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Jacquet modules of the Weil representations and families of relative trace identities

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Abstract

In this paper we show how to predict relative trace identities from the computation of Jacquet modules of the Weil representations. Many previously considered special cases of relative trace identities fit the principle we develop here, including those with important applications on L-functions. We also show how to prove these identities using the Weil representation. We give a proof of the relative trace identities between the distributions on $SO(n+1,n)$ and $\widetilde{Sp}(m)$ $(n \geqslant m)$. The proof should serve as a model to the other cases conjectured in the paper.

1. Introduction

Let G and G' be a dual reductive pair (see [How90]). Let ω_ψ be the Weil representation of the group $G \times G'$ (see [Wei64]). Let $H \subset G$ be a subgroup, and χ be a representation of H. Then the space of (H,χ)-covariants of ω_ψ has a G'-action. A natural question is to describe this G'-module. In the first part of this paper, we study this question in several cases. It turns out that this question is closely related to the theory of the relative trace formula. The main goal of the paper is to explore this relation.

The relative trace formula identities have been studied in many papers. They are a tool in the theory of Langlands' functoriality, and recently have found many other applications in number theory. We study here several families of such identities. We arrive at these identities through consideration of covariants of the Weil representation. Included in the families are the generalizations of many cases considered before, including for example the identities conjectured or proved in [Fli93, FM04, FJ96, Jac87, Mao92, MR99a, MR99b, Zin98]. We will give the proof for one family of identities. The proof for other families of identities can be done similarly.

1.1 Definition of a distribution

Let F be a number field, with $\mathbf{A}$ its adele ring. We use v to denote a place of F. Let G be a reductive group.

In studying the relative trace formula, one considers a distribution of the following type: for $f \in \mathcal{S}(G(\mathbf{A}))$ (the space of Schwartz functions on $G(\mathbf{A})$), let

$$I_G(f : H_1, \chi_1, H_2, \chi_2) = \int_{H_1(F)\backslash H_1(\mathbf{A})} \int_{H_2(F)\backslash H_2(\mathbf{A})} K_f(h_1, h_2)\chi_1(h_1)\chi_2(h_2)\, dh_1\, dh_2. \tag{1.1}$$

Here H_1 and H_2 are two closed subgroups of G, χ_i $(i = 1, 2)$ is a character of $H_i(\mathbf{A})$ trivial on

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$H_i(F)$, and $K_f(x,y)$ is the kernel function for the representation $\rho(f)$ acting on $L^2(G(F)\backslash G(\mathbf{A}))$; more explicitly

$$K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

1.2 Relative trace identity

One of the applications of the relative trace formula is in the study of Langlands' functoriality theory. Assume G and G' are two reductive groups such that there is a homomorphism between the L-groups of G and G'. Then Langlands' philosophy predicts that there is a correspondence between the automorphic representations of G and G'. The method of the relative trace formula is to establish a relation between two distributions as defined by (1.1) over the groups G and G'. The spectral decomposition of these distributions will give a correspondence between the automorphic representations of G and G'. For an example of such an application, see [Jac87].

The relation between the two distributions is what we call a *relative trace identity*. Explicitly, we say there is a relative trace identity

$$I_G(f : H_1, \chi_1, H_2, \chi_2) = I_{G'}(f' : H_1', \chi_1', H_2', \chi_2') \tag{1.2}$$

if the following are true.

a) There exists maps $\epsilon_v : \mathcal{S}(G_v) \to \mathcal{S}(G_v')$ for all places v of F.

b) There is a finite set S_0 of bad places, such that for any S a finite set of places containing S_0, we have for any $f = \bigotimes_{v\in S} f_v \bigotimes_{v\notin S} f_v$, with $f_v \in \mathcal{S}(G_v)$ when $v \in S$ and f_v a Hecke function when $v \notin S$, that Equation (1.2) holds for $f' = \epsilon_v(f_v) \bigotimes_{v\notin S} \lambda_v(f_v)$. Here λ_v is the local Hecke algebra homomorphism between G_v and G_v' given by the Satake isomorphism and the homomorphism between the L-groups of G and G'.

We will say that f and f' *match* if the identity (1.2) holds for f and f'.

From the identity (1.2), we can expect (roughly) to get an identity of the following type: For π of G and π' of G' two corresponding cuspidal automorphic representations,

$$\sum_{\{\phi_i\}} P(\pi(f)\phi_i : H_1, \chi_1) P(\bar{\phi_i} : H_2, \chi_2) = \sum_{\{\phi_i'\}} P(\pi'(f')\phi_i' : H_1', \chi_1') P(\bar{\phi_i'} : H_2', \chi_2'), \tag{1.3}$$

where $\{\phi_i\}$ and $\{\phi_i'\}$ are orthonormal bases of the spaces of π and π'; f and f' match, and the notation $P(\phi : H, \chi)$ denotes the period integral

$$P(\phi : H, \chi) = \int_{H(F)\backslash H(\mathbf{A})} \phi(h)\chi(h)\, dh. \tag{1.4}$$

An identity like (1.3) has other applications. For example it is used in [Guo96] to show that $L(\pi, 1/2) \geqslant 0$ for any cuspidal representation π of $PGL(2)$.

1.3 An example

We discuss here a basic example of identity (1.2). Let $G = PGL(2)$ and $G' = \widetilde{SL}(2)$ be the double cover of $SL(2)$. Let H_1 be the subgroup of the diagonal matrices in G, and let $H_1' = H_2' = H_2$ be the group of upper triangular matrices with unit diagonal in G. We note that there is a homomorphism from H_1' to G' as the covering splits over this subgroup of $SL(2)$. Thus we may consider $H_1' = H_2'$ as a subgroup of G'. Let χ_1 be a quadratic character χ_τ of $\mathbf{A}$ associated to the quadratic extension $F[\sqrt{\tau}]$; it can be considered as a character of H_1. Fix a non-trivial additive character ψ of $\mathbf{A}/F$. For

$$u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in H_2,$$

let $\chi_2(u(x)) = \psi(x)$, $\chi_1'(u(x)) = \chi_2'(u(-x)) = \psi(\tau x/2)$. In [Jac87], Jacquet showed that there is a relative trace identity (1.2) for the above choice of data.

In this example, $P(\phi : H_1, \chi_1)$ for $\phi \in \pi$ is related to $L(\pi \otimes \chi_\tau, 1/2)$, while the right-hand side of (1.3) is related to the 'Fourier coefficient' of π'. The identity (1.3) states roughly that $L(\pi \otimes \chi_\tau, 1/2)$ equals the square of the norm of the τth Fourier coefficient of π'. For a more precise statement and relation with the corresponding result on modular forms, see [KZ81] and [BM04].

A natural question is to generalize this identity. A generalization would have implications in the identities for the values of L-functions, and problems like Böcherer's conjecture [Böc86, FS99].

1.4 Orbital integral

One of the difficulties in the theory of the relative trace formula is to determine the choice of data (H_i, χ_i) and (H_i', χ_i') $(i = 1, 2)$. In § 1.6, we suggest a principle of choosing the data in some cases, which in particular would lead to a generalization of the identity in § 1.3. In this and the next subsection, we will give some motivation for the principle.

Let us recall how the map ϵ_v in condition a of Equation (1.2) is defined. Assume $f = \otimes f_v$, and the distribution $I_G(f : H_1, \chi_1, H_2, \chi_2)$ decomposes into a sum of orbital integrals (see § 5 for an example)

$$\sum_{o\in\mathcal{O}} I_G(f, o : H_1, \chi_1, H_2, \chi_2) = \sum_{o\in\mathcal{O}} c(o) \prod_v I_{G,v}(f_v, o : H_1, \chi_1, H_2, \chi_2).$$

Here $\mathcal{O}$ is the set of representatives of orbits, a subset of $G(F)$; $c(o)$ is a positive coefficient that equals some volume; $I_{G,v}(f_v, o : H_1, \chi_1, H_2, \chi_2)$ is the local orbital integral which takes the form

$$\int_{H_{2,v}\cap o^{-1}H_{1,v}o\backslash H_{2,v}} \int_{H_{1,v}} f_v(h_1^{-1} o h_2)\chi_1(h_1)\chi_2(h_2)\, dh_1\, dh_2.$$

A similar decomposition holds for $I_{G'}(f' : H_1', \chi_1', H_2', \chi_2')$:

$$I_{G'}(f' : H_1', \chi_1', H_2', \chi_2') = \sum_{o'\in\mathcal{O}'} c(o') \prod_v I_{G',v}(f_v', o' : H_1', \chi_1', H_2', \chi_2').$$

We will assume there is a bijection ι between the set of orbits $\mathcal{O}$ and $\mathcal{O}'$ (for an example where this is true, see § 4). Then we will define ϵ_v by requiring that, for $f_v' = \epsilon_v(f_v)$, the following identity of orbital integrals holds:

$$I_{G,v}(f_v, o : H_1, \chi_1, H_2, \chi_2) = I_{G',v}(f_v', \iota(o) : H_1', \chi_1', H_2', \chi_2')\Delta_v(o). \tag{1.5}$$

Here $\Delta_v(o)$ is some transfer factor independent of f_v, satisfying $c(\iota(o)) \prod_v \Delta_v(o) = c(o)$.

If the maps ϵ_v exist, and if moreover we have the fundamental lemma, i.e. the identity (1.5) holds for $f_v' = \lambda(f_v)$ where $v \notin S$ and f_v is a Hecke function, then it follows immediately that the relative trace identity (1.2) holds. An example of identity (1.5) and the fundamental lemma is given in §§ 5–7.

1.5 Orbital integral as a linear functional

We consider the orbital integral $I_{G,v}(f_v, o : H_1, \chi_1, H_2, \chi_2)$ as a linear functional on $\mathcal{S}(G_v)$. Assume v is a p-adic place. The map

$$f_v(g) \to \int_{H_{1,v}} f_v(h_1^{-1} g)\chi_1(h_1)\, dh_1$$

is a projection from $\mathcal{S}(G(F_v))$ onto $\mathrm{ind}_{H_{1,v}}^{G_v} \chi_1$ (all inductions are set to be compact inductions in this paper). Thus for fixed o, the functional $I_{G,v}(f_v, o : H_1, \chi_1, H_2, \chi_2)$ gives a linear functional

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$I_{G,v}(\phi_v, o : H_2, \chi_2)$ on the space of $\mathrm{ind}_{H_{1,v}}^{G_v} \chi_1$, satisfying

$$I_{G,v}(\rho(h_2)\phi_v - \chi_2^{-1}(h_2)\phi_v, o : H_2, \chi_2) = 0, \quad \forall \phi_v \in \mathrm{ind}_{H_{1,v}}^{G_v} \chi_1,$$

where ρ is the right regular representation.

Thus the local orbital integral is a linear functional on the set of covariants $\mathrm{ind}_{H_{1,v}}^{G_v} \chi_1[H_{2,v}, \chi_2^{-1}]$. Here we adopt the following notations: for a space of representation E of ρ of G, for (H, χ) as before, let $E(H, \chi)$ be the space spanned by $\rho(h)v - \chi(h)v$, $v \in E, h \in H$; let $E[H, \chi] = E/E(H, \chi)$.

A similar argument works for local orbital integrals of G'. The identity (1.5) is a comparison between linear functionals on the space $\mathrm{ind}_{H_{1,v}}^{G_v} \chi_1[H_{2,v}, \chi_2^{-1}]$ and linear functionals on the space $\mathrm{ind}_{H'_{1,v}}^{G'_v} \chi'_1[H'_{2,v}, \chi_2'^{-1}]$. To make such a comparison possible, it is natural to choose the data $(H_i, \chi_i, H'_i, \chi'_i)$ $(i = 1, 2)$ so that there is an isomorphism

$$\mathrm{ind}_{H_{1,v}}^{G_v} \chi_1[H_{2,v}, \chi_2^{-1}] \cong \mathrm{ind}_{H'_{1,v}}^{G'_v} \chi'_1[H'_{2,v}, \chi_2'^{-1}]. \tag{1.6}$$

In the next subsection, we describe how to arrive at such a choice of data in the cases when G, G' is a dual reductive pair.

1.6 The case of a dual pair

We continue to consider the local situation; fix a p-adic place v, and drop it in the notations. Assume now that G, G' is a dual reductive pair inside a metaplectic group $\widetilde{Sp}(M)$ for some M. Fix a non-trivial additive character ψ of $\mathbf{A}/F$. Let ω_ψ be the local Weil representation of $\widetilde{Sp}(M)$ associated to the character ψ. It acts on the space of Schwartz functions $\mathcal{S}(V)$ on an M-dimensional space V.

Let H be a closed subgroup of G, with χ its character. We consider the space $\omega_\psi[H, \chi]$. The group G' acts on this space via the Weil representation. Similarly if H' is a closed subgroup of G' with character χ', we can define a representation space $\omega_\psi[H', \chi']$ of G. In the first part of this paper, we will prove some isomorphisms of the following type:

$$\omega_\psi[H_2, \chi_2^{-1}] \cong \mathrm{ind}_{H'_1}^{G'} \chi'_1, \tag{1.7}$$

$$\omega_\psi[H'_2, \chi_2'^{-1}] \cong \mathrm{ind}_{H_1}^{G} \chi_1. \tag{1.8}$$

Here the isomorphisms are as G'-modules or G-modules, and the inductions are all compact inductions.

We claim the isomorphisms (1.7) and (1.8) imply the isomorphism (1.6). This follows from the isomorphisms

$$\omega_\psi[H_2 \times H'_2, \chi_2^{-1} \otimes \chi_2'^{-1}] \cong \omega_\psi[H_2, \chi_2^{-1}][H'_2, \chi_2'^{-1}] \cong \omega_\psi[H'_2, \chi_2'^{-1}][H_2, \chi_2^{-1}]. \tag{1.9}$$

In view of the discussion in § 1.5, we will set the data for trace identity (1.2) to be $(H_i, \chi_i, H'_i, \chi'_i)$ whenever the isomorphisms (1.7) and (1.8) hold. We come up with several families of relative trace identities in § 2 using this principle. We have checked that all these identities hold. In the second part of this paper, we provide a proof for one family of identities.

1.7 Remark on the proofs

The above discussion also indicates a way to prove the trace identities. As stated in § 1.4, to establish a trace identity, one needs to show the orbital integral identity (1.5). From the isomorphism (1.9), we see that the orbital integrals $I_{G,v}(f_v, o : H_1, \chi_1, H_2, \chi_2)$ and $I_{G',v}(f'_v, \iota(o) : H'_1, \chi'_1, H_2, \chi'_2)$ give linear functionals of functions in the space of the Weil representation. The idea is to show that they give the same functional by using the properties of the Weil representation. The same idea

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can be used to prove the fundamental lemma. It turns out all one needs is the already proven local unramified Howe duality conjecture.

1.8 A specific family of relative trace identities

We discuss in detail the family of relative trace identities that is proved here.

