta_2 & \delta_1 \end{bmatrix} \middle| \delta_1^2 - \nu\delta_2^2 = \delta\right\} \cup \left\{\begin{bmatrix} \sigma_1 & \nu\sigma_2 \\ \sigma_2 & \sigma_1 \end{bmatrix}\begin{bmatrix} 1 & \mu_1 \\ 0 & \mu_2 \end{bmatrix}\begin{bmatrix} \alpha & \nu\beta \\ \beta & \alpha \end{bmatrix} \middle| \sigma_1^2 - \nu\sigma_2^2 = \sigma; \alpha^2 - \nu\beta^2 = 1\right\}\right\}$$

where

(i) μ_1, μ_2 *are fixed elements of F such that* $\Delta = (\mu_1^2 - \nu\mu_2^2 + \nu)^2 - 4\nu\mu_1^2$ *is a non-square in F.*

(ii) σ *is a fixed non-zero element of F and*

$$T(\sigma) = \{[\lambda^2 - 2\mu_1\lambda + \mu_1^2 - \nu\mu_2^2]\sigma/(\lambda^2 - \nu)|\lambda \in F\}.$$

(iii) $\delta \in F$, $\delta \neq 0$, *and* $\delta \notin T(\sigma)$. *If* σ *is chosen so that* $1 \notin T(\sigma)$, *then S contains the identity matrix I.*

It is instructive to contrast S with a spread set representation for André subregular spreads [**2**, p. 492; **7**, p. 33], which may be exhibited as the set

$$\left\{0 \cup \left\{\begin{bmatrix} \delta_1 & \nu\delta_2 \\ \delta_2 & \delta_1 \end{bmatrix} \middle| \delta_1^2 - \nu\delta_2^2 = \delta\right\} \cup \left\{\begin{bmatrix} \epsilon_1 & -\nu\epsilon_2 \\ \epsilon_2 & -\epsilon_1 \end{bmatrix} \middle| \epsilon_1^2 - \nu\epsilon_2^2 = \epsilon\right\}\right\}$$

where ϵ and δ are in two disjoint sets whose union is $F - 0$. The contrast suggests, but does not prove, that $\mathscr{S}_m$ is not in general an André subregular spread. It can be shown however, by a comparison of indicator sets, that for every case except $q = 3$, $\mathscr{S}_m$ is not a subregular spread. Therefore in particular $\mathscr{S}_m$ is not an André subregular spread. The proof is straightforward, but rather long.

The spreads $\mathscr{S}_m$ will of course define translation planes. Any plane defined by a particular choice of $\mathscr{S}_m$ is of dimension two over its kernel, and has the property that its $q^2 + 1$ parallel classes of lines is the union $M \cup N_1 \cup N_2$ of three sets M, N_1 and N_2 consisting of 2, $(q - 3)(q + 1)/2$ and $(q + 1)^2/2$ parallel classes respectively. These correspond to the three divisions of the indicator points $\mathscr{I}_m$: the single point $(0, 0)$ (which indicates a parallel class of M, the other arising from l_∞), the $(q - 3)/2$ reguli $\mathscr{R}_\gamma$ on $y = xt$, and the $(q + 1)/2$ reguli $\mathscr{R}^i$. The linear translation complement [**8**, p. 197] of the resulting plane contains an abelian group G of order $(q + 1)^2/2$ which fixes both parallel classes of M. Also G permutes the parallel classes of N_1 in $(q - 3)/2$ orbits, each of length $q + 1$, and is sharply transitive on the parallel classes of N_2. Each of the orbits in N_1 determines a derivable net. Moreover, N_2 can be expressed in two different ways as a union of disjoint derivable nets.

At least three of the planes defined from particular choices of $\mathscr{S}_m$ are known. They are the nearfield plane of order 9, and the irregular nearfield planes of orders 25 and 49. That $\mathscr{S}_m$, when $q = 3$, yields the nearfield plane of order 9 is of course trivial: $\mathscr{S}_m$ yields a non-Desarguesian translation plane, and the nearfield plane is the only non-Desarguesian translation plane of order 9.

