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A note on planes of characteristic three

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A projective plane has characteristic three if in every ternary ring coordinatizing it all elements $\neq 0$ of the additive loop have order 3. We show that if a finite plane of characteristic three is coordinatized by a near-field, then the plane is desarguesian.

1. Introduction

Lombardo-Radice [7] has defined a projective plane π to have characteristic m, m a positive integer, if in every ternary ring coordinatizing π all elements of the additive loop except the identity 0 have the same order m. The only known finite planes with characteristic are the desarguesian ones. It has been conjectured that there are no others. This has been proved only of the case $m = 2$ (Gleason [3]).

The case $m = 3$ has been investigated by Keedwell [5, 6], Lombardo-Radice [7, 8], and Zappa [9]. Lombardo-Radice shows that a plane has characteristic 3 if and only if every quadrangle generates a subplane isomorphic to the plane of order 3. Zappa investigates the question of existence of characteristic 3 planes. He proves that if the plane is coordinatized by a near-field of dimension 2 or less over its center, then the plane is desarguesian. He also shows that no Hughes plane has characteristic 3.

Keedwell derives several configuration theorems associated with the idea of characteristic and does a detailed investigation of them.

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In this paper we extend Zappa's first result to all finite near-fields; that is, we prove:

MAIN THEOREM. *If a finite plane* π *of characteristic* 3 *is coordinatized by a near-field, then* π *is desarguesian.*

The idea of the proof is as follows. Every near-field, except the seven exceptional ones, is a generalized André system. Using this fact we are able to prove (Theorem A) that if a near-field satisfies

$$(1+x)x = x + x^2 ,$$

then it is a field. We then show (Theorem B) that this identity is satisfied in a near-field coordinatizing a plane of characteristic 3 .

We assume the reader is completely familiar with the theory of projective planes as given in Chapters 3-5 of Dembowski [1].

2. Proof of the main theorem

A generalized André system [2] is a quasi-field $(Q, + \cdot)$ defined as follows: $(Q, +) = \left(\mathrm{GF}\left(p^r\right), +\right)$ and "$\cdot$" is defined by

(1)
$$m \cdot x = mx^{p^{\lambda(m)}} , \quad m \neq 0 ,$$
$$0 \cdot x = 0 ,$$

where λ is a mapping from the multiplicative group $\mathrm{GF}\left(p^r\right) - \{0\}$ into the additive group Z_r and multiplication on the right in (1) is multiplication in $\mathrm{GF}\left(p^r\right)$. λ also satisfies the two restrictions:

(i) $\lambda(1) = 0$;

(ii) if $x = \omega^i$, $y = \omega^j$ with $i \equiv j \pmod{p^t-1}$ where $t = \text{g.c.d.}\left(r, \lambda(x)-\lambda(y)\right)$ then $x = y$. (Here ω is a fixed primitive element of $\mathrm{GF}\left(p^r\right)$.)

The non-exceptional finite near-fields are all generalized André systems ([2], p. 383). Note that a generalized André system is a near-field if and only if the mapping λ satisfies

(2)
$$\lambda(a \cdot b) = \lambda(a) + \lambda(b)$$

for all $a, b \in GF(p^r) - \{0\}$. Also if $\lambda(a) = 0$ for all $a \neq 0$ then Q is the field $GF(p^r)$.

THEOREM A. *If* $(Q, +, \cdot)$ *is a finite near-field in which*

$$(1+x)x = x + x^2$$

for all $x \in Q$ *, then* $(Q, +, \cdot)$ *is a field.*

Proof. Assume first that $(Q, +, \cdot)$ is one of the exceptional finite near-fields. If u is the element satisfying $u^2 = -1$ (and there is such an element in each of the seven exceptional near-fields - Hall [4]), then $(1+u)u \neq u + u^2$. This follows from an easy calculation using the identities in Hall [4].

Assume now that $(Q, +, \cdot)$ is a non-exceptional near-field. Then $(Q, +, \cdot)$ is a generalized André system in which (2) is satisfied. For all $a, x \in Q - \{0\}$ we have

$$[a\cdot(1+x)]\cdot x = a \cdot [(1+x)\cdot x] ,$$

which becomes

$$\left[a+ax^{p^{\lambda(a)}}\right]x^{p^{\lambda(a)+\lambda(1+x)}} = a\left(x+xx^{p^{\lambda(x)}}\right)^{p^{\lambda(a)}}$$

$$a\left(1+x^{p^{\lambda(a)}}\right)x^{p^{\lambda(a)+\lambda(1+x)}} = a\left(1+x^{p^{\lambda(x)+\lambda(a)}}\right)x^{p^{\lambda(a)}} ,$$

where multiplication is now in $GF(p^r)$. Hence we have

$$\left[1+x^{p^{\lambda(a)}}\right]x^{p^{\lambda(1+x)}} = 1 + x^{p^{\lambda(x)+\lambda(a)}}$$

for all $a, x \in GF(p^r) - \{0\}$.

Assume there exists an $a \in GF(p^r) - \{0\}$ with $\lambda(a) \neq 0$. If $\lambda(x) = 0$ then

$$x^{p^{\lambda(1+x)}} = 1 ,$$

which implies $x = 1$. Since (2) implies λ is a (group) homomorphism from the multiplicative group of Q into the additive group Z_r this last equation implies λ is an isomorphism. Since $|Q-\{0\}| = p^r - 1$ and

$|Z_r| = r$ we must have $p^r - 1 \leq r$ and this occurs only if $p = 2$, $r = 1$. However if $p^r = 2$, $\lambda(a) = 0$ always.

Thus we have shown that $\lambda(a) = 0$ for all $a \in \mathrm{GF}(p^r) - \{0\}$ and this implies Q is $\mathrm{GF}(p^r)$.