Let $G_n = SO(n+1, n)$ be the split special orthogonal group. Let $G'_m = Sp(m)$ be the symplectic group, and $\tilde{G}_m$ its double cover. Assume $m \leqslant n$. Denote an element in $\tilde{G}_m$ by (g, ϵ) with $g \in G'_m$, $\epsilon \in \{\pm 1\}$. If the covering splits over a subgroup H of G'_m, for $h \in H$, we write $\tilde{h}$ for the image in $\tilde{G}_m$ under the splitting map. The consideration in § 1.6 and the results in § 2 lead us to consider the following distributions.

Let N'_m be the subgroup of upper triangular matrices with unit diagonal in G'_m. Then the covering of G'_m splits over N'_m. We define a distribution on $\tilde{G}_m(\mathbf{A})$: for $\tilde{f} \in \mathcal{S}(\tilde{G}_m(\mathbf{A}))$,

$$J_m(\tilde{f}) = \int_{N'_m(F)\backslash N'_m(\mathbf{A})} \int_{N'_m(F)\backslash N'_m(\mathbf{A})} K_{\tilde{f}}(\tilde{n}_1, \tilde{n}_2)\theta'(n_1 n_2^{-1})\, dn_1\, dn_2. \tag{1.10}$$

Here for $n \in N'_m$, $\tilde{n} = (n, 1)$, and

$$\theta'(n) = \psi\left(n_{1,2} + \cdots + n_{m-1,m} + \frac{n_{m,m+1}}{2}\right). \tag{1.11}$$

We now define a distribution $I_{m,n}(f)$ on $G_n(\mathbf{A})$. In § 2.1, we introduce the subgroups $R_{m,n}$ and $U_{m,n}$ of G_n and the characters χ and μ on them. We define

$$I_{m,n}(f) = \int_{R_{m,n}(F)\backslash R_{m,n}(\mathbf{A})} \int_{U_{m,n}(F)\backslash U_{m,n}(\mathbf{A})} K_f(r, u)\chi^{-1}(r)\mu^{-1}(u)\, dr\, du. \tag{1.12}$$

The distributions $I_{m,n}$ and J_m can be written as sums of orbital integrals. The groups $R_{m,n}$ and $U_{m,n}$ act on G_n by $(r, u) : g \to r^{-1}gu$. For $o \in G_n$, let $(R_{m,n} \times U_{m,n})_o$ be the set of all pairs (r, u) satisfying $r^{-1}ou = o$. We say the orbit of o under the action of $R_{m,n} \times U_{m,n}$ is *relevant* if we have $\chi(r)\mu(u) \equiv 1$ for $(r, u) \in (R_{m,n} \times U_{m,n})_o$. Then we have (§ 5)

$$I_{m,n}(f) = \sum_{\{o\}} \prod_v I_o(f_v), \quad f = \otimes f_v, \tag{1.13}$$

where the sum is taken over the set of representatives for the relevant orbits, and the orbital integral is defined as

$$I_o(f_v) = \int_{R_{m,n,v}} \int_{o^{-1}R_{m,n,v}o \cap U_{m,n,v}\backslash U_{m,n,v}} f_v(r^{-1}ou)\chi^{-1}(r)\mu^{-1}(u)\, dr\, du. \tag{1.14}$$

Similarly, the group $N'_m \times N'_m$ acts on G'_m by $(n_1, n_2) : g \to n_1^{-1}gn_2$. We say an orbit o' is relevant if $n_1^{-1}o'n_2 = o'$ implies that $\theta'(n_1 n_2^{-1}) = 1$. Then

$$J_m(\tilde{f}) = \sum_{\{o'\}} \prod_v J_{o'}(\tilde{f}_v), \quad \tilde{f} = \otimes \tilde{f}_v, \tag{1.15}$$

where the sum is taken over representatives of relevant orbits and the orbital integral $J_{o'}(\tilde{f}_v)$ is defined to be

$$\int_{N'_{m,v}} \int_{N'_{m,v} \cap o'^{-1}N'_{m,v}o' \backslash N'_{m,v}} \tilde{f}_v(\tilde{n}_1^{-1}(o', 1)\tilde{n}_2)\theta'(n_1 n_2^{-1})\, dn_1\, dn_2. \tag{1.16}$$

We will show the identity (1.5) between orbital integrals in this case.

Theorem 1.1. *There is a bijection ι from the set of relevant orbits $\{o\}$ in $G_n(F)$ to the set of relevant orbits $\{o'\}$ in $G'_m(F)$, such that there exists a map ϵ_v from $C_c^\infty(G_{n,v})$ to $\mathcal{S}(\tilde{G}_{m,v})$ for all*

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places v, and transfer factors $\Delta_v(o)$ satisfying $\prod_v \Delta_v(o) \equiv 1$, $o \in G_n(F)$, with

$$J_{\iota(o)}(\epsilon(f_v)) = \Delta_v(o) I_o(f_v). \tag{1.17}$$

We note that in the statement we used $C_c^\infty(G_{n,v})$ instead of $\mathcal{S}(G_{n,v})$. This is a technical point which appears in the proof of Lemma 5.2.

We now state the fundamental lemma in this case. Let v be a p-adic place with odd residue characteristics. Let $\mathcal{O}_v$ be the ring of integers. Let $\mathcal{H}_{n,v}$ be the Hecke algebra of $G_{n,v}$. It is the set of compactly supported functions that are biinvariant under $G_{n,v}(\mathcal{O}_v)$. Recall that the double cover of $G'_{m,v}$ splits over $G'_{m,v}(\mathcal{O}_v)$. We let $\tilde{\mathcal{H}}_{m,v}$ be the set of compactly supported functions $\tilde{f}$ on $\tilde{G}_{m,v}$ that are biinvariant under $G'_{m,v}(\mathcal{O}_v)$, and satisfying $\tilde{f}((g,1)) = -\tilde{f}((g,-1))$ (that is, $\tilde{f}$ is a genuine function). It is the Hecke algebra of $\tilde{G}_{m,v}$. There is a Hecke algebra homomorphism λ from $\mathcal{H}_{n,v}$ to $\tilde{\mathcal{H}}_{m,v}$ (see § 7). Fix the measures so that $G'_{m,v}(\mathcal{O})$ and $G_{n,v}(\mathcal{O})$ have volumes 1. We prove the following theorem.

Theorem 1.2. *Let ι and Δ_v be as in Theorem 1.1. Then if ψ is of order 0 at v, we have*

$$J_{\iota(o)}(\lambda(f_v)) = \Delta_v(o) I_o(f_v) \tag{1.18}$$

whenever $f_v \in \mathcal{H}_{n,v}$.

As stated in § 1.4, taking into account the identities (1.13) and (1.15), the above two theorems imply the following.

Theorem 1.3. *There is a relative trace identity (in the sense of equation (1.2) and Theorem 1.1):*

$$I_{m,n}(f) = J_m(\tilde{f}). \tag{1.19}$$

1.9 Remarks

1) When $m = n = 1$, the identity (1.19) is Jacquet's identity (when $\tau = 1$) that we mentioned in § 1.3.
2) When $m = n$, $G_n = SO(n+1,n)$, $\tilde{G}_n = \widetilde{Sp}(n)$, the correspondence of automorphic representations was considered in [Fur95]. In our formula, the group $R_{n,n}$ is the Bessel group defined in [Fur95], while $U_{n,n}$ is a maximal unipotent subgroup. The trace identity should give a correspondence between the generic automorphic representations of $SO(n+1,n)$ with non-vanishing Bessel periods and the generic automorphic representations of $\widetilde{Sp}(n)$. The Bessel periods are closely related to the value $L(\pi,1/2)$ (see [Gin90, Sou93]). In particular, the identity (1.19) should give an identity for the value of $L(\pi,1/2)$. When $n=1$ this identity is given in [BM04].
3) The identity (1.19) can be considered as a tower of identities. Fix n, and we get a family of formulas, which describe the correspondence between $SO(n+1,n)$ and a family of metaplectic groups $\tilde{G}_m$ with $m = 1,\dots,n$. Fix m, and we get a correspondence between $\widetilde{Sp}(m)$ and a family of orthogonal groups $SO(n+1,n)$ with $n \geqslant m$. The tower principle in the correspondence between orthogonal and metaplectic groups is described in [Ral84].
4) The case $m = 1$ is also studied in [MR99b]. Here $R_{m,n}$ is isomorphic to the group $SO(n,n)$, and $U_{m,n}$ is an abelian unipotent group. The particular case $m = 1$, $n = 2$ is also considered in [FM04] and [Zin98].
5) The space $\omega_\psi[H,\chi]$ is the Jacquet module of the Weil representation with respect to the data (H,χ), thus the title of the paper.
6) Similar arguments can be made when we consider the minimal representations of the exceptional groups in place of the Weil representation. The computation of the Jacquet module there should yield relative trace identities for the dual pairs inside the exceptional groups.

1.10 Structure of the paper

This paper is organized as follows. In § 2, we state the isomorphisms of type (1.7) and (1.8) in various cases. In some cases, the isomorphisms are only true for a subspace of the Jacquet module, but it is good enough for the isomorphism (1.6) to hold. In § 3, we provide the proofs of the results in § 2. The proof of the results in § 1.8 starts in § 4, where we classify the relevant orbits. In § 5, we express the orbital integrals as linear functionals on the space of Weil representations. We show Theorem 1.1 in § 6, Theorem 1.2 and thus Theorem 1.3 in § 7. In § 8, we make some remarks on other cases of trace identities, and the relation between the identities considered in this paper and those in previous works on relative trace identities.

1.11 Notation

The symbols m and n denote integers. We denote the set of $m \times n$ matrices by $M_{m,n}$. We denote the (i,j)th entry in a matrix A by $A_{i,j}$. We use 1_m to denote an $m \times m$ identity matrix, and $\mathbf{0}$ to denote a matrix of suitable size with all entries being 0. We use $*$ to denote a block of a matrix that can have any entry. All vectors in a vector space are written as column vectors, with matrices acting by left multiplication.

We will let $\sigma_m \in GL_m$ be the permutation matrix with '1's on the antidiagonal. Denote by $\mathcal{S}_m \subset M_{m,m}$ the set of matrices A where $A\sigma_m$ is symmetric. Given $g \in GL_m$, we will let $g^* = \sigma_m \, {}^{\mathrm{t}}g^{-1}\sigma_m$.

For $\mathbf{v} \in F^n$ a vector, we let $\mathbf{v}_i$ be its ith coordinate.

The homomorphism ρ is defined in § 2.1, δ is defined in Equation (3.4), and the homomorphisms $\eta, \eta', \rho', \rho''$ are defined in § 4.

2. Families of relative trace identities

In this section, F is a p-adic field. We state isomorphisms (1.7) and (1.8) for various pairs of groups G and G', and state the corresponding relative trace identities.

2.1 When $G = SO(n+1,n)$, $G' = \widetilde{Sp}(m)$, with $n \geqslant m$

Let $e_1, \dots, e_{2n+1}$ be the standard basis of F^{2n+1}. Let $SO(n+1,n)$ be the special orthogonal group fixing the symmetric bilinear form $\langle\,,\rangle$ given by $\langle e_i, e_j\rangle = 2$ when $i + j = 2n+2, i \neq j$, $\langle e_{n+1}, e_{n+1}\rangle = 1$, and $\langle e_i, e_j\rangle = 0$ otherwise. For $m \leqslant n$, let $V_{m,n}$ be the subspace of F^{2n+1} spanned by $\{e_1, \dots, e_{m-1}, e_{n+1}\}$, and let $W_{m,n}$ be the subspace spanned by $\{e_m, \dots, e_{2n+2-m}\}$. Denote by $\mathrm{Pr}_{W_{m,n}} \mathbf{v}$ the orthogonal projection of $\mathbf{v} \in F^{2n+1}$ in $W_{m,n}$. We define some subgroups of $SO(n+1,n)$.

Let $R'_{m,n}$ be the subgroup of $SO(n+1,n)$ fixing $V_{m,n}$:

$$R'_{m,n} = \{r \mid r(e_1, \dots, e_{m-1}, e_{n+1}) = (e_1, \dots, e_{m-1}, e_{n+1})\}. \tag{2.1}$$

Let $R^0_{m,n}$ be the subgroup of $R'_{m,n}$:

$$R^0_{m,n} = \{r \in R'_{m,n} \mid \mathrm{Pr}_{W_{m,n}}(r\mathbf{v}) = \mathbf{v},\ \forall \mathbf{v} \in W_{m,n}\}. \tag{2.2}$$

Then $R^0_{m,n}$ is a normal subgroup of $R'_{m,n}$, with $R'_{m,n}/R^0_{m,n} \cong SO(n-m+1, n-m+1)$.

Let N_l denote the subgroup of upper triangular matrices with unit diagonal in GL_l. Then N_m acts on $V_{m,n}$. Define $R_{m,n}$ to be the subgroup of $SO(n+1,n)$:

$$R_{m,n} = \{r \in SO(n+1,n) \mid \exists n \in N_m,\ r\mathbf{v} = n\mathbf{v},\ \forall \mathbf{v} \in V_{m,n}\}. \tag{2.3}$$

When $m = n$, this is the Bessel group defined in [Fur95] for $SO(n+1,n)$. When $m = 1$, it is isomorphic to $SO(n,n)$.

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The group $R'_{m,n}$ is a normal subgroup of $R_{m,n}$ with an isomorphism $\rho : R_{m,n}/R'_{m,n} \cong N_m$ given by $r \mapsto n$ in Equation (2.3).

Let $U_{m,n}$ be the subgroup of $R_{m+1,n}$:

$$U_{m,n} = \{u \in R_{m+1,n} \mid \mathrm{Pr}_{W_{m+1,n}}(u\mathbf{v}) = \mathbf{v},\ \forall \mathbf{v} \in W_{m+1,n}\}. \tag{2.4}$$

Then it is a unipotent radical of a standard parabolic subgroup of $SO(n+1,n)$, whose Levi part is isomorphic to $GL_1^m \times SO(n-m+1,n-m)$. When $m=n$, $U_{n,n}$ is a maximal unipotent subgroup in $SO(n+1,n)$. The group $R^0_{m+1,n}$ is a normal subgroup of $U_{m,n}$ with the isomorphism $\rho : U_{m,n}/R^0_{m+1,n} \cong N_{m+1}$.

Fix ψ a non-trivial additive character of F. We define the characters on $R_{m,n}$ and $U_{m,n}$. First we define an additive character θ on N_l:

$$\theta(n) = \psi(n_{1,2} + n_{2,3} + \cdots + n_{l-1,l}), \quad n \in N_l. \tag{2.5}$$

Then we define a character χ on $R_{m,n}$ by $\chi(r) = \theta(\rho(r))$, and a character μ on $U_{m,n}$ by $\mu(u) = \theta^{-1}(\rho(u))$.

Let N'_m be the group of upper triangular matrices in $Sp(m)$ with unit diagonal. Denote an element in G' by (g,ϵ) with $g \in Sp(m)$ and $\epsilon = \pm 1$. The group N'_m can be considered as a subgroup of G' via the embedding $n \to \tilde{n} = (n,1)$, $n \in N'$. Define a character θ' of N'_m by

$$\theta'(n) = \psi\left(n_{1,2} + \cdots + n_{m-1,m} + \frac{n_{m,m+1}}{2}\right), \tag{2.6}$$

where $n = (n_{i,j}) \in N'_m$.

