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LATTICE ISOMORPHISMS OF MODULAR INVERSE SEMIGROUPS

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1. Introduction

For an inverse semigroup S we will consider the lattice of inverse subsemigroups of S, denoted $L(S)$. A major problem in algebra has been that of finding to what extent an algebra is determined by its lattice of subalgebras. (See, for example, the survey article [**9**]). By a *lattice isomorphism* (L-isomorphism, structural isomorphism, or projectivity) of an inverse semigroup S onto another T we shall mean an isomorphism Φ of $L(S)$ onto $L(T)$. A mapping ϕ from S to T is said to *induce* Φ if $A\Phi = A\phi$ for all A in $L(S)$. We say that S is *strongly determined* by $L(S)$ if every lattice isomorphism of S onto T is induced by an isomorphism of S onto T.

In previous papers, P. R. Jones and the author have investigated the structure of $LF(S)$, the lattice of *full* inverse subsemigroups of S [**2, 3, 4**]. We call S *modular* [*distributive*] if $LF(S)$ is. In [**5**] Jones showed that any simple distributive inverse semigroup S is strongly determined by $L(S)$. Indeed, each lattice isomorphism is induced by a *unique* isomorphism of S. Here we will extend this result and show that any simple *modular* inverse semigroup S is strongly determined by $L(S)$ and is induced by a unique isomorphism.

Throughout this paper we will assume that S and T are inverse semigroups with semilattices of idempotents E_S and E_T respectively, and Φ an *L-isomorphism of* S onto T. For basic properties of inverse semigroups the reader is referred to [**1, 8**]. If $(P, \leqq)$ is a poset, we write $p \,\|\, q$ if p and q are incomparable, and $p \nparallel q$ otherwise. We now state a preliminary result.

Result 1 ([**6**, Theorem 2.1]). *Let S be a simple inverse semigroup. If $L(S) \simeq L(T)$ then T is also simple. Furthermore, $E_S \simeq E_T$ under the mapping $\phi_E\colon E_S \to E_T$ given by $\{e\}\Phi = \{e\phi_E\}$ for all $e \in E_S$.*

2. Simple modular inverse semigroups

We now consider the lattice LF, and basic facts about modular inverse semigroups needed later. An element x of S is called *strictly right* (*left*) *regular* if $xx^{-1} > x^{-1}x\,[x^{-1}x > xx^{-1}]$. Such an element generates a bicyclic subsemigroup. E_S is *Archimedean* in S if for any $e \in E_S$ and strictly right regular element x, $x^{-n}x^n \leqq e$ for some positive integer n. We let $K_S = \ker \sigma$, where σ is the minimum group congruence on S.

Result 2 ([2]). *A simple inverse semigroup S which is not a group is modular if and only if*

(*i*) *S is combinatorial,*
(*ii*) *E_S is Archimedean in S,*
(*iii*) *S/σ is locally cyclic, and*
(*iv*) *the poset of idempotents of each D-class of S is either a chain or contains exactly one pair of incomparable elements, each of which is maximal.*

In the proof that simple distributive inverse semigroups are strongly determined by their lattice isomorphisms, much use was made of the fact that these semigroups are E-unitary, and hence $K_S = E_S$. This is not necessarily true in the modular case, so the kernel of σ, K_S, will be of interest.

Result 3 ([2, Propositions 3.1 and 4.2]). *If S is a simple modular inverse semigroup which is not a group, then $K_S = \{x: xx^{-1} \| x^{-1}x\} \cup E_S = \{x: x^2 = x^3\}$, and for each $e \in E_S$, $|K_S \cap R_e| \leqq 2$.*

If S is a simple modular inverse semigroup which is not a group and Φ a lattice isomorphism of S onto T, then by Result 1, $E_S\Phi = E_T$. Thus Φ restricts naturally to an isomorphism of $LF(S)$ onto $LF(T)$, and therefore T cannot be a group. T must therefore also be a modular inverse semigroup, not a group, which is simple by Result 1, meaning it satisfies the hypotheses of Result 2. Since S is combinatorial, we can now construct a bijection from S onto T.

Result 4 ([6, Proposition 1.6]). *If S is combinatorial, for each x in S there is a unique element y of T such that $\langle x \rangle \Phi = \langle y \rangle$, $(xx^{-1})\phi_E = yy^{-1}$ and $(x^{-1}x)\phi_E = y^{-1}y$.*

We may now define $\phi: S \to T$ by letting $x\phi$ be the unique element y as above. Properties of ϕ are given in the next result.