When $q = 5$ we can chose $\nu = -2$, $m = 2 - t$, and $\sigma = -1$. Then the spread set S of $\mathscr{S}_m$ is

$$\left\{0 \cup \left\{\begin{bmatrix} \alpha & -2\beta \\ \beta & \alpha \end{bmatrix} \middle| \alpha^2 + 2\beta^2 = 1\right\} \cup \left\{\begin{bmatrix} \sigma_1 & -2\sigma_2 \\ \sigma_2 & \sigma_1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} \alpha & -2\beta \\ \beta & \alpha \end{bmatrix} \middle| {\sigma_1}^2 + 2{\sigma_2}^2 = -1; \alpha^2 + 2\beta^2 = 1\right\}\right\}.$$

Letting $R = \begin{bmatrix} -2 & -2 \\ 1 & -2 \end{bmatrix}$ we have that S is the set $\{0 \cup \{R^i\} \cup \{R^iTR^j\}\}$, where $i, j = 0, \ldots, 5$, and $T = \begin{bmatrix} 2 & -1 \\ 0 & -2 \end{bmatrix}$. Now $R^3 = T^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $(RT)^3 = I$, from which it follows that every element of the group $N = \langle R, T\rangle$, generated by R and T, can be expressed either as R^i or as R^iTR^j $(i, j = 0, \ldots, 5)$. Therefore $N \subset S$. Examining N further, we note that it is equally well generated by T and $TR = \begin{bmatrix} 0 & -2 \\ -2 & -1 \end{bmatrix}$. Transforming N by the matrix $U = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, we have that N^U is generated by $A = U^{-1}TU = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = U^{-1}TRU = \begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix}$. A and B are the generators of the multiplicative group of the irregular nearfield of order 25 as given in [**6**, p. 231] (with an error corrected). Therefore $S = N \cup 0$, and S defines the irregular nearfield plane of order 25 [**6**, p. 220]. Thus for the choices of ν, m and σ taken above, $\mathscr{S}_m$ defines the irregular nearfield plane of order 25.

The situation is a little different when $q = 7$. Here we choose $\nu = -1$, $m = -3 - 2t$, and $\sigma = 3$. Then the spread set S of $\mathscr{S}_m$ is

$$\left\{0 \cup \left\{\begin{bmatrix} \delta_1 & -\delta_2 \\ \delta_2 & \delta_1 \end{bmatrix} \middle| {\delta_1}^2 + {\delta_2}^2 = \pm 1\right\} \cup \left\{\begin{bmatrix} \sigma_1 & -\sigma_2 \\ \sigma_2 & \sigma_1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \middle| {\sigma_1}^2 + {\sigma_2}^2 = 3, \alpha^2 + \beta^2 = 1\right\}\right\}.$$

Letting $R = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$, we have that S is the set

$$\left\{0 \cup \left\{\begin{bmatrix} \delta_1 & -\delta_2 \\ \delta_2 & \delta_1 \end{bmatrix} \middle| {\delta_1}^2 + {\delta_2}^2 = -1\right\} \cup \left\{R^i\right\} \cup \left\{R^iTR^j\right\}\right\},$$

where $i, j = 0, \ldots, 7$ and $T = \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 1 & 3 \end{bmatrix}$. If we reverse the regulus given in S by $\left\{\begin{bmatrix} \delta_1 & -\delta_2 \\ \delta_2 & \delta_1 \end{bmatrix} \middle| {\delta_1}^2 + {\delta_2}^2 = -1\right\}$ we get a spread $\mathscr{S}_m'$, derived from $\mathscr{S}_m$, whose spread set S' is $\{0 \cup \{TR^2TR^i\} \cup$

$\{R^i\} \cup \{R^iTR^j\}\}$. Now $R^4 = T^2 = (RT)^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and it follows that every element of the group $N = \langle R, T\rangle$ can be expressed either as R^i, or as TR^2TR^i, or as R^iTR^j $(i, j = 0, \ldots, 7)$. Therefore $N \subset S$. If we now choose generators T and $T^{-1}R$ of N, and transform by $U = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$, we identify N^U with the multiplicative group of the irregular nearfield of order 49 as given in [**6**, p. 231]. Therefore $S = N \cup 0$ and S defines the irregular nearfield plane of order 49. Thus for the choices of ν, m and σ taken above, the irregular nearfield plane of order 49 is defined by a spread derived from $\mathscr{S}_m$ by the reversal of the regulus $\mathscr{R}_{-1}$.