Proposition 2.1. *As a representation of $G = SO(n+1,n)$,*

$$\omega_\psi[N'_m, \theta'] \cong \mathrm{ind}^G_{R_{m,n}} \chi^{-1}. \tag{2.7}$$

As a representation of $G' = \widetilde{Sp}(m)$,

$$\omega_\psi[U_{m,n}, \mu] \cong \mathrm{ind}^{G'}_{N'_m} \theta'. \tag{2.8}$$

The proposition suggests the following relative trace identity:

$$I_G(f : R_{m,n}, \chi^{-1}, U_{m,n}, \mu^{-1}) = I_{G'}(f' : N'_m, \theta', N'_m, \theta'^{-1}). \tag{2.9}$$

This formula will be established in this paper.

We look at a generalization of the identity (2.9). Let $\tau \in F^\times$. Consider the character θ'_τ of N'_m:

$$\theta'_\tau(n) = \psi\left(n_{1,2} + \cdots + n_{m-1,m} + \frac{\tau}{2} n_{m,m+1}\right). \tag{2.10}$$

All non-degenerate characters of N'_m are in the orbit of a θ'_τ for some τ. We will state a result for $\omega_\psi[N'_m, \theta'_\tau]$.

Let $V^\tau_{m,n}$ be the subspace of F^{2n+1} spanned by $e_1, \ldots, e_{m-1}$ and $e_m + (\tau/2)e_{2n+2-m}$. We define the subgroups $R^\tau_{m,n}$, $R'^\tau_{m,n}$, $R^{0\tau}_{m,n}$ and $U^\tau_{m,n}$ as before with $V_{m,n}$ replaced by $V^\tau_{m,n}$. (However in the definition of $U^\tau(m,n)$ we need the assumption $n > m$.) Then again there is an isomorphism $\rho : R^\tau_{m,n}/R'^\tau_{m,n} \cong N_m$, and we can again define character χ^τ on $R^\tau_{m,n}$ by setting $\chi^\tau(r) = \theta(\rho(r))$, and (when $n > m$) character μ^τ on $U^\tau_{m,n}$ by $\mu^\tau(u) = \theta^{-1}(\rho(u))$.

We note that $U_{m,n}$ and $U^\tau_{m,n}$ are the same groups.

Proposition 2.2. *As a representation of $G = SO(n+1,n)$,*

$$\omega_\psi[N'_m, \theta'_\tau] \cong \mathrm{ind}^G_{R^\tau_{m,n}} (\chi^\tau)^{-1}. \tag{2.11}$$

When $n > m$, as a representation of $G' = \widetilde{Sp}(m)$,

$$\omega_\psi[U_{m,n}, \mu^\tau] \cong \mathrm{ind}^{G'}_{N'_m} \theta'_\tau. \tag{2.12}$$

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This proposition together with Proposition 2.1 suggests the following relative trace identities:

$$I_G(f : R^{\tau}_{m,n}, (\chi^{\tau})^{-1}, U_{m,n}, \mu^{-1}) = I_{G'}(f' : N'_m, \theta', N'_m, (\theta'_{\tau})^{-1}), \tag{2.13}$$

and when $n > m$

$$I_G(f : R^{\tau}_{m,n}, (\chi^{\tau})^{-1}, U_{m,n}, (\mu^{\tau})^{-1}) = I_{G'}(f' : N'_m, \theta'_{\tau}, N'_m, (\theta'_{\tau})^{-1}). \tag{2.14}$$

Remark 1. In the case $m = n = 1$, the implication of (2.13) is the following: the period $P(\pi, R^{\tau}_{m,n}, \chi^{\tau})$ defined in Equation (1.4) for a cuspidal representation π of PGL_2, if not 0, should roughly be the product of two different Fourier coefficients of the lifting $\tilde{\pi}$ of π to $\widetilde{SL}_2$. A modular form version of this statement is proved in [Koh85]. Combining this with the statement for $L(\pi, 1/2)$ and $L(\pi \otimes \chi_{\tau}, 1/2)$ coming from Jacquet's identity in § 1.3, we get that $L(\pi, 1/2)L(\pi \otimes \chi_{\tau}, 1/2)$ is roughly the square of $P(\pi, R^{\tau}_{m,n}, \chi^{\tau})$, assuming $P(\pi, R^{\tau}_{m,n}, \chi^{\tau}) \neq 0$. Extending this argument to the general case, we should get a general version of Böcherer's conjecture for generic representations of $SO(n+1, n)$.

Remark 2. The generalization (2.14) of (2.9) is not a superficial one. As an example, when $n = 2, m = 1$, the group $R_{1,2}$ is isomorphic to $SO(2,2)$ while the group $R^{\tau}_{1,2}$ is isomorphic to $SO(3,1)$ for τ not a square. In particular, while the cuspidal representations do not appear in the spectral decomposition of $I_G(f : R_{1,2}, \chi^{-1}, U_{1,2}, \mu^{-1})$, they do appear in the spectral decomposition of $I_G(f : R^{\tau}_{1,2}, (\chi^{\tau})^{-1}, U_{1,2}, (\mu^{\tau})^{-1})$ for τ not a square. The formula (2.14) in this case should give the Saito–Kurukawa lifting.

2.2 When $G = SO(n+1, n)$, $G' = \widetilde{Sp}(m)$ with $m > n$

Let U_n be the subgroup of upper triangular matrices in G with unit diagonal. Define μ a character of U_n by

$$\mu(u) = \psi^{-1}(u_{1,2} + \cdots + u_{n,n+1}), \quad n = (n_{i,j}) \in U_n. \tag{2.15}$$

Let $e_1, \ldots, e_{2m}$ be the standard basis of F^{2m} where $Sp(m)$ acts. Let $V'_{m,n}$ be the subspace of F^{2m} spanned by $\{e_1, \ldots, e_n\}$ and $W'_{m,n}$ be the subspace spanned by $\{e_n, \ldots, e_{2m+1-n}\}$. We define some subgroups of G'.

Let H'_1 be the group

$$H'_1 = \{(h, \pm 1) \in G' \mid \exists n \in N_n,\ h\mathbf{v} = n\mathbf{v},\ \forall \mathbf{v} \in V'_{m,n}\}. \tag{2.16}$$

Let $U'_{m,n}$ be the subgroup of $Sp(m)$:

$$U'_{m,n} = \{u \in Sp(m) \mid (u, 1) \in H'_1,\ \mathrm{Pr}_{W'_{m,n}} u\mathbf{v} = \mathbf{v},\ \forall \mathbf{v} \in W'_{m,n}\}. \tag{2.17}$$

As the covering splits over any unipotent subgroup, $U'_{m,n}$ can be considered a subgroup of H'_1, and it is a normal subgroup. Define another subgroup of H'_1 to be the inverse image $J'_{m,n}$ in the covering of the set of

$$j(g, X, Y, z) = \begin{pmatrix} 1_n & & \\ & g & \\ & & 1_n \end{pmatrix} \begin{pmatrix} 1_{n-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & 1 & X & Y & z & \mathbf{0} \\ & & 1_{m-n} & \mathbf{0} & \sigma_{m-n}\,{}^{\mathrm{t}}Y & \mathbf{0} \\ & & & 1_{m-n} & -\sigma_{m-n}\,{}^{\mathrm{t}}X & \mathbf{0} \\ & & & & 1 & \mathbf{0} \\ & & & & & 1_{n-1} \end{pmatrix} \tag{2.18}$$

where $g \in Sp(m-n)$, ${}^{\mathrm{t}}X$, ${}^{\mathrm{t}}Y \in F^{m-n}$ and $z \in F$. Then H'_1 is the semidirect product of $U'_{m,n}$ and $J'_{m,n}$. Recall that there is an oscillator representation χ'_1 defined on $J'_{m,n}$, explicitly.

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For $\Phi(Z) \in \mathcal{S}(F^{m-n})$:

$$\chi'_1(j(g,\mathbf{0},\mathbf{0},\mathbf{0}),\pm 1)\Phi(Z) = \omega_\psi(g,\pm 1)\Phi(Z);$$
$$\chi'_1(j(1_{2m-2n},X,\mathbf{0},z),1)\Phi(Z) = \psi(z/2)\Phi(Z+{}^{\mathrm{t}}X);$$
$$\chi'_1(j(1_{2m-2n},\mathbf{0},Y,\mathbf{0}),1)\Phi(Z) = \psi(Y\cdot Z)\Phi(Z).$$

We define a character μ' on $U'_{m,n}$ as follows: for each $u \in U'_{m,n}$, from (2.16) there is an $n \in N_n$ associated to it. We define $\mu'(u)$ to be $\theta(n)$ where θ is defined by (2.5). As $J'_{m,n}$ stabilizes this character μ' on $U'_{m,n}$, we can extend μ' and χ'_1 to a representation χ'_1 on H'_1: for any $h = uj$ with $j \in J'_{m,n}$ and $u \in U'_{m,n}$, let

$$\chi'_1(h) = \mu'(u)\chi'_1(j).$$

Let H'_2 be the semidirect product of $U'_{n+1,m}$ and

$$H_{m,n} = \{(j(1_{2m-2n-2},\mathbf{0},Y,z),\pm 1) \mid {}^{\mathrm{t}}Y \in F^{n-m-1}, z \in F\}.$$

For $h = u(j(1_{2m-2n-2},\mathbf{0},Y,z),1) \in H'_2$ with $u \in U'_{n+1,m}$, let

$$\chi'_2(\rho'(n)h) = \mu'(u)\psi(z/2).$$

As $H_{m,n}$ stabilizes the character μ', χ'_2 is a character on the unipotent group H'_2.

Proposition 2.3. *As a representation of $G = SO(n+1,n)$,*

$$\omega_\psi[H'_2,\chi'_2] \cong \mathrm{ind}_{U_n}^{G}\,\mu. \tag{2.19}$$

As a representation of $G' = \widetilde{Sp}(m)$,

$$\omega_\psi[U_n,\mu] \cong \mathrm{ind}_{H'_1}^{G'}\,\chi'_1. \tag{2.20}$$

We expect that there is a relative trace identity:

$$I_G(f : U_n,\mu,U_n,\mu^{-1}) = I_{G'}(f' : H'_1,\chi'_1,H'_2,\chi_2^{\prime -1}). \tag{2.21}$$

Here since χ'_1 is no longer a character, we define $I_{G'}(f' : H'_1,\chi'_1,H'_2,\chi_2^{\prime -1})$ as follows:

$$\int_{H'_1(F)\backslash H'_1(\mathbf{A})}\int_{H'_2(F)\backslash H'_2(\mathbf{A})} K_{f'}(h_1,h_2)\Theta^{\psi}_{\Phi}(h_1)\chi'_2(h_2^{-1})\,dh_2\,dh_1, \tag{2.22}$$

where

$$\Theta^{\psi}_{\Phi}(h_1) = \sum_{Z\in F^{m-n}} \chi'_1(h_1)\Phi(Z).$$

This type of *coperiod* trace formula has appeared in for example [MR99c]. We refer to that paper for the precise meaning of the relative trace identities in this situation.

2.3 When $G = O(n,n)$, $G' = Sp(m)$

The situation is similar to that of §§ 2.1 and 2.2. We will only state the results without giving the proof.

When $n > m$, consider G as a subgroup of $SO(n+1,n)$ mapping e_{n+1} to $\pm e_{n+1}$. Let $R^{\tau}_{m,n}$ and $U_{m,n}$ be the intersection between G and the corresponding groups defined in § 2.1. Let χ^{τ} and μ^{τ} be the characters defined in § 2.1. Let $N'_m \subset G'$ and θ'_{τ} be as defined in § 2.1.

Proposition 2.4. *As a representation of $G = O(n,n)$,*

$$\omega_\psi[N'_m,\theta'_\tau] \cong \mathrm{ind}_{R^{\tau}_{m,n}}^{G}(\chi^{\tau})^{-1}. \tag{2.23}$$

As a representation of $G' = Sp(m)$,

$$\omega_\psi[U_{m,n},\mu^{\tau}] \cong \mathrm{ind}_{N'_m}^{G'}\,\theta'_\tau. \tag{2.24}$$

We expect the relative trace identities (2.13) and (2.14) to hold in this setting.

When $1 < n \leqslant m$, let U_n be the intersection of G and the corresponding group in § 2.2. Define μ on U_n by letting

$$\mu(u) = \psi^{-1}(u_{1,2} + \cdots + u_{n-1,n} + u_{n-1,n+1}), \quad u \in U_n. \tag{2.25}$$

Define the subgroup $U'_{m,n}$ as in (2.17). Let $H'_1 \subset G'$ be the semidirect product of $U'_{m,n-1}$ and the group with elements of the form $j(g, \mathbf{0}, \mathbf{0}, z)$ where $g \in Sp(m-n+1)$. (Recall that $j(g, X, Y, z)$ is defined in (2.18).) For $h = uj(g, \mathbf{0}, \mathbf{0}, z) \in H'_1$ with $u \in U'_{m,n-1}$, let

$$\chi'_1(h) = \mu'(u)\psi(z).$$

Let $H'_2 \subset G'$ be the semidirect product of $U'_{m,n}$ and the group with elements of the form $j(g, \mathbf{0}, Y, z)$ with ${}^{\mathrm{t}}Y \in F^{m-n}$, $z \in F$ and

$$g = \begin{pmatrix} 1_{m-n} & Z \\ & 1_{m-n} \end{pmatrix} \in Sp(m-n).$$

For $h = uj(g, \mathbf{0}, Y, z) \in H'_2$ with $u \in U'_{m,n}$, let

$$\chi'_2(h) = \mu'(u)\psi(z).$$

PROPOSITION 2.5. *As a representation of $G = O(n,n)$,*

$$\omega_\psi[H'_2, \chi'_2] \cong \mathrm{ind}_{U_n}^{G} \mu. \tag{2.26}$$

As a representation of $G' = Sp(m)$,

$$\omega_\psi[U_n, \mu] \cong \mathrm{ind}_{H'_1}^{G'} \chi'_1. \tag{2.27}$$

We expect the identity (2.21) to be true in this setting.

2.4 When $G = GL_n$, $G' = GL_m$ with $n > m$

Here G and G' form a dual reductive pair in $\widetilde{Sp}_{mn}$.

Let N_m be the subgroup of G' defined in § 2.1 and θ its character defined in (2.5). Let $H_{m,n} \subset G$ be the following subgroup: it consists of elements

$$\begin{pmatrix} n & \mathbf{0} \\ * & * \end{pmatrix}, \quad n \in N_m.$$

For $h \in H_{m,n}$ of above type, let $\chi(h) = \theta^{-1}(n)$; then χ is a character of $H_{m,n}$. Let $U_{m,n} \subset G$ be the following subgroup: it consists of elements

$$\begin{pmatrix} n & * \\ \mathbf{0} & 1_{n-m} \end{pmatrix}, \quad n \in N_m.$$

For $u \in U_{m,n}$ of above type, let $\mu(u) = \theta(n)\psi(u_{m,m+1})$; then μ is a character of $U_{m,n}$.