Result 5 ([6, Proposition 1.7]). *If S is combinatorial then so is T, and ϕ is a one-to-one map of S onto T which extends ϕ_E. Further, ϕ and ϕ^{-1} are R- and L-preserving. If θ is a homorphism of S onto T which induces Φ, then $\theta = \phi$.*

We can now show that ϕ preserves the kernel of σ, K_S, in modular inverse semigroups. *Throughout the remainder of this paper we will assume that S is not a group.*

Theorem 6. *If S is a simple modular inverse semigroup, then $K_S\Phi = K_T$.*

Proof. Take any $u \in K_T$. If $u \in E_T$, then $u \in E_S\Phi \subseteq K_S\Phi$. So assume $u \notin E_T$. Now $u = x\phi$ for some $x \in S$ by Result 5, and from Result 3, $uu^{-1} \| u^{-1}u$. It follows that $(xx^{-1})\phi \| (x^{-1}x)\phi$ by Result 4, and since ϕ is an isomorphism on E_S, $xx^{-1} \| x^{-1}x$. Applying Result 3 again $x \in K_S$ yielding $K_T \subseteq K_S\Phi$.

Similarly, $K_S \subseteq K_T\Phi^{-1}$, and the proof is complete.

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In a simple modular inverse semigroup S, every element x not in K_S is either strictly right or left regular, and therefore generates a bicyclic subsemigroup. Thus properties of $L(B)$ for the bicyclic semigroup B are bound to play an important role.

Result 7 ([5, 6]). *The bicyclic semigroup is strongly determined by its lattice of inverse subsemigroups. In fact every lattice isomorphism of B onto T is induced by a unique isomorphism of B onto T.*

If S is a simple modular inverse semigroup and x is an element not in K_S, then Φ restricted to $\langle x\rangle$ is a lattice isomorphism of $\langle x\rangle$ onto $\langle x\phi\rangle$, which from Results 4, 5 and 7 is induced by ϕ, and ϕ must be the isomorphism from $\langle x\rangle$ onto $\langle x\phi\rangle$. Note that Result 4 shows that an element x is strictly right [left] regular if and only if $x\phi$ is.

The following generalization of Lemma 4.6 of [5] will be useful.

Lemma 8. *Let S be a simple modular inverse semigroup. (i) If a is a strictly right [left] regular element of S, then so is ea for every idempotent $e \leqq aa^{-1}$. (ii) If $a\sigma b$ for $a, b \in s$, then a is strictly right [left] regular if and only if b is.*

Proof. (i) Take any strictly right regular element a of S and any idempotent $e<aa^{-1}$. Then $a\notin K_S$ by Result 3 and so also $ea\notin K_S$. Clearly $eaRe$. Since E_S is Archimedean, $a^{-n}a^n<e$ for some $n>0$, whence $a^{-n}ea^n \leqq a^{-n}a^n<e$. By way of contradiction, assume ea is *left* regular; that is, $a^{-1}ea=(ea)^{-1}(ea)>e$. Then $a^{-2}ea^2=a^{-1}(a^{-1}ea)a>a^{-1}ea>e$, and by induction, $a^{-n}ea^n>e$ which is impossible. Thus ea is strictly right regular. The proof for strictly left regular is similar.

(ii) Take $a, b\in S$ with $a\sigma b$ and suppose a is strictly right regular. Now $a\notin K_S=\ker\sigma$ so also $b\notin K_S$, hence b must be strictly right or left regular. Since $a\sigma b$ there is an $e\in E_S$ such that $ea=eb$, and without loss of generality we may take $e\leqq aa^{-1}$. By (i) above, ea is strictly right regular and thus so is eb. Again using (i), b cannot be strictly left regular, so the desired result follows.

Before proceeding further we will need several technical results.

Result 9 ([2, Proposition 2.9]). *If S is a simple modular inverse semigroup, then S/σ is a torsion-free abelian group.*

Result 10 ([2, Lemma 1.8]). *Let S be an inverse semigroup and $b\in S$, $b\notin E_S$. If x is a nonidempotent in $\langle E_S, b\rangle$ then $x=xx^{-1}b^n$ for some non-zero integer n.*

Lemma 11. *If x and z are strictly right regular elements of a simple modular inverse semigroup S and $\langle x\rangle\sigma=\langle z\rangle\sigma$, then $x\sigma z$.*

Proof. We must have $x\sigma w$ for some $w\in\langle z\rangle\subseteq\langle E_S, Z\rangle$. By Result 10, $w=ww^{-1}z^n$ for some non-zero integer n, so $x\sigma z^n$. Similarly, $z\sigma x^m$ for some $m\neq 0$, and it follows that $x\sigma x^{mn}$. Since $x\notin K_S$,$\langle x\rangle$ is torsion-free by Result 9, so $x\sigma=(x\sigma)^{mn}$ implies $mn=1$, whence $n=\pm 1$. This gives us $x\sigma z^{\pm 1}$, but since z^{-1} is strictly left regular, we cannot have $x\sigma z^{-1}$ by Lemma 8(ii). It now follows that $x\sigma z$.