We examine one more feature of $\mathscr{S}_m$. There are of course many different spreads $\mathscr{S}_m$, corresponding to different choices of m, for any particular value of q. Two spreads $\mathscr{S}_m$ and $\mathscr{S}_{m'}$ $m' \neq m$ are usually not equivalent in the sense that $\mathscr{I}_m$ and $\mathscr{I}_{m'}$ are images under the group Λ of affinities. Nor are the planes defined through $\mathscr{S}_m$ and $\mathscr{S}_{m'}$ necessarily isomorphic. The properties that distinguish $\mathscr{S}_m$ from $\mathscr{S}_{m'}$ (or $\mathscr{I}_m$ from $\mathscr{I}_{m'}$) in the case that they are not equivalent do not appear to be strikingly different, except in certain special cases. One very interesting special case is that in which $q \equiv -1 \pmod 4$, $\nu = -1$, and $m^{q+1} = -1$. For in this case, and only in this case, there is a rotation in the group H of Section 6 that takes slope m into slope m^q, namely the rotation $R^{(q+1)/4} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. This means that the regulus $\mathscr{R}^{(q+1)/4}$ is the set of points $\{(xm, xm^{q+1}) | x^{q+1} = \sigma\} = \{(z, zm^q) | z^{q+1} = -\sigma\}$ on the line $y = xm^q$. Thence, by Lemma 7, $\mathscr{R}'^{(q+1)/4}$, the opposite regulus to $\mathscr{R}^{(q+1)/4}$, is the set of points $\{(z, zm) | z^{q+1} = -\sigma\}$ on $y = xm$, the same line which contains $\mathscr{R}$. Note however that $\mathscr{R}$ and $\mathscr{R}'^{(q+1)/4}$ are different reguli, having norms σ and $-\sigma$ respectively. If similarly we reverse each regulus $\mathscr{R}^k$, where $k = (q + 1)/4, (q + 3)/4, \ldots, (q - 1)/2$, we obtain from $\mathscr{I}_m$ the interesting indicator set $\mathscr{I}_m'$, which has a set of $q - 1$ disjoint reguli arranged in such a way that $(q - 3)/2$ lie on the line $y = xt$, and the remaining $(q + 1)/2$ lie in pairs on $(q + 1)/4$ other lines through $(0, 0)$.

We emphasize that the above case occurs only when $q \equiv -1 \pmod 4$ and $m^{q+1} = \nu = -1$. The case is remarkable because it provides an instance in which $\mathscr{T}(\mathscr{R})$, the set of reguli $\mathscr{R}_\gamma$ on $\underline{y} = tx$ that are incompatible with $\mathscr{R}$, is identical with $\mathscr{T}(\bar{\mathscr{R}})$, where $\mathscr{R}$ and $\bar{\mathscr{R}}$ are different reguli on the same line $y = xm$ ($\bar{\mathscr{R}}$ of course is the regulus $\mathscr{R}'^{(q+1)/4}$). In fact, further examination of $\mathscr{T}(\mathscr{R})$ reveals that the above example is the *only* case in which this phenomenon can occur. The necessary and sufficient condition for this occurance can be shown to be that $\mathscr{T}(\mathscr{R})$ contains $\mathscr{R}_{-\gamma}$ whenever it contains $\mathscr{R}_\gamma$.

The properties of the set $\mathscr{T}(\mathscr{R})$ for arbitrary choice of $m \in K - F$ are likewise interesting. We have worked out several, using the formula given in Lemma 9 as a basic tool. We shall not present these results here; we are convinced however that a thorough analysis of $\mathscr{T}(\mathscr{R})$ is the key to the discovery of more *md*-spreads (if they exist) and possibly to a complete classification of *md*-spreads.

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