PROPOSITION 2.6. *As a representation of $G' = GL_m$,*

$$\omega_\psi[U_{m,n}, \mu] \cong \mathrm{ind}_{N_m}^{G'} \theta^{-1}. \tag{2.28}$$

We have an injective homomorphism between GL_n-modules: $\mathrm{ind}_{H_{m.n}}^{G} \chi \mapsto \omega_\psi[N_m, \theta]$. This injection induces a vector space isomorphism

$$\mathrm{ind}_{H_{m,n}}^{G} \chi[U_{m,n}, \mu] \cong \omega_\psi[N_m, \theta][U_{m,n}, \mu]. \tag{2.29}$$

The proof of Proposition 2.6 is similar to that of Proposition 2.1 and will be skipped.

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From Proposition 2.6, the argument in § 1.6 shows that the spaces $\mathrm{ind}_{H_{m,n}}^{G}\chi[U_{m,n},\mu]$ and $\mathrm{ind}_{N_m}^{G'}\theta^{-1}[N_m,\theta]$ are isomorphic. We expect that there is a relative trace identity:

$$I_G(f;H_{m,n},\chi,U_{m,n},\mu^{-1}) = I_{G'}(f';N_m,\theta^{-1},N_m,\theta^{-1}). \tag{2.30}$$

Remark 3. Note that $I_G(f';N_m,\theta^{-1},N_m,\theta^{-1})$ is a Kuznietsov trace formula on GL_m.

Remark 4. When $m=n$, we would need to set $H_{m,m}$ and $U_{m,m}$ to be N_m. Of course one gets a trivial trace identity here.

In the case $m=2$, we can get another family of identities which is studied in [Fli93] and [Mao92]. We consider $H=GL_{n-1}$ as a subgroup of G; let $U\subset G$ be a unipotent subgroup consisting of matrices of the form

$$u(\mathbf{v},\mathbf{w},z) = \begin{pmatrix} 1 & {}^t\mathbf{v} & z \\ & 1_{n-2} & \mathbf{w} \\ & & 1 \end{pmatrix}, \tag{2.31}$$

where $\mathbf{v}$ and $\mathbf{w}$ are vectors in F^{n-2}. Define a character μ of U by letting $\mu(u(\mathbf{v},\mathbf{w}),z)=\psi(\mathbf{v}_1+\mathbf{w}_1)$. The following proposition was mentioned to us by W. Gan.

Proposition 2.7. *As a module of $G=GL_n$:*

$$\omega_\psi[N_2,\theta] \cong \mathrm{ind}_H^G 1. \tag{2.32}$$

As a module of $G'=GL_2$:

$$\omega_\psi[U,\mu] \cong \mathrm{ind}_{N_2}^{G'}\theta^{-1}. \tag{2.33}$$

Thus one expects a relative trace identity:

$$I_G(f:H,1,U,\mu^{-1}) = I_{G'}(f':N_2,\theta^{-1},N_2,\theta^{-1}). \tag{2.34}$$

In [Fli93], a similar identity is conjectured. However, the character μ of U is chosen there as $\psi(\mathbf{v}_1+\mathbf{w}_{n-2})$, which is an incorrect choice. With the choice of μ in [Fli93], the right-hand side of (2.34) should be replaced by $I_{G'}(f':N_2,1,N_2,\theta^{-1})$ (when $n>3$).

3. Proof of results in § 2

We will make use of the following lemma [BZ86, Lemma 2.23].

Lemma 3.1. *Assume H is exhausted by compact subgroups (i.e. any compact subset of H is contained in a compact subgroup). Let χ be a character of H. Let π be a representation of H acting on E. Then a vector $\xi\in E$ lies in $E(H,\chi)$ if and only if there exists a compact subgroup $H_c\subset H$, such that*

$$\int_{H_c}\chi^{-1}(h)\pi(h)\xi\,dh = 0.$$

We give in detail the proofs of Proposition 2.1, and will only sketch or skip the proofs of the other results, as the proofs are similar.

3.1 Proof of the isomorphism (2.7)

We recall the model of the representation ω_ψ. It acts on the space $\mathcal{S}(M_{2n+1,m})$ such that

$$\omega_\psi(g,\delta(h))\Phi(Z) = |\det(h)|^{n+1/2}\frac{\gamma(1,\psi)}{\gamma(\det(h)^{2n+1},\psi)}\Phi(g^{-1}Zh), \tag{3.1}$$

$$\omega_\psi(1,\sigma_l')\Phi(Z) = \gamma(1,\psi)^{-(2n+1)l/2}\Phi^{\wedge l}(Z), \tag{3.2}$$

$$\omega_\psi(1,(\begin{pmatrix}1_m & V\\ & 1_m\end{pmatrix},1))\Phi(Z) = \psi(\mathrm{tr}({}^{\mathrm{t}}Z\sigma_{2n+1}ZV\sigma_m)/2)\Phi(Z). \tag{3.3}$$

Here $g \in G$, $h \in GL_m$, σ_l is the matrix with '1's on the antidiagonal and the '0's elsewhere,

$$\delta(h) = \left(\begin{pmatrix}h & \\ & h^*\end{pmatrix},1\right), \quad \sigma_l' = \left(\begin{pmatrix} & & -\sigma_l\\ & 1_{2m-2l} & \\ \sigma_l & & \end{pmatrix},1\right), \tag{3.4}$$

$\gamma(a,\psi)$ denotes the Weil constant, and for $Z=[Z_1,Z_2] \in M_{2n+1,m}$, where $Z_1 \in M_{2n+1,l}$, let

$$\Phi^{\wedge l}(Z) = \int_{M_{2n+1,l}} \psi(\mathrm{tr}({}^{\mathrm{t}}Z'\sigma_{2n+1}Z_1))\Phi([Z',Z_2])\,dZ'. \tag{3.5}$$

We compute the projection from $\mathcal{S}(M_{2n+1,m})$ to $\omega_\psi[N_m',\theta']$ in two steps. Note that N_m' is the direct product of $N_{m,L}'$ and $N_{m,U}'$, where $N_{m,L}' = \delta(N_m)$ is the intersection of N_m' with the Levi part of the Siegel parabolic subgroup of $Sp(m)$, and $N_{m,U}'$ is the intersection with the unipotent radical. Thus as a G-module,

$$\omega_\psi[N_m',\theta'] \cong (\omega_\psi[N_{m,U}',\theta'])[N_{m,L}',\theta']. \tag{3.6}$$

We first compute $\omega_\psi[N_{m,U}',\theta']$.

LEMMA 3.2. *The space $\omega_\psi(N_{m,U}',\theta')$ spanned by $\omega_\psi(n)\Phi - \theta'(n)\Phi$ $(n \in N_{m,U}',\ \Phi \in \omega_\psi)$ is $\mathcal{S}(M_{2n+1,m}\backslash Y)$, where*

$$Y = \{[z_1,\ldots,z_m] \mid \langle z_m,z_m\rangle = 1,\ \langle z_i,z_j\rangle = 0 \text{ for other combinations of } i,j\}.$$

Proof. From Lemma 3.1, we need to check that there exists a compact subgroup N'' of $N_{m,U}'$ such that

$$\int_{N''}\omega_\psi(1,(n,1))\Phi(Z)\theta'(n^{-1})\,dn \equiv 0$$

if and only if Φ is a function supported away from Y. The integral equals

$$\int \psi(-V_{1,m}/2)\psi(\mathrm{tr}({}^{\mathrm{t}}Z\sigma_mVZ)/2)\Phi(Z)\,dV,$$

where the domain of integration is over the group of $V \in \mathcal{S}_m$ satisfying

$$\begin{pmatrix}1_m & V\\ & 1_m\end{pmatrix} \in N''.$$

For $Z \in Y$, the integral equals $\mathrm{vol}(N'')\Phi(Z)$. If Φ is supported away from Y, there exists a large enough compact set N'' such that the above integral vanishes for all Z in the support of Φ. Thus the claim is proved. $\square$

From Lemma 3.2, $\omega_\psi[N_{m,U}',\theta'] \cong \mathcal{S}(Y)$ as a $(G,N_{m,L}')$-module. Here the action is given as

$$(g,\delta(n)) : \Phi(Z) \mapsto \Phi(g^{-1}Zn), \quad g \in G,\ n \in N_m. \tag{3.7}$$

Let Y_0 be the subset of Y consisting of matrices of rank m. Then $\mathcal{S}(Y_0)$ is a submodule of $\mathcal{S}(Y)$.

Lemma 3.3. *As G-modules,*

$$\mathcal{S}(Y_0)[N'_{m,L},\theta'] \cong \mathcal{S}(Y)[N'_{m,L},\theta']. \tag{3.8}$$

Proof. The group $G\times N_m$ acts on Y by $(g,n): Z\mapsto g^{-1}Zn$, $g\in G, n\in N_m$. By Witt's theorem, Y has a finite number of $G\times N_m$ orbits, represented by $[a_1e_1,a_2e_2,\dots,a_{m-1}e_{m-1},e_{n+1}]$, where $a_i=0$ or 1. Consider an orbit o with such a representative where one of the a_i equals 0. Then the stabilizer of the orbit would contain a group $\{1_{2n+1}\}\times N_{m,o}$, where $N_{m,o}$ is a subgroup of N_m and the character θ' is non-trivial on $\delta(N_{m,o})$. Thus for this orbit o, $\mathcal{S}(o)[N'_{m,L},\theta']$ is a trivial space. Since Y_0 is the orbit with representative $a_i\equiv 1$, we see $\mathcal{S}(Y\backslash Y_0)[N'_{m,L},\theta']$ is a trivial space. The lemma follows from the exact sequence:

$$1\mapsto\mathcal{S}(Y_0)\mapsto\mathcal{S}(Y)\mapsto\mathcal{S}(Y\backslash Y_0)\mapsto 1. \qquad\square$$

From the above two lemmas and (3.6), we get

$$\omega_\psi[N'_m,\theta']\cong\mathcal{S}(Y_0)[N'_{m,L},\theta']. \tag{3.9}$$

As Y_0 is a $G\times N_m$ orbit, we see that $\mathcal{S}(Y_0)$ is a (G,N_m)-module with action given by (3.7). Then as G-modules,

$$\mathcal{S}(Y_0)[N'_{m,L},\theta']\cong\mathcal{S}(Y_0)[N_m,\theta]. \tag{3.10}$$

Define an action of $R_{m,n}$ on $\mathrm{ind}_{R'_{m,n}}^G 1$ by

$$r:\phi(g)\mapsto\phi(r^{-1}g). \tag{3.11}$$

Lemma 3.4. *As G-modules, $\mathcal{S}(Y_0)[N_m,\theta]\cong\mathrm{ind}_{R'_{m,n}}^G 1[R_{m,n},\chi]$.*

Proof. By Witt's theorem, $g\mapsto Z_g=g^{-1}[e_1,\dots,e_{m-1},e_{n+1}]$ gives a bijection of $R'_{m,n}\backslash G$ and Y_0. Thus $\Phi\mapsto\phi_\Phi(g)=\Phi(Z_g)$ defines an isomorphism between $\mathrm{ind}_{R'_{m,n}}^G 1$ and $\mathcal{S}(Y_0)$. Observe that

$$r[e_1,\dots,e_{m-1},e_{n+1}]=[e_1,\dots,e_{m-1},e_{n+1}]\rho(r),\quad r\in R_{m,n} \tag{3.12}$$

where ρ is the isomorphism from $R_{m,n}/R'_{m,n}$ to N_m in § 2.1. Thus $\phi_\Phi(r^{-1}g)=\Phi(Z_g\rho(r))$. As $\chi(r)=\theta(\rho(r))$, $\Phi\mapsto\phi_\Phi$ defines the isomorphism in the lemma. $\square$

Since as G-modules $\mathrm{ind}_{R'_{m,n}}^G 1[R_{m,n},\chi]\cong\mathrm{ind}_{R_{m,n}}^G\chi^{-1}$, the isomorphism in (2.7) follows from Lemma 3.4 and (3.9) and (3.10).

3.2 Proof of the isomorphism (2.8)

To prove the isomorphism (2.8), we use the mixed model of the Weil representation. In this model, ω_ψ acts on $\mathcal{S}(M_{2m,n}\times F^m)$. The action is given as follows: for a function $\Phi\otimes\Phi_0(Z,Z_0)=\Phi(Z)\Phi_0(Z_0)$ in the set $(Z\in M_{2m,n},\ Z_0\in F^m)$, for $g'\in Sp(m)$,

$$\omega_\psi(1,(g',1))\Phi\otimes\Phi_0(Z,Z_0)=\Phi(g'^{-1}Z)\omega_\psi(g',1)\Phi_0(Z_0), \tag{3.13}$$

the second ω_ψ being the Weil representation of G' acting on $\mathcal{S}(F^m)$. Let $U^1_{m,n},U^2_{m,n},U^3_{m,n}$ be subgroups of $U_{m,n}$ consisting respectively of

$$u_1(n)=\begin{pmatrix} n & & \\ & 1 & \\ & & n^* \end{pmatrix},\quad u_2(v)=\begin{pmatrix} 1_n & v & -vv^*/2 \\ & 1 & v^* \\ & & 1_n \end{pmatrix},\quad u_3(V)=\begin{pmatrix} 1_n & \mathbf{0} & V \\ & 1 & \mathbf{0} \\ & & 1_n \end{pmatrix}.$$

Here in the definition of $u_2(v)$, we understand v as a vector in F^m written as a column vector in F^n with the last $n-m$ entries being 0. Then $U_{m,n}$ is the semidirect product of $U^1_{m,n}U^2_{m,n}$ and $U^3_{m,n}$,

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and $U^1_{m,n}U^2_{m,n}$ is the semidirect product of $U^1_{m.n}$ and $U^2_{m,n}$. The action of $U_{m,n}$ on the mixed model can be described as follows:

$$\omega_\psi(u_1(n),1)\Phi\otimes\Phi_0(Z,Z_0)=\Phi(Zn)\Phi_0(Z_0), \tag{3.14}$$

$$\omega_\psi(u_3(V),1)\Phi\otimes\Phi_0(Z,Z_0)=\psi(\mathrm{tr}({}^{\mathrm{t}}ZV\sigma_nZ)/2)\Phi\otimes\Phi_0(Z,Z_0). \tag{3.15}$$

We only need to know the value of $\omega_\psi(u_2(v),1)\Phi\otimes\Phi_0(Z,Z_0)$ for some special choice of Z and Z_0:

$$\omega_\psi(u_2(v),1)\Phi\otimes\Phi_0\left(\begin{pmatrix} n & * \\ \mathbf{0} & \mathbf{0}\end{pmatrix},Z_0\right)=\psi({}^{\mathrm{t}}Z_0nv)\Phi\left(\begin{pmatrix} n & * \\ \mathbf{0} & \mathbf{0}\end{pmatrix}\right)\Phi_0(Z_0). \tag{3.16}$$

where $n\in N_m$ and $v\in F^m$.