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Lemma 12. *If S is a simple modular inverse semigroup and $z=fx$ where $x, z \in S \backslash K_S$, then $(\langle z\rangle\Phi)\sigma=(\langle x\rangle\Phi)\sigma$.*

Proof. From Results 4 and 5, $c=z\phi$ and $a=x\phi$ are such that $c, a \in T\backslash K_T$. Now since $z=fx$, $\langle z\rangle \subseteq \langle E_S, x\rangle = E_s \vee \langle x\rangle$, so $\langle c\rangle=\langle z\rangle\Phi \subseteq E_S\Phi \vee \langle x\rangle\Phi = E_T \vee \langle a\rangle = \langle E_T, a\rangle$. Thus by Result 10, $c=cc^{-1}a^n$, for some non-zero n, and hence $c\sigma a^n$. Let $y=(a^n)\phi^{-1} \in \langle a^n\rangle\Phi^{-1} \subseteq \langle x\rangle$. Now $\langle y\rangle \subseteq \langle x\rangle$ and $\langle z\rangle=\langle c\rangle\Phi^{-1} \subseteq \langle E_T, a^n\rangle\Phi^{-1}=E_T\Phi^{-1}=E_T\Phi^{-1} \vee \langle a^n\rangle\Phi^{-1}=\langle E_S, y\rangle$. Applying Result 10 again, $z=zz^{-1}y^m$, for some non-zero m, and so $x\sigma z\sigma y^m$.

Since the mapping ϕ restricted to $\langle x\rangle$ is an isomorphism onto $\langle a\rangle$, $y=(a^n)\phi^{-1}=(a\phi^{-1})^n=x^n$, and combining this with $x\sigma y^m$ gives $x\sigma x^{mn}$. We know that $\langle x\sigma\rangle$ is torsion-free since $x \notin K_S$, so as in the previous proof, $n=\pm 1$. From this we get $a=a^{\pm 1}$, so $\langle a^n\rangle=\langle a\rangle$, and finally $(\langle z\rangle\Phi)\sigma=\langle c\rangle\sigma=\langle a^n\rangle\sigma=\langle a\rangle\sigma=(\langle x\rangle\Phi)\sigma$ as desired.

Notice that if $a\sigma b$ in a simple modular inverse semigroup S, and $a, b \notin K_S$, then $ea=eb \notin K_S$ for some idempotent e. The above lemma shows that $(\langle ea\rangle\Phi)\sigma=(\langle a\rangle\Phi)\sigma=(\langle b\rangle\Phi)\sigma$, or equivalently, $\langle a\phi\rangle\sigma=\langle b\phi\rangle\sigma$. Applying Lemma 11, $(a\phi)\sigma(b\phi)$. Thus we have proved:

Corollary 13. *If S is a simple modular inverse semigroup and $a, b \in S$, then $a\sigma b$ implies $(a\phi)\sigma(b\phi)$.*

We know that ϕ is R-, L- and σ-preserving on a simple modular inverse semigroup S, and that ϕ is an isomorphism on E_S. These results will be crucial in proving our main theorems.

Theorem 14. *If S is is a simple modular inverse semigroup, then ϕ is a homomorphism.*

Proof. Take any $a, b \in S$ and put $f=(a^{-1}a)(bb^{-1})$. Then $(ab)R(af)L(f)R(fb)L(ab)$, and since ϕ is R- and L-preserving.

$$(ab)\phi R(af)\phi L(f\phi)R(fb)\phi L(ab)\phi.$$

By a result of Miller and Clifford [7],

$$(ab)\phi H(af)\phi(fb)\phi,$$

and since T is combinatorial,

$$(ab)\phi=(af)\phi(fb)\phi. \tag{1}$$

But $(fb)\sigma b$, so $(fb)\phi\sigma(b\phi)\sigma(f\phi)(b\phi)$, and since ϕ is an isomorphism on E_S and is R-preserving, $f\phi \leq (bb^{-1})\phi=(b\phi)(b\phi)^{-1}$, so that

$$((fb)\phi, (f\phi)(b\phi)) \in \boldsymbol{R} \cap \sigma. \tag{2}$$

We will show that $(fb)\phi=(f\phi)(b\phi)$. To do this we now consider two cases.