THEOREM B (Zappa [9]). *Let* π *be a finite plane of characteristic* 3 *which is coordinatized by a near-field* $(Q, +, \cdot)$ *with respect to the points* U, V, O, I. *For all* $u \in Q$ *we have*

$$(1+u)u = u + u^2 .$$

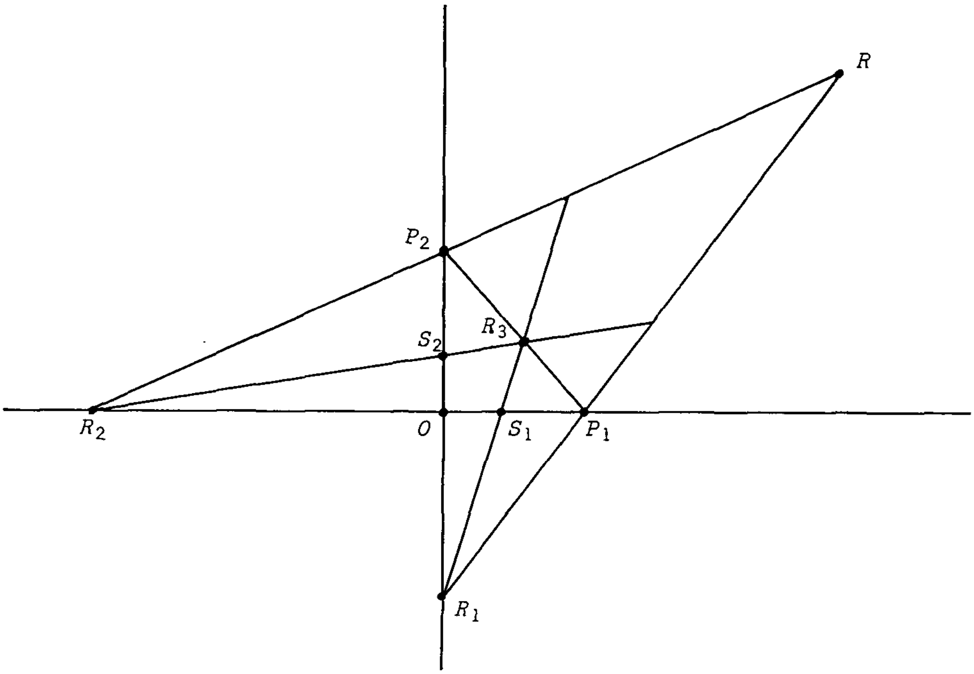
Proof. We remark that for all $y \in Q$, $y + y + y = 0$ or $y + y = -y$. We assume now that Q has dimension $r > 1$ over $\mathrm{GF}(3)$; that is, π has order 3^r, $r > 1$. We need only prove the theorem for $u \neq 0, 1, -1$. Consider the quadrangle with vertices $O = (0, 0)$, $P_1 = (1, 0)$, $P_2 = (0, 1)$, $R = (u, u)$. The points on the lines OP_1, OP_2, OR, P_1P_2, P_1R, P_2R satisfy the following equations

$$OP_1 : y = 0 , \quad OP_2 : x = 0 , \quad OR : y = x ,$$

$$P_1P_2 : y = -x + 1 ,$$

$$P_1R : y = u(u-1)^{-1}x - u(u-1)^{-1} ,$$

$$P_2R : y = (u-1)u^{-1}x + 1 ,$$



Let $R_1 = P_1R \cap OP_2$, $R_2 = OP_1 \cap P_2R$, $R_3 = OR \cap P_1P_2$. Then we have

$$R_1 = \left(0, -u(u-1)^{-1}\right), \quad R_2 = \left(-u(u-1)^{-1}, 0\right),$$

$$R_3 = (-1, -1).$$

The lines R_1R_3 and R_2R_3 have the equations

$$R_1R_3 : y = \left[1-u(u-1)^{-1}\right]x - u(u-1)^{-1},$$

$$R_2R_3 : y = \left[1-u(u-1)^{-1}\right]x + \left[(u-1)u^{-1}-1\right]^{-1}.$$

If $S_1 = OP_1 \cap R_1R_3$ and $S_2 = OP_2 \cap R_2R_3$, then

$$S_1 = \left(\left[(u-1)u^{-1}-1\right]^{-1}, 0\right), \quad S_2 = \left(0, \left[(u-1)u^{-1}-1\right]^{-1}\right).$$

Thus S_1S_2 has the equation

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$$S_1S_2 : y = -x + [(u-1)^{-1}-1]^{-1} .$$

Since O, P_1, P_2, R generate a subplane isomorphic to the plane of order 3, we know the point R lies on the line S_1S_2. This gives

$$u = -u + [(u-1)u^{-1}-1]^{-1} ,$$

which becomes

$$u = [1-(u-1)u^{-1}]^{-1} ,$$

$$u^{-1} = 1 - (u-1)u^{-1} ,$$

$$(u-1)u^{-1} = 1 - u^{-1} .$$

Because $(Q, +, \cdot)$ is a near-field we have

$$u^{-1}(u-1) = 1 - u^{-1} .$$

Thus

$$(u-1)u^{-1} = u^{-1}(u-1) ,$$

which gives

$$u(u-1) = (u-1)u .$$

If $v = u - 1$, then we have

$$(v+1)v = v(v+1) = v^2 + v$$

for all $v \in Q$ with $v \neq 0, 1, -1$. This proves the theorem.

We can now prove the main theorem. By Theorem B the near-field $(Q, +, \cdot)$ satisfies the identity $(1+u)u = u + u^2$ for all $u \in Q$. By Theorem A, $(Q, +, \cdot)$ is a field. Hence π is desarguesian.

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