Similar to the computation of $\omega_\psi[N'_{m,U},\theta']$, we see as $(U^1_{m,n}U^2_{m,n},G')$-module

$$\omega_\psi[U^3_{m,n},\mu]\cong\mathcal{S}(Y'\times F^m), \tag{3.17}$$

where

$$Y'=\{[f_1,\dots,f_n]\mid\langle f_i,f_j\rangle'=0,\ i\leqslant n\text{ or }j\leqslant n\},$$

$\langle\,,\rangle'$ being the alternating form on the symplectic space. We will let Y'_0 be the subspace of Y' consisting of the matrices with $f_1,\dots,f_m$ being of rank m. Similar to the proof of Lemma 3.3, we can prove that as G'-modules

$$\mathcal{S}(Y'\times F^m)[U^1_{m,n}U^2_{m,n},\mu]\cong\mathcal{S}(Y'_0\times F^m)[U^1_{m,n}U^2_{m,n},\mu]. \tag{3.18}$$

From Witt's theorem, the map

$$(g,B)\mapsto g^{-1}\begin{pmatrix}1_m & B\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$$

is a bijection between $N'_{m,U}\backslash Sp(m)\times M_{m,n-m}$ and Y'_0. Thus $\mathcal{S}(Y'_0\times F^m)$ can be identified with $\mathcal{S}(N'_{m,U}\backslash Sp(m)\times M_{m,n-m}\times F^m)$ under this bijection. Given $F_1\otimes F_2\otimes F_3(g,B,Z_0)$ in the space, the action of $G'\times U^1_{m,n}U^2_{m,n}$ (again denoted ω_ψ) is given as follows:

$$\omega_\psi(1,(h,1))F_1\otimes F_2\otimes F_3(g,B,Z_0)=F_1(gh)F_2(B)\omega_\psi(h,1)F_3(Z_0),\quad h\in Sp(m), \tag{3.19}$$

$$\omega_\psi(u_1(n),1)F_1\otimes F_2\otimes F_3(g,B,Z_0)=F_1\left(\begin{pmatrix}n_1^{-1} & \\ & n_1^{*-1}\end{pmatrix}g\right)F_2(n_1^{-1}(B+C))F_3(Z_0), \tag{3.20}$$

$$\omega_\psi(u_2(v),1)F_1\otimes F_2\otimes F_3(g,B,Z_0)=F_1(g)F_2(B)F_3^{g,v}(Z_0), \tag{3.21}$$

where

$$n=\begin{pmatrix}n_1 & C\\ & 1_{n-m}\end{pmatrix}\in N_n$$

and $F_3^{g,v}(Z_0)=\omega_\psi(g,1)^{-1}F'_3(Z_0)$ with $F'_3(Z_0)=\psi({}^{\mathrm{t}}Z_0v)\omega_\psi(g,1)F_3(Z_0)$ (the function is determined by the $N'_{m,U}$-coset of g).

Given $F_1\otimes F_2\otimes F_3$ as above, we define a genuine function F on $G'\times M_{m,n-m}$ by

$$F((g,1),B)=F_1(g)F_2(B)\omega_\psi(g,1)F_3(e_m). \tag{3.22}$$

Lemma 3.5. *The map $\delta:F_1\otimes F_2\otimes F_3\mapsto F$ defines a $(G,U^1_{m,n})$-module isomorphism*

$$\bar{\delta}:\mathcal{S}(N'_{m,U}\backslash Sp(m)\times M_{m,n-m}\times F^m)[U^2_{m,n},\mu]\cong\mathrm{ind}^{G'}_{N'_{m,U}}\theta'\otimes\mathcal{S}(M_{m,n-m}).$$

The action on the second space (denoted η) is given by

$$\eta((h,1))F(g,B)=F(g(h,1),B) \tag{3.23}$$

$$\eta(u_1(n))F(g,B)=F\left(\left(\begin{pmatrix}n_1^{-1} & \\ & n_1^{*-1}\end{pmatrix},1\right)g,n_1^{-1}(B+C)\right). \tag{3.24}$$

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Here

$$n = \begin{pmatrix} n_1 & C \\ & 1_{n-m} \end{pmatrix} \in N_n.$$

Proof. From (3.21) it is easy to see that $\mathcal{S}(N'_{m,U}\backslash Sp(m)\times M_{m,n-m}\times F^m)(U^2_{m,n},\mu)$ lies in the kernel of δ. It is also clear that the image of δ lies in $\mathrm{ind}_{N'_{m,U}}^{G'}\theta'\otimes\mathcal{S}(M_{m,n-m})$. Thus $\bar\delta$ is a well defined map between two spaces in the lemma. The fact that it is a module homomorphism is also clear from the definition of the actions (3.19) and (3.20) and the action of η.

To show the map $\bar\delta$ is surjective, we only need to show that, for any compact open set $\Omega\in M_{m,n-m}$ and any small enough compact neighborhood K_0 of identity in $Sp(m)$, there exists $F_1\otimes F_2\otimes F_3$ such that its image $F(g,B)$ is given by $\theta'(u)\epsilon$ whenever $B\in\Omega$ and $g=(uh,\epsilon)$, with $\epsilon=\pm1$, $u\in N'_{m,U}$ and $h\in K_0$, and $F(g,B)=0$ otherwise. To find such a function $F_1\otimes F_2\otimes F_3$, we simply set F_1 to be the characteristic function of $N'_{m,U}K_0$, F_2 to be the characteristic function of Ω, and F_3 to be a function fixed under the Weil representation of $(k,1)$ with $k\in K_0$, and $F_3(e_m)=1$. A direct computation shows the image $F(g,B)$ is the function specified above.

To show injectivity of the map $\bar\delta$, we will apply Bernstein's localization principle, stated as follows.

Lemma 3.6 [Ber84, p. 58]. *Let $q: X\mapsto T$ be a continuous map of l-spaces. For $t\in T$, let $X_t=q^{-1}(t)$ and consider the space $S^*(X_t)$ as a subspace of $S^*(X)$ (here S^* denotes the space of distributions). Let W be a closed subspace of $S^*(X)$ which is an $\mathcal{S}(T)$-submodule. Then the sum of $W^t=W\cap S^*(X_t)$ over $t\in T$ is dense in W.*

We now return to the proof of Lemma 3.5.

We apply Lemma 3.6 to the case where $X=N'_{m,U}\backslash Sp(m)\times M_{m,n-m}\times F^m$, $T=N'_{m,U}\backslash Sp(m)\times M_{m,n-m}$, and W is the space of the distributions D on X satisfying $\omega_\psi(u_2(v))D=\psi(v_m)D$ for $v\in F^m$. We check W is an $\mathcal{S}(T)$-submodule. Let $F'_1\otimes F'_2\in\mathcal{S}(T)$, then for $D\in W$,

$$\begin{aligned}\omega_\psi(u_2(v))F'_1\otimes F'_2\cdot D(F_1\otimes F_2\otimes F_3) &= D(F'_1\otimes F'_2\omega_\psi(u_2(v))F_1\otimes F_2\otimes F_3)\\ &= D(\omega_\psi(u_2(v))F_1F'_1\otimes F_2F'_2\otimes F_3)\\ &= \psi(v_m)F'_1\otimes F'_2\cdot D(F_1\otimes F_2\otimes F_3).\end{aligned}$$

Here the first equation follows from the definition, the second follows from Equation (3.21), and the third from the fact that $D\in W$. It follows that $F'_1\otimes F'_2\cdot D\in W$, i.e. W is an $\mathcal{S}(T)$-submodule.

Next we consider the space W^t for each $t\in T$. Such a t can be written as (g,B) with $g\in N'_{m,U}\backslash Sp(m)$ and $B\in M_{m,n-m}$. As $X_t\cong F^m$, we will identify $S^*(X_t)$ with the space of distributions on F^m. For $D\in W^t$, let $D'(F_3)=D(\omega_\psi(g,1)^{-1}F_3)$, $F_3(Z_0)\in\mathcal{S}(F^m)$. Then from (3.21) and the fact that $\omega_\psi(u_2(v))D(F_3)=\psi(v_m)D(F_3)$, we get $\psi(v_m)D'(F_3(Z_0))=D'(\psi({}^{\mathrm{t}}Z_0v)F_3(Z_0))$. Thus $D'(F_3)=c_tF_3(e_m)$ for some constant c_t. Therefore $D(F_3)=c_t\omega_\psi(g,1)F_3(e_m)$ whenever $D\in W^t$.

Recall that any function lying in the kernel of δ has the form $\sum_i F_1^i\otimes F_2^i\otimes F_3^i$ with

$$\sum_i F_1^i(g)F_2^i(B)\omega_\psi(g,1)F_3^i(e_m)\equiv 0.$$

Thus for $D\in W^t$ considered as a distribution in W, for any function as above in the kernel of δ,

$$D\biggl(\sum_i F_1^i\otimes F_2^i\otimes F_3^i\biggr)=c_t\sum_i F_1^i(g)F_2^i(B)\omega_\psi(g,1)F_3^i(e_m)=0.$$

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From the localization principle, we see that for any $D \in W$,

$$D\bigg(\sum_i F_1^i \otimes F_2^i \otimes F_3^i\bigg) = 0$$

whenever $\sum_i F_1^i \otimes F_2^i \otimes F_3^i$ lies in the kernel of δ. As W only vanishes on the space $\mathcal{S}(N'_{m,U}\backslash Sp(m) \times M_{m,n-m} \times F^m)(U^2_{m,n}, \mu)$, we see that the space contains the kernel of δ. We get the injectivity of $\bar{\delta}$. The proof of Lemma 3.5 is done. □

We now consider $\operatorname{ind}_{N'_{m,U}}^{G'} \theta' \otimes \mathcal{S}(M_{m,n-m})[U^1_{m,n}, \mu]$. Define an action of N'_m on $\operatorname{ind}_{N'_{m,U}}^{G'} \theta'$ by

$$n : \phi(g) \mapsto \phi(n^{-1}g). \tag{3.25}$$

Lemma 3.7. *As G'-module,* $\operatorname{ind}_{N'_{m,U}}^{G'} \theta' \otimes \mathcal{S}(M_{m,n-m})[U^1_{m,n}, \mu] \cong \operatorname{ind}_{N'_{m,U}}^{G'} \theta'[N'_m, \theta'^{-1}]$.

Proof. We can write an element in $U^1_{m,n}$ as $u_1(n)$ with

$$n = \begin{pmatrix} n_1 & \\ & 1_{n-m} \end{pmatrix} \begin{pmatrix} 1_m & C \\ & 1_{n-m} \end{pmatrix}.$$

Thus $U^1_{m,n}$ is the semidirect product of two subgroups $U^{11}_{m,n}$ and $U^{12}_{m,n}$ with $U^{11}_{m,n} \cong N_m$ and $U^{12}_{m,n} \cong M_{m,n-m}$. It is clear from Lemma 3.1 that the map

$$F(g, B) \mapsto F_0(g) = \int_{M_{m,n-m}} F(g, B)\, dB \tag{3.26}$$

defines a (G', N_m)-module isomorphism between $\operatorname{ind}_{N'_{m,U}}^{G'} \theta' \otimes \mathcal{S}(M_{m,n-m})[U^{12}_{m,n}, \mu]$ and $\operatorname{ind}_{N'_{m,U}}^{G'} \theta'$, where the action of N_m on the second space is given by

$$\eta'(n_1)F_0(g) = F_0\left(\left(\begin{pmatrix} n_1^{-1} & \\ & n_1^{*-1} \end{pmatrix}, 1\right) g\right).$$

As $N'_m/N'_{m,U} \cong N_m$, it is then clear that

$$\operatorname{ind}_{N'_{m,U}}^{G'} \theta' \otimes \mathcal{S}(M_{m,n-m})[U^1_{m,n}, \mu] \cong \operatorname{ind}_{N'_{m,U}}^{G'} \theta'[N_m, \theta^{-1}] \cong \operatorname{ind}_{N'_{m,U}}^{G'} \theta'[N'_m, \theta'^{-1}]$$

as G'-modules. □

As $\operatorname{ind}_{N'_{m,U}}^{G'} \theta'[N'_m, \theta'^{-1}] \cong \operatorname{ind}_{N'_m}^{G'} \theta'$, the isomorphism (2.8) follows from Lemmas 3.5 and 3.7 and isomorphisms (3.17) and (3.18).

The proof of Proposition 2.2 is similar to that of Proposition 2.1 and will be skipped.

3.3 When $G = SO(n+1, n)$, $G' = \widetilde{Sp}(m)$, $m > n$

The proof of Proposition 2.3 is similar to that of Proposition 2.1. We will only give a sketch. We will let U^i_n be the groups $U^i_{n,n}$ $(i = 1, 2, 3)$ in § 3.2.

To prove the isomorphism (2.19), we use the model of ω_ψ given by (3.1)–(3.3). Let $H'_{2,U}$ be the intersection of H'_2 with the unipotent radical of the Siegel parabolic subgroup of G', and $H'_{2,L}$ the intersection with its Levi part. Then H'_2 is a semidirect product of $H'_{2,U}$ and $H'_{2,L}$. As in § 3.1, we get an isomorphism of $(G, H'_{2,L})$-modules,

$$\omega_\psi[H'_{2,U}, \chi'_2] \cong \mathcal{S}(Y),$$

where

$$Y = \{(y_1, \dots, y_m) \mid \langle y_i, y_j \rangle = 0, i \leqslant n+1 \text{ except } \langle y_{n+1}, y_{n+1} \rangle = 1\},$$

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and $(G, H'_{2,L})$ acts on $\mathcal{S}(Y)$ by

$$(g,(h,1)) : \Phi(Z) \mapsto \Phi(g^{-1}Zh), \quad g \in G, (h,1) \in H'_{2,L}.$$

As before

$$\mathcal{S}(Y)[H'_{2,L}, \chi'_2] \cong \mathcal{S}(Y_0)[H'_{2,L}, \chi'_2],$$

where Y_0 consists of matrices where rank $[y_1, \dots, y_{n+1}] = n+1$. From Witt's theorem, we see there is a bijection between $U_n^3\backslash G \times M_{n,m-n-1}$ and Y_0 given by

$$(g, B) \mapsto g^{-1}[e_1, \dots, e_{n+1}, B], \quad g \in G, B \in M_{n,m-n-1},$$

where we consider B to be a matrix in $M_{2n+1,m-n-1}$ whose last $n+1$ rows are 0. We will identify $\mathcal{S}(Y_0)$ with $\mathcal{S}(U_n^3\backslash G \times M_{n,m-n-1})$ through this bijection.

Similar to the proof of Lemma 3.7, we can establish an isomorphism of G-modules:

$$\mathcal{S}(Y_0)[H'_{2,L}, \chi'_2] \cong \operatorname{ind}_{U_n^3}^G 1[U_n, \mu^{-1}],$$

where U_n acts on $\operatorname{ind}_{U_n^3}^G 1$ by $u : \phi(g) \mapsto \phi(u^{-1}g)$. As $\operatorname{ind}_{U_n^3}^G 1[U_n, \mu^{-1}] \cong \operatorname{ind}_{U_n}^G \mu$, we get the isomorphism (2.19).