Case I. The element b is in K_S. Then $fb\in K_S, b\phi\in K_T$, and so $(fb)\phi$ and $(f\phi)(b\phi)$ are in K_T. Suppose $(fb)\phi=f\phi$.

Notice that if x is any element of K_S (or K_T), Result 3 tells us that $x^2=x^3$. From this we can see that x^2 is idempotent, for

$$(x^2)^2=x^3x=x^2x=x^3=x^2.$$

Also note that

$$x^2x^{-1}=x^{-2}x^{-1}=x^{-3}=x^3=x^2, \text{ and thus}$$

$$x^{-1}x^2x^{-1}=(x^{-1}x)(xx^{-1})=x^2.$$

Applying this to b and $b\phi$ we have

$$(b^2)\phi=[(b^{-1}b)(bb^{-1})]\phi=(b^{-1}b)\phi(bb^{-1})\phi$$

$$=(b\phi)^{-1}(b\phi)(b\phi)(b\phi)^{-1}=(b\phi)^{-1}(b\phi)^2(b\phi)^{-1}=(b\phi)^2.$$

We previously assumed that $(fb)=f\phi$, and since ϕ is a bijection, $fb=f$, and we can calculate $fb^2=fb=f$. Therefore $(f\phi)(b\phi)=(fb^2)\phi(b\phi)=(f\phi)(b^2\phi)(b\phi)=(f\phi)(b\phi)^2(b\phi)=(f\phi)(b\phi)^3\in E_T$ so that $(f\phi)(b\phi)=(fb)\phi=f\phi$, as required.

If we assume rather that $(f\phi)(b\phi)=f\phi$, then let $e=f\phi$ and $c=b\phi$. Then $(f\phi)(b\phi)=f\phi$ becomes $ec=e$, and (2) is

$$((ec)\phi^{-1},(e\phi^{-1})(c\phi^{-1}))\in R\cap\sigma$$

since ϕ^{-1} is R- and σ-preserving. Thus the same argument as above, applied to ϕ^{-1}, yields $(ec)\phi^{-1}=(e\phi^{-1})(c\phi^{-1})$, in other words $f=fb$ as before. Thus, $(fb)\phi=(f\phi)(b\phi)$ in this case also.

Since, by Result 3 $|K_T\cap R_{f\phi}|\leqq 2$, if neither $(fb)\phi$ nor $(f\phi)(b\phi)=f\phi$, then $(fb)\phi$ and $(f\phi)(b\phi)$ must be equal.

Case II. The element b is *not* in K_s. Let $U=\langle E_S,b\rangle$. Now $U\cap K_S=E_S$, for if x is a nonidempotent in $\langle E_S,b\rangle$, $x=xx^{-1}b^n$ for some non-zero integer n by Result 10. Then the subgroup $\langle x\sigma\rangle=\langle b^n\sigma\rangle$ of S/σ is nontrivial since $b\sigma\neq 1$ and S/σ is torsion-free, and so $x\notin K_S$. From this it follows that U is E-unitary. Recall that $((fb)\phi,(f\phi)(b\phi))\in R\cap\sigma$ (in T), and it easy to verify that also $((fb)\phi,(f\phi)(b\phi))\in R_{U\phi}\cap\sigma_{U\phi}$.

By [6, Theorem 3.4] $U\phi$ is E-unitary. From [8, Proposition III.7.2], that $U\phi$ is E-unitary implies that $R_{U\phi}\cap\sigma_{U\phi}$ is the identical relation on $U\phi$. Hence we get get $(fb)\phi=(f\phi)(b\phi)$, as desired. A similar argument yields $(af)\phi=(a\phi)(f\phi)$.

Now we have that (1) is equivalent to

$$(ab)\phi=(a\phi)(f\phi)(b\phi). \tag{3}$$

But since $f=(a^{-1}a)(bb^{-1})$, $f\phi=(a^{-1}a)\phi(bb^{-1})\phi=(a\phi)^{-1}(a\phi)(b\phi)(b\phi)^{-1}$, and replacing $f\phi$ with this in (3) gives $(ab)\phi=(a\phi)(b\phi)$.

Combining this last theorem with Result 5 we can now state the main result of this paper.

Theorem 15. *Let S be a simple modular inverse semigroup which is not a group. Then each isomorphism of $L(S)$ onto $L(T)$ is induced by a unique isomorphism of S onto T.*

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