To prove the isomorphism (2.20), we use the model given by (3.13)–(3.16). Then as $(G, U_n^1U_n^2)$-modules,

$$\omega_\psi[U_n^3, \mu] \cong \mathcal{S}(Y' \times F^m),$$

where

$$Y' = \{[f_1, \dots, f_n] \mid \langle f_i, f_j\rangle' = 0, \ \forall i, j\}.$$

As before, we can replace Y' by Y'_0 consisting of the matrices of rank n.

Let $J_{m,n}$ be the subgroup of $J_{m,n}$ fixing the space $V'_{m,n}$ defined in § 2.2. Let $\tilde{J}_{m,n}$ be its inverse image in G'. Then $\tilde{J}_{m,n}$ is a normal subgroup of H'_1. There is an isomorphism $H'_1/\tilde{J}_{m,n} \mapsto N_n$ defined by (2.16).

From Witt's theorem, there is a bijection from $J_{m,n}\backslash Sp(m)$ to Y'_0 given by $g \mapsto g^{-1}[e_1, \dots, e_n]$. Thus

$$\omega_\psi[U_n, \mu] \cong \mathcal{S}(J_{m,n}\backslash Sp(m) \times F^m)[U_n^1U_n^2, \mu].$$

Given $F_1 \otimes F_2(g, Z) \in \mathcal{S}(J_{m,n}\backslash Sp(m) \times F^m)$, we define a genuine function F on $G' \times F^{m-n}$ by setting

$$F((g,1), Z^-) = F_1(g)\omega_\psi(g,1)F_2\left(\begin{pmatrix} e_m \\ Z^- \end{pmatrix}\right), \quad Z^- \in F^{m-n}. \tag{3.27}$$

As in Lemma 3.5, we can show that this map gives an isomorphism between G'-modules:

$$\mathcal{S}(J_{m,n}\backslash Sp(m) \times F^m)[U_n^2, \mu] \cong \operatorname{ind}_{\tilde{J}_{m,n}}^{G'} \chi'_1.$$

Moreover if we define the action of H'_1 on $\operatorname{ind}_{\tilde{J}_{m,n}}^{G'} \chi'_1$ as $h : \phi(g) \mapsto \phi(h^{-1}g)$, we can get an isomorphism between G'-modules:

$$\mathcal{S}(J_{m,n}\backslash Sp(m) \times F^m)[U_n^1U_n^2, \mu] \cong \operatorname{ind}_{\tilde{J}_{m,n}}^{G'} \chi'_1[H'_1, {\chi'_1}^{-1}].$$

As $\operatorname{ind}_{\tilde{J}_{m,n}}^{G'} \chi'_1[H'_1, {\chi'_1}^{-1}] \cong \operatorname{ind}_{H'_1}^{G'} \chi'_1$ as G'-modules, we get the isomorphism (2.20).

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3.4 When $G = GL_n$, $G' = GL_2$

To prove Proposition 2.7, we will use the following model for the Weil representation: for $\Phi \in \mathcal{S}(M_{n,2})$,

$$\omega_\psi(g,1)\Phi[\mathbf{v},\mathbf{w}] = \Phi[g^{-1}\mathbf{v},\, {}^{\mathrm{t}}g\mathbf{w}], \quad g \in G, \tag{3.28}$$

$$\omega_\psi\left(1, \begin{pmatrix} 1 & \\ & a \end{pmatrix}\right)\Phi[\mathbf{v},\mathbf{w}] = \Phi[\mathbf{v}, a\mathbf{w}], \tag{3.29}$$

$$\omega_\psi\left(1, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right)\Phi[\mathbf{v},\mathbf{w}] = \int \Phi[\mathbf{v}',\mathbf{w}']\psi({}^{\mathrm{t}}\mathbf{v}'\mathbf{v} + {}^{\mathrm{t}}\mathbf{w}'\mathbf{w})\, d\mathbf{v}'\, d\mathbf{w}', \tag{3.30}$$

$$\omega_\psi\left(1, \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right)\Phi[\mathbf{v},\mathbf{w}] = \Phi[\mathbf{v},\mathbf{w}]\psi(x^t\mathbf{v}\mathbf{w}). \tag{3.31}$$

From (3.31), we see that $\omega_\psi[N_2,\theta]$ is isomorphic as a G-module to the space of Schwartz functions on the closed subset $M_{n,2}^2$ of $M_{n,2}$ consisting of $[\mathbf{v},\mathbf{w}]$ with ${}^{\mathrm{t}}\mathbf{v}\mathbf{w} = 1$, the G-action being described as in (3.28). The group G acts on this closed set transitively by $g \circ [\mathbf{v},\mathbf{w}] = [g^{-1}\mathbf{v},\, {}^{\mathrm{t}}g\mathbf{w}]$, and the stabilizer of $[e_1, e_1]$ is H. Thus if we define, for a function $\Phi \in \mathcal{S}(M_{n,2}^2)$, $\eta(\Phi)(g) = \Phi[g^{-1}e_1,\, {}^{\mathrm{t}}ge_1]$, then η is a G-module isomorphism from $\mathcal{S}(M_{n,2}^2)$ to $\mathrm{ind}_H^G 1$. This gives isomorphism (2.32).

To show the isomorphism (2.33), we use another model for the Weil representation ω_ψ: for $g \in G$, $g' \in G'$, and $\Phi \in \mathcal{S}(M_{2,n})$,

$$\omega_\psi(g,g')\Phi[X] = \Phi(g'^{-1}Xg), \quad X \in M_{2,n}, \tag{3.32}$$

where $A \in M_{m,m}$ and $B \in M_{m,n-m}$. The proof is similar to the computations done in the proof of Proposition 2.1, and will be skipped.

4. The relevant orbits

We start the proof of the relative trace identity (2.9). The results are stated in § 1.8. We will use the notations introduced in §§ 1.8 and 2.1.

In this section, we give a description of the set of representatives of the relevant $R_{m,n} \times U_{m,n}$ orbits in $G_n = SO(n+1,n)$, and match the orbits with the relevant $N'_m \times N'_m$ orbits in $G'_m = Sp(m)$.

To classify the relevant orbits in $SO(n+1,n)$, we solve an equivalent problem of classifying the relevant orbits in the subset Y_0 (introduced in § 3.1) of the set of $(2n+1) \times m$ matrices. We make a change of notation and denote Y_0 by X. Explicitly, X is the image of $SO(n+1,n)$ under the map

$$\phi : g \to Z_g = g^{-1}[e_1, \dots, e_{m-1}, e_{n+1}].$$

The group $U_{m,n} \times N_m$ acts on X by

$$(u,n) : x \mapsto u^{-1}xn; \quad u \in U_{m,n},\ n \in N_m.$$

We say an orbit of $x \in X$ under the action of $U_{m,n} \times N_m$ is relevant if $u^{-1}xn = x$ implies $\mu(u)\theta(n) = 1$.

Lemma 4.1. *The map ϕ induces a bijection between the $R_{m,n} \times U_{m,n}$ orbits in $SO(n+1,n)$ and the $U_{m,n} \times N_m$ orbits in X. It also induces a bijection between the relevant orbits.*

Proof. Let g_1 and g_2 be in $SO(n+1,n)$. Recall that $R_{m,n}/R'_{m,n} \cong N_m$ with the projection ρ from $R_{m,n}$ to N_m (see § 2.1). From the definition of ϕ and (3.12), we see for $r \in R_{m,n}$, $u \in U_{m,n}$,

$$r^{-1}g_1u = g_2 \quad \text{if and only if } u^{-1}\phi(g_1)\rho(r) = \phi(g_2). \tag{4.1}$$

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Thus ϕ gives a bijection of orbits. Set $g_1 = g_2 = g$ in (4.1). We see (r, u) lies in the stabilizer of g if and only if $(u, \rho(r))$ fixes $\phi(g)$. Since $\chi(r)\mu(u) = \theta(\rho(r))\mu(u)$ by definition, we see $\chi(r)\mu(u)$ being trivial on the stabilizer of g is equivalent to $\theta(n)\mu(u)$ being trivial on the stabilizer (u, n) of $\phi(g)$. The map ϕ induces a bijection between relevant orbits. □

We now describe the relevant orbits in X. The group $N_l \times N_l$ acts on GL_l by $(n_1, n_2) : g \to n_1^{-1} g n_2$. We say the orbit of g is relevant if $n_1^{-1} g n_2 = g$ implies $\theta(n_1 n_2) = 1$. Let S_l be a complete set of representatives for these relevant orbits. For $g \in S_l$, $l < m$, let $s^{\pm}(g)$ be an element in X given by

$$s^{\pm}(g) = \begin{pmatrix} \mathbf{0} & \pm I_{m-l} \\ g & \mathbf{0} \end{pmatrix}, \tag{4.2}$$

where

$$I_{m-l} = [e_{l+2}, e_{l+3}, \ldots, e_m, e_{n+1}]. \tag{4.3}$$

If $l = m$, then let

$$s(g) = \begin{pmatrix} \hat{g} \\ \mathbf{0} \\ g \end{pmatrix} \in X, \quad \text{where } g \in GL_m, \quad \hat{g} = \begin{pmatrix} \mathbf{0} & 0 \\ \mathbf{0} & 1/2 \end{pmatrix} g^{-1} \sigma_m.$$

Then we have the following proposition.

Proposition 4.2. *A complete set of representatives for the relevant $U_{m,n} \times N_m$ orbits in X is given by*

$$\bigcup_{l=0}^{m-1} \{s^{\pm}(g) \mid g \in S_l\} \cup \{s(g) \mid g \in S_m\}.$$

Although we do not need a description of the complete set S_l of representatives of the relevant orbits, such a description is well known (see [JR92]). It is given by $\{I'_l \sigma_l w_\nu \mathbf{a}_\nu\}$, where I'_l is the diagonal matrix $\mathrm{diag}[1, -1, \ldots, (-1)^{l+1}]$, ν is a partition of l, w_ν is the longest Weyl element for the standard Levi subgroup of GL_l corresponding to ν, and $\mathbf{a}_\nu$ is an element in the center of this standard Levi subgroup.

The $N'_m \times N'_m$ relevant orbits in $Sp(m)$ have been classified in [Mao93]. We have our next proposition.

Proposition 4.3. *A complete system of representatives of the relevant orbits in $Sp(m)$ is given by*

$$\bigcup_{l=0}^{m-1} \left\{ t^{\pm}(g) = \left. \begin{pmatrix} & & -g^* \\ & \pm 1_{2m-2l} & \\ g & & \end{pmatrix} \right| g \in S_l \right\} \cup \left\{ t(g) = \left. \begin{pmatrix} & -g^* \\ g & \end{pmatrix} \right| g \in S_m \right\}.$$

From Propositions 4.2 and 4.3 it is easy to describe a bijection between the relevant orbits. We denote by $\{o\}$ the orbit containing o as a representative. Lemma 4.1 gives a bijection ϕ from the set of relevant orbits in $SO(n+1, n)$ to the set of relevant orbits in X. Let ϕ' be a map from the set of orbits in X to the set of orbits in $Sp(m)$, such that $\phi'(\{s^{\pm}(g)\}) = \{t^{\pm}(g)\}$ and $\phi'(\{s(g)\}) = \{t(g)\}$. Then from Propositions 4.2 and 4.3 we obtain the following.

Proposition 4.4. *The map ϕ' is a bijection of relevant orbits. The map $\iota = \phi' \circ \phi$ defines a bijection from the set of relevant $R_{m,n} \times U_{m,n}$-orbits in $SO(n+1, n)$ to the set of relevant $N'_m \times N'_m$-orbits in $Sp(m)$.*

The proofs of Propositions 4.2 and 4.3 involve the careful consideration of the stabilizer of the orbits. To save space, the proofs are not included here. Instead we simply describe the stabilizer of the relevant orbits.

For $\{o\}$ a relevant orbit in $SO(n+1,n)$, let $U_{m,n,o} = o^{-1}R_{m,n}o \cap U_{m,n}$. For $\{o'\}$ a relevant orbit in $Sp(m)$, let $N'_{m,o'} = o'^{-1}N'_m o \cap N'_m$.

For $l \leqslant m$, let ρ' and η' be injections from N_l and $Sp(l)$ respectively to $Sp(m)$ given by

$$\rho'(u) = \begin{pmatrix} u & & \\ & 1_{2m-2l} & \\ & & u^* \end{pmatrix}, \quad \eta'(h) = \begin{pmatrix} 1_{m-l} & & \\ & h & \\ & & 1_{m-l} \end{pmatrix}. \tag{4.4}$$

For $l \leqslant n$, let ρ'' and η be injections from N_l and $SO(l+1,l)$ respectively to $SO(n+1,n)$ given by

$$\rho''(u) = \begin{pmatrix} u & & \\ & 1_{2n+1-2l} & \\ & & u^* \end{pmatrix}, \quad \eta(g) = \begin{pmatrix} 1_{n-l} & & \\ & g & \\ & & 1_{n-l} \end{pmatrix}. \tag{4.5}$$

For $g \in GL_l$, let $U_g = N_l \cap g^{-1}N_l g$. Let $U^1_{m,n}$ be the subgroup of $U_{m,n}$ consisting of elements satisfying

$$\rho(u) = \begin{pmatrix} 1 & \mathbf{0} \\ & n' \end{pmatrix}$$

with $n' \in N_m$.

LEMMA 4.5. *When $o' = t^{\pm}(g)$, $N'_{m,o'}$ is the direct product $\eta'(N'_{n-l})\rho'(U_g)$. When $o' = t(g)$, $N'_{m,o'} = \rho'(U_g)$.*

When $\phi(o) = s^{\pm}(g)$, $U_{m,n,o}$ is the direct product $\eta(U^1_{m-l,n-l})\rho''(U_g)$. When $\phi(o) = s(g)$, $U_{m,n,o} = \rho''(U_g)$.

5. The orbital integrals

5.1 Expansion of distributions into orbital integrals

We prove the identities (1.13) and (1.15).

PROPOSITION 5.1. *Choose the measure so that $\mathbf{A}/F$ has volume 1, and let the Haar measures on $U_{m,n}(\mathbf{A})$ and $N'_m(\mathbf{A})$ be the products of additive measures on $\mathbf{A}$. Then if $f = \otimes f_v$, $I_{m,n}(f) = \sum_o \prod_v I_o(f_v)$. If $\tilde{f} = \otimes \tilde{f}_v$, $J_m(\tilde{f}) = \sum_{o'} \prod_v J_{o'}(\tilde{f}_v)$.*

Proof. Let $I_{m,n}(f)$ be the integral defined in (1.12). From the definition of K_f, we get $I_{m,n}(f)$ equals

$$\sum_o \int_{R_{m,n}(\mathbf{A})} \int_{o^{-1}R_{m,n}(F)o\cap U_{m,n}(F)\backslash U_{m,n}(\mathbf{A})} f(r^{-1}ou)\chi^{-1}(r)\mu^{-1}(u)\,dr\,du.$$

The sum is taken over the $R_{m,n} \times U_{m,n}$ orbits in $SO(n+1,n)(F)$. The above integral has a factor

$$\int_{o^{-1}R_{m,n}(F)o\cap U_{m,n}(F)\backslash o^{-1}R_{m,n}(\mathbf{A})o\cap U_{m,n}(\mathbf{A})} \mu^{-1}(u)\,du.$$

If o is not relevant, this integral is 0. If the orbit o is relevant, the above integrand is 1. The integration equals the volume of

$$o^{-1}R_{m,n}(F)o \cap U_{m,n}(F)\backslash o^{-1}R_{m,n}(\mathbf{A})o \cap U_{m,n}(\mathbf{A}),$$

which equals 1. The integral $I_{m,n}(f)$ equals

$$\sum_o \int_{R_{m,n}(\mathbf{A})} \int_{o^{-1}R_{m,n}(\mathbf{A})o\cap U_{m,n}(\mathbf{A})\backslash U_{m,n}(\mathbf{A})} f(r^{-1}ou)\chi^{-1}(r)\mu^{-1}(u)\,dr\,du$$

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where the sum is taken over the relevant orbits. When $f = \otimes f_v$, the integration factors into local orbital integrals defined in (1.14). We get the identity (1.13). Similarly one can show the identity (1.15). The formal argument can be justified using the convergence proved in [Jac95]. $\square$

From the above proposition, to prove the relative trace identity (2.9), we need to compare the orbital integrals $I_o(f_v)$ and $J_{o'}(\tilde{f}_v)$ when $o' = \iota(o)$. We will relate the orbital integrals $I_o(f_v)$ and $J_{o'}(\tilde{f}_v)$ with linear functionals in the space of Weil representations. From now on, we will fix a local place v and drop it from the notation.

5.2 The orbital integral $I_o(f)$

Lemma 5.2. *Given $f \in C_c^\infty(SO(n+1,n))$, there exists $\Phi = \Phi_f \in C_c^\infty(M_{2n+1,m})$, such that*

$$\Phi(g^{-1}[e_1,\dots,e_{m-1},e_{n+1}]) = \int_{R'_{m,n}} f(r^{-1}g)\,dr. \tag{5.1}$$

Proof. By definition $R'_{m,n}$ is the subgroup that fixes the vectors $e_1,\dots,e_{m-1},e_{n+1}$. Thus the identity (5.1) defines a function $\Phi_0 \in C_c^\infty(X)$ (here X is the variety Y_0 defined in § 3.1). Since X is an open subset of the closed variety Y defined in § 3.1, $C_c^\infty(X)$ embeds in $C_c^\infty(Y)$. The map $C_c^\infty(M_{2n+1,m}) \mapsto C_c^\infty(Y)$ through restriction is surjective. Thus the lemma is proved. $\square$

As a corollary, we have the following.

Lemma 5.3. *Given $f \in C_c^\infty(SO(n+1,n))$, there exists $\Phi = \Phi_f \in C_c^\infty(M_{2n+1,m})$, such that*

$$\int_{N_m} \Phi(g^{-1}[e_1,\dots,e_{m-1},e_{n+1}]n)\theta(n^{-1})\,dn = \int_{R_{m,n}} f(r^{-1}g)\chi^{-1}(r)\,dr. \tag{5.2}$$

Proof. The integral on the right-hand side can be unwound to $\int_{R_{m,n}/R'_{m,n}}\int_{R'_{m,n}}$. The lemma follows from the fact that $R_{m,n}/R'_{m,n} \cong N_m$ and the identity (3.12). $\square$

Let f and Φ be as in Lemma 5.3. Then $I_o(f)$ equals

$$\int_{o^{-1}R_{m,n}o\cap U_{m,n}\backslash U_{m,n}} \int_{N_m} \Phi(u^{-1}\phi(o)n)\theta^{-1}(n)\mu^{-1}(u)\,du\,dn. \tag{5.3}$$

which we will denote by $I_{\phi(o)}(\Phi)$.

Using the model for Weil representation (Equations (3.1)–(3.3)), we can rewrite the above integral $I_{\phi(o)}(\Phi)$ as follows.

Lemma 5.4. *Let f and Φ satisfy the identity (5.2), then*

$$I_o(f) = I_{\phi(o)}(\Phi) = \int_{U_{m,n,\phi(o)}\backslash U_{m,n}} \int_{N_m} \omega_\psi(u,\delta(n))\Phi(\phi(o))\theta^{-1}(n)\mu^{-1}(u)\,du\,dn. \tag{5.4}$$

Here we use the notation

$$\delta(g) = \left(\begin{pmatrix} g & \\ & g^* \end{pmatrix}, 1\right) \in \widetilde{Sp}(m), \quad g \in GL_m.$$

5.3 The orbital integral $J_{o'}(\tilde{f})$

Recall that we define $I_m \in M_{2n+1,m}$ to be the element given by (4.3) when $l = 0$.

Lemma 5.5. *Given $\Phi \in \mathcal{S}(M_{2n+1,m})$, there is a function $\tilde{f} = \tilde{f}_\Phi \in \mathcal{S}(\widetilde{Sp}(m))$, such that*

$$\int_{N'_m} \tilde{f}(\tilde{n}^{-1}g)\theta'(n)\,dn = \int_{R^0_{m+1,n}\backslash U_{m,n}} \omega_\psi(u,g)\Phi(I_m)\mu^{-1}(u)\,du. \tag{5.5}$$

Proof. Denote the right-hand side of (5.5) as $F(g)$. Let $u \in U_{m,n}$, then $\rho(u) \in N_{m+1}$ (§ 2.1). Let $n \in N_m$ and fix a $u = u_n$ such that

$$\rho(u_n) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & n^{-1} \end{pmatrix}.$$

Then $u_n^{-1} I_m = I_m n$. Thus from the identity (3.1),

$$\omega_\psi(u_n u, g)\Phi(I_m) = \omega_\psi(u, \delta(n)g)\Phi(I_m) = \omega_\psi(u, g)\Phi(I_m n).$$

A simple change of variable gives $F(\delta(n)g) = \mu(u_n)F(g) = \theta(n)F(g)$ for $n \in N_m$. From Equation (3.3), we see further that if

$$\tilde{n} = \left(\begin{pmatrix} 1_n & V \\ & 1_n \end{pmatrix}, 1 \right) \in N'_m,$$

then

$$\omega_\psi(u, \tilde{n}g)\Phi(I_m) = \psi(V_{m,1}/2)\omega_\psi(u, g)\Phi(I_m) = \theta'(n)\omega_\psi(u, g)\Phi(I_m).$$

We get in this case $F(\tilde{n}g) = \theta'(n)F(g)$ also. Thus for $n \in N'_m$, $F(\tilde{n}g) = \theta'(n)F(g)$.

We next show that the function $F(g)$ is also a Schwartz function on $N'_m \backslash \widetilde{Sp}(m)$. To do so we consider $F(\mathbf{a}, 1)$, where $\mathbf{a} = \operatorname{diag}[a_1, \ldots, a_m, a_m^{-1}, \ldots, a_1^{-1}]$. Assume $\Phi = \otimes \Phi_{i,j}$ where $\Phi_{i,j}$ is a function on the (i,j)th entry in $M_{2n+1,m}$. A direct computation using (3.1) shows that $F(\mathbf{a}, 1)$ equals

$$\prod_{i=1}^{m} \left[|a_i|^{-i} \left(\prod_{j=1}^{i-1} \hat{\Phi}_{j,i}(0) \right) \left(\prod_{j=i+2}^{2n+1} \Phi_{j,i}(0) \right) \Phi_{i+1,i}(a_i) \hat{\Phi}_{i,i}(a_i^{-1}) \right] \left(\frac{\Phi_{n+1,m}(a)}{\Phi_{m+1,m}(a)} \right),$$

where $\hat{\Phi}_{i,j}$ denotes the Fourier transform of $\Phi_{i,j}$. The above is clearly a Schwartz function on $(F^\times)^n$. Our claim now follows from the next lemma. ◻

LEMMA 5.6. *Let $F(g)$ be a Schwartz function on $N'_m \backslash \widetilde{Sp}(m)$ satisfying $F(\tilde{n}g) = \theta'(n)F(g)$. Let $g = n\mathbf{a}k$ be the Iwasawa decomposition. Define $\tilde{f}(g)$ so that $\tilde{f}(\tilde{n}(\mathbf{a}k, 1)) = \lambda(n)F(\mathbf{a}k, 1)$ where λ is any Schwartz function on N'_m such that*

$$\int \lambda(n^{-1})\theta'(n)\, dn = 1.$$

Then the function $\tilde{f}(g)$ is a Schwartz function on $\widetilde{Sp}(m)$, satisfying

$$\int_{N'_m} \tilde{f}(\tilde{n}^{-1}g)\theta'(n)\, dn = F(g).$$

Proof. When $g = \tilde{n}_0(\mathbf{a}k, 1)$, the integral is just

$$\int \lambda(\tilde{n}^{-1}\tilde{n}_0)F(\mathbf{a}k, 1)\theta'(n)\, dn.$$

A change of variable $n \mapsto nn_0$ gives the equation in the lemma. We now show $\tilde{f}(g) \in \mathcal{S}(\widetilde{Sp}(m))$. This is clear when the place v is a p-adic place, as in this case $\mathcal{S}(\widetilde{Sp}(m)) = C_c^\infty(\widetilde{Sp}(m))$. Now assume v is archimedean. We may as well consider the statement for $Sp(m)$ instead of the covering group $\widetilde{Sp}(m)$. From the definition of the Schwartz function [Cas89], we need to show for all X differential operators in the enveloping Lie algebra of $Sp(m)$, the function

$$Xf(g) = \frac{d}{dt} f(g \exp(tX))|_{t=0}$$

has norm bounded by $\|g\|^{-r}$ for any $r > 0$ (here we choose the algebraic norm $\|g\|$ on $Sp(m)$ so that $\|gh\| \leqslant \|g\|\|h\|$, see [Cas89]). Fix k in the maximal compact subgroup K' of $Sp(m)$ and consider elements of the form $g = n_0\mathbf{a}k$. Given X, there exists a finite number of operators X_i and polynomials $f_i(k)$ on K', such that

$$\mathrm{Ad}(k)X = \sum_i f_i(k)X_i.$$

Then

$$\begin{aligned} Xf(n_0\mathbf{a}k) &= \frac{d}{dt}f(n_0\mathbf{a}k\exp(tX))|_{t=0} \\ &= \frac{d}{dt}f(n_0\mathbf{a}\exp(t\,\mathrm{Ad}(k)X)k)|_{t=0} \\ &= \sum_i f_i(k)\frac{d}{dt}f(n_0\mathbf{a}\exp(tX_i)k)|_{t=0}. \end{aligned}$$

For X any operator in the enveloping Lie algebra, let

$$X\circ f(n_0\mathbf{a}k) = \frac{d}{dt}f(n_0\mathbf{a}\exp(tX)k)|_{t=0}.$$

Thus we only need to show for any operator X, and any $r > 0$, that

$$|X\circ f(n_0\mathbf{a}k)| = \left|\frac{d}{dt}f(n_0\mathbf{a}\exp(tX)k)|_{t=0}\right| \ll \|n_0\mathbf{a}\|^{-r}. \tag{5.6}$$

If X is in the Lie algebra of the K', then the function $X\circ f(n_0\mathbf{a}k)$ has the form $\lambda(n)F_X(\mathbf{a}k)$ where F_X is a Schwartz function. If X is in the Lie algebra of the torus of $Sp(m)$, then again $X\circ f(n_0\mathbf{a}k)$ has the form $\lambda(n)F_X(\mathbf{a}k)$ where F_X is a Schwartz function. If X is in the Lie algebra of N'_m, then $X = X_\alpha$ where α is a positive root of $Sp(m)$. Then $X\circ(fn_0\mathbf{a}k) = \alpha(\mathbf{a})F(\mathbf{a}k)X\lambda(n_0)$. Thus we get $X\circ f$ always has the form $\lambda_X(n_0)F_X(\mathbf{a}k)$ where both λ_X and F_X are Schwartz functions. We get

$$|X\circ f(n_0\mathbf{a}k)| = |\lambda_X(n_0)F_X(\mathbf{a}k)| \ll \|n_0\|^{-r}\|\mathbf{a}\|^{-r} \ll \|n_0\mathbf{a}\|^{-r}.$$

We have shown Lemma 5.6. □

From Lemma 5.5 we get immediately the following relation between the orbital integrals on $\widetilde{Sp}(m)$ and the orbital integrals on X.

Lemma 5.7. *Let Φ, $\tilde{f}$ satisfy the identity (5.5) in Lemma 5.5. Then for any relevant orbit $\{o'\}$, $J_{o'}(\tilde{f}) = J_{o'}(\Phi)$ where*

$$J_{o'}(\Phi) = \int_{N'_{m,o'}\backslash N'_m}\int_{R^0_{m+1,n}\backslash U_{m,n}} \omega_\psi(u,(o',1)\tilde{n})\Phi(I_m)\mu^{-1}(u)\theta'(n^{-1})\,dn\,du. \tag{5.7}$$

The fixator $N'_{m,o'}$ is described in Lemma 4.5. It turns out that the expression (5.7) almost equals (5.4) when $\iota(o) = o'$. This is the key relation for the proofs of Theorems 1.1, 1.2 and 1.3. We prove this relation in the next section.

6. Comparison of orbital integrals

We prove Theorem 1.1 in this section. Fix a local place v, and fix the Haar measure on F_v to be self dual with respect to the additive character ψ. We will drop the reference to v in the notations. The theorem follows from the following proposition.

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Proposition 6.1. *There is a function $\Delta(o)$ defined for $o = s^{\pm}(g)$ or $s(g)$, such that for any $\Phi \in \mathcal{S}(M_{2n+1,m})$, we have*

$$J_{t^{\pm}(g)}(\Phi) = \Delta(s^{\pm}(g)) I_{s^{\pm}(g)}(\Phi), \quad g \in S_l,\ l < m, \tag{6.1}$$

$$J_{t(g)}(\Phi) = \Delta(s(g)) I_{s(g)}(\Phi), \quad g \in S_m. \tag{6.2}$$

The definition of $\Delta(o)$ is given in the proof, which is broken down into subsections in the following. We treat here only the case relating $J_{t^+(g)}(\Phi)$ with $I_{s^+(g)}(\Phi)$. The other cases can be done similarly.

6.1 Simplification of the problem

We first need to simplify the problem at hand. We let $U^0_{m,n,s^+(g)}$ be the group

$$U^0_{m,n,s^+(g)} = \{u \in U_{m,n} \mid u^{-1}s^+(g) = s^+(g)\}.$$

Then in fact $U^0_{m,n,s^+(g)} = \eta(R^0_{m-l+1,n-l})$. Let $N_{m,s^+(g)}$ be the group

$$N_{m,s^+(g)} = \{n \in N_m \mid \exists u \in U_{m,n}, u^{-1}s^+(g) = s^+(g)n\}.$$

It is clear that for $\Phi \in \mathcal{S}(M_{2n+1,m})$,

$$\int_{N_{m,s^+(g)}} \omega_\psi(1, \delta(n))\Phi(s^+(g))\theta^{-1}(n)\,dn = \int_{U^0_{m,n,s^+(g)} \backslash U_{m,n,s^+(g)}} \omega_\psi(u,1)\Phi(s^+(g))\mu^{-1}(u)\,du.$$

Thus we get from (5.4)

$$I_{s^+(g)}(\Phi) = \int_{\eta(R^0_{m-l+1,n-l})\backslash U_{m,n}} \int_{N_{m,s^+(g)}\backslash N_m} \omega_\psi(u, \delta(n))\Phi'(s^+(g))\theta^{-1}(n)\mu^{-1}(u)\,du\,dn.$$

The group $U_{m,n}$ has a normal subgroup $U_{l,n}$ with the quotient group isomorphic to $\eta(U_{m-l,n-l})$. The above equation can be written as

$$\begin{aligned} I_{s^+(g)}(\Phi) = \int_{u_2 \in R^0_{m-l+1,n-l}\backslash U_{m-l,n-l}} \int_{N_{m,s^+(g)}\backslash N_m} \int_{u_1 \in U_{l,n}} \omega_\psi(u_1\eta(u_2), \delta(n)) \\ \times \Phi(s^+(g))\theta^{-1}(n)\mu^{-1}(u_1\eta(u_2))\,du_1\,du_2\,dn. \end{aligned} \tag{6.3}$$

Recall that $\tilde{N}'_m = \delta(N_m)N'_{m,U}$ where $N'_{m,U}$ is the Siegel unipotent subgroup of N'_m. Let $N'_{m,t^+(g),U} = N'_{m,t^+(g)} \cap N'_{m,U}$. Then a direct calculation shows that $N'_{m,t^+(g),U}\backslash N'_{m,t^+(g)} \cong \delta(N_{m,s^+(g)})$. Observe also that the intersection of $R^0_{m+1,n}$ with $U_{l,n}$ is $R^0_{l+1,n}$; and the intersection of $R^0_{m+1,n}$ with $\eta(U_{m-l,n-l})$ is just $\eta(R^0_{m-l+1,n-l})$. Thus Equation (5.7) can be written as

$$\begin{aligned} J_{t^+(g)}(\Phi) = \int_{u_2 \in R^0_{m-l+1,n-l}\backslash U_{m-l,n-l}} \int_{n \in N_{m,s^+(g)}\backslash N_m} \int_{n' \in N'_{m,t^+(g),U}\backslash N'_{m,U}} \\ \int_{u_1 \in R^0_{l+1,n}\backslash U_{l,n}} \omega_\psi(u_1\eta(u_2), (t^+(g),1)\tilde{n'}\delta(n)) \\ \times \Phi(I_m)\mu^{-1}(u_1\eta(u_2))\theta'(n'^{-1})\theta^{-1}(n)\,dn\,dn'\,du_1\,du_2. \end{aligned} \tag{6.4}$$

Thus to prove Proposition 6.1, we only need to show that the following key identity holds for

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any $\Phi \in \mathcal{S}(M_{2n+1,m})$:

$$\int_{N'_{m,t^+(g),U}\backslash N'_{m,U}} \int_{R^0_{l+1,n}\backslash U_{l,n}} \omega_\psi(u,(t^+(g),1)\tilde{n'})\Phi(I_m)\theta'(n'^{-1})\mu^{-1}(u)\,du\,dn'$$

$$= \Delta^+(g)\int_{u\in U_{l,n}} \omega_\psi(u,\tilde{1}_{2m})\Phi(s^+(g))\mu^{-1}(u)\,du. \tag{6.5}$$

6.2 Two necessary lemmas

We now state two lemmas that would imply Equation (6.5). Assume $g \in S_l$ with $l < m$. The element $(t^+(g),1)$ equals

$$\sigma'_l\delta\left(\begin{pmatrix}\sigma_l g & \\ & 1_{m-l}\end{pmatrix}\right)\Delta_1(g), \tag{6.6}$$

where $\Delta_1(g) \in \{\pm 1\}$ comes from cocycle computation (σ'_l is defined in (3.4)).

Let $\mathcal{S}_l \subset M_{l,l}$ be the set of matrices p such that $p\sigma_l$ is symmetric. For $p \in \mathcal{S}_l$, let

$$\xi(p) = \begin{pmatrix} p\sigma_l \\ \mathbf{0} \\ \sigma_l \end{pmatrix}.$$

For $q \in M_{l,m-l}$, let

$$\xi'(q) = \begin{pmatrix} q \\ \mathbf{0}\end{pmatrix} \in M_{2n+1,m-l}.$$

It is convenient to write an element in $M_{2n+1,m}$ as $[A,B]$ where $A \in M_{2n+1,l}$ and $B \in M_{2n+1,m-l}$. Equation (6.5) follows from the following two lemmas.

Lemma 6.2. *For any $\Phi \in \mathcal{S}(M_{2n+1,m})$,*

$$\int_{R^0_{l+1,n}\backslash U_{l,n}} \omega_\psi(u,\sigma'_l)\Phi(I_m)\mu^{-1}(u)\,du$$

$$= \Delta_2 \int_{U_{l,n}}\int_{p\in\mathcal{S}_l}\int_{q\in M_{l,m-l}} \omega_\psi(u,\tilde{1}_{2m})\Phi([\xi(p),\xi'(q)+I_{m-l}])\,dp\,dq\mu^{-1}(u)\,du, \tag{6.7}$$

where $\Delta_2 = |2|^{l(l-1)/2}\gamma(1,\psi)^{-(2n+1)l/2}$.

Lemma 6.3. *For any $\Phi \in \mathcal{S}(M_{2n+1,m})$,*

$$\int_{\tilde{n}\in N'_{m,t^+(g),U}\backslash N'_{m,U}}\int_{p\in\mathcal{S}_l}\int_{q\in M_{l,m-l}} \omega_\psi\left(1_{2n+1},\delta\left(\begin{pmatrix}\sigma_l g & \\ & 1_{m-l}\end{pmatrix}\right)\tilde{n}\right)$$

$$\times\, \Phi([\xi(p),\xi'(q)+I_{m-l}])\,dp\,dq\theta'(n^{-1})\,dn = \Delta_3(g)\Phi(s^+(g)), \tag{6.8}$$

where

$$\Delta_3(g) = |\det(g)|^{n-m-1/2}\frac{\gamma(1,\psi)}{\gamma(\det(g)^{2n+1},\psi)}.$$

From the two lemmas, we see that Equation (6.5) holds with $\Delta^+(g) = \Delta_1(g)\Delta_2\Delta_3(g)$.

To prove the two Lemmas, we apply the model of Weil representation (Equations (3.1)–(3.3)) and the Fourier inversion formula. Because of the generality of the cases we consider, the proof appears more complicated than it really is. A similar proof for a special case has appeared in [MR99a], where the notations are not as complicated as here.

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6.3 Proof of Lemma 6.2

For $u \in U_{l,n}$, uI_m has the following form:

$$uI_m = [Z'(\mathbf{v}^l, n_1), \xi'(q) + I_{m-l}], \tag{6.9}$$

where $\mathbf{v}^l$ is a vector in F^l with the coordinates $\mathbf{v}_i^l = \mathbf{v}_i$ for $1 \leqslant i \leqslant l$, $n_1 \in N_l$, $q \in M_{l,m-l}$ and

$$Z'(\mathbf{v}^l, n_1) = \begin{pmatrix} {}^{\mathrm{t}}\mathbf{v}^l \\ n_1 \\ \mathbf{0} \end{pmatrix}.$$

The character $\mu^{-1}(u) = \theta(n_1)\psi(\mathbf{v}_1)$. Thus the left-hand side of (6.7) is

$$\int_{n_1 \in N_l} \int_{\mathbf{v}^l \in F^l} \int_{q \in M_{l,m-l}} \omega_\psi(1, \sigma_l')\Phi([Z'(\mathbf{v}^l, n_1), \xi'(q(u)) + I_{m-l}])\theta^{-1}(n_1)\psi^{-1}(\mathbf{v}_1^l)\, dn_1\, d\mathbf{v}^l\, dq.$$

From Equation (3.2), this integral equals:

$$\begin{aligned}\gamma(1,\psi)^{-(2n+1)l/2} \int_{n_1 \in N_l} \int_{\mathbf{v}^l \in F^l} \int_{q \in M_{l,m-l}} \int_{Z \in M_{2n+1,l}} \psi(\mathrm{tr}({}^{\mathrm{t}}Z\sigma_{2n+1}Z'(\mathbf{v}^l, n_1)))\\ \times \Phi([Z, \xi'(q) + I_{m-l}])\, dZ\, \theta^{-1}(n_1)\psi^{-1}(\mathbf{v}_1^l)\, d\mathbf{v}^l\, dn_1\, dq.\end{aligned}$$

We can apply the Fourier inversion formula for the integrals over Z, n_1 and $\mathbf{v}^l$. Let $V_{l,n}'$ be the subspace of $M_{2n+1,l}$:

$$\left\{ Y' \in M_{2n+1,l} \,\middle|\, Y' = \begin{pmatrix} Y_0' \\ n(Y')\sigma_l \end{pmatrix},\ n(Y') \in N_l,\ Y_0' \in M_{2n+1-l,l} \right\}.$$

The above integral equals

$$\gamma(1,\psi)^{-(2n+1)l/2} \int_{q \in M_{l,m-l}} \int_{Y' \in V_{l,n}'} \Phi([Y', \xi'(q) + I_{m-l}])\theta(n(Y'))\, dY'\, dq. \tag{6.10}$$

We study the $U_{l,n}$-orbit of $V_{l,n}'$.

Lemma 6.4. *There is a bijection from $U_{l,n} \times \mathcal{S}_l$ to $V_{l,n}'$, given by $(u,p) \to Y' = u\xi(p)$. Moreover, $\mu(u) = \theta^{-1}(n(Y'))$ (here μ is the character defined on $U_{m,n}$ instead of $U_{l,n}$).*

Proof. The fact that $u\xi(p)$ lies in $V_{l,n}'$ is clear, so is the identity on the character $\mu(u)$. If $u\xi(p) = \xi(p')$ with $p, p' \in \mathcal{S}_l$, the fact $u \in O(n+1,n)$ implies that ${}^{\mathrm{t}}\xi(p)\sigma_{2n+1}\xi(p) = {}^{\mathrm{t}}\xi(p')\sigma_{2n+1}\xi(p')$, which is just $p = p'$, and thus also $u = 1$. Therefore the above map is an injection. On the other hand, if

$$Y' = \begin{pmatrix} Z_1 \\ Z_2 \\ n\sigma_l \end{pmatrix} \quad \text{for some } n \in N_l,$$

let

$$u = \begin{pmatrix} n^* & * & * \\ & 1_{2n+1-2l} & Z_2\sigma \\ & & n \end{pmatrix} \in U_{l,n}.$$

Then $Y' = u\xi(g)$ for some $g \in M_{l,l}$. Let $p = (g\sigma_l + {}^{\mathrm{t}}g\sigma_l)/2$, and

$$u' = \begin{pmatrix} 1_l & & g\sigma_l - p \\ & 1_{2n+1-2l} & \\ & & 1_l \end{pmatrix} \in U_{l,n},$$

then $Y' = uu'\xi(p)$ with $p \in \mathcal{S}_l$. The map is a surjection. This completes the proof of Lemma 6.4 ☐

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We now return to the proof of Lemma 6.2.

We also observe that $dY' = |2|^{l(l-1)/2}\,du\,dp$ with the above bijection. Note that the group $U_{l,n}$ stabilizes the space of $\xi'(q)$ with $q \in M_{l,m-l}$, where the action is by left multiplication. Thus with Lemma 6.4 and Equation (3.1), we can write (6.10) as

$$|2|^{l(l-1)/2}\gamma(1,\psi)^{-(2n+1)l/2}\int_{U_{l,n}}\int_{q\in M_{l,m-l}}\int_{\mathcal{S}_l}\omega_\psi(u,\tilde{1}_{2m})\Phi([\xi(p),\xi'(q)I_{m-l}])\mu(u)\,dp\,du\,dq.$$

This finishes the proof of Lemma 6.2.

6.4 Proof of Lemma 6.3

The group $N'_{m,t^+(g),U}$ is independent of g; it is just $\eta'(N'_{m-l})\cap N'_{m,U}$. As $\delta\left(\begin{pmatrix}\sigma_l g & \\ & 1_{m-l}\end{pmatrix}\right)$ stabilizes under conjugation by the groups $N'_{m,U}$ and $\eta'(N'_{m-l})$, we can make a change of variable

$$\tilde{n} \mapsto \delta\left(\begin{pmatrix}\sigma_l g & \\ & 1_{m-l}\end{pmatrix}\right)^{-1}\tilde{n}\delta\left(\begin{pmatrix}\sigma_l g & \\ & 1_{m-l}\end{pmatrix}\right).$$

Under this change of variable, the conjugation also stabilizes the character θ' restricted to $N'_{m,U}$; the integral in Lemma 6.3 becomes

$$\begin{aligned}|\det(g)|^{-m-1}&\int_{\tilde{n}\in\eta'(N'_{m-l})\cap N'_{m,U}\backslash N'_{m,U}}\int_{\mathcal{S}_l}\int_{q\in M_{l,m-l}}\omega_\psi\left(1_{2n+1},\tilde{n}\delta\left(\begin{pmatrix}\sigma_l g & \\ & 1_{m-l}\end{pmatrix}\right)\right)\\ &\times\Phi([\xi(p),\xi'(q)+I_{m-l}])\,dp\,dq\,\theta'(n^{-1})\,dn.\end{aligned}$$

Let

$$\Phi_g = \omega_\psi\left(1_{2n+1},\delta\left(\begin{pmatrix}\sigma_l g & \\ & 1_{m-l}\end{pmatrix}\right)\right)\Phi.$$

An element $\tilde{n}$ in $N'_{m,U}$ has the form

$$\tilde{n} = \nu(S) = \left(\begin{pmatrix}1_m & S\\ & 1_m\end{pmatrix},1\right) \tag{6.11}$$

where

$$S = \begin{pmatrix}S_1 & S_2\\ S_3 & \sigma_{m-l}\,{}^{\mathrm{t}}S_1\sigma_l\end{pmatrix},\quad S_1\in M_{l,m-l},\quad S_2\in\mathcal{S}_l,\quad S_3\in\mathcal{S}_{m-l}.$$

Then from (3.3), the above integral is

$$\begin{aligned}|\det(g)|^{-m-1}\int_{S_1\in M_{l,m-l}}\int_{S_2\in\mathcal{S}_l}\int_{\mathcal{S}_l}\int_{q\in M_{l,m-l}}&\Phi_g([\xi(p),\xi'(q)+I_{m-l}])\psi(\mathrm{tr}({}^{\mathrm{t}}\xi(p)\sigma_{2n+1}\xi(p)S_2\sigma_l)/2\\ &+\mathrm{tr}({}^{\mathrm{t}}(\xi'(q)+I_{m-l})\sigma_{2n+1}\xi(p)S_1\sigma_{m-l}))\,dp\,dq\,dS_1\,dS_2.\end{aligned}$$

From the Fourier inversion formula, the above integral equals

$$|\det(g)|^{-m-1}\Phi_g([\xi(\mathbf{0}),I_{m-l}]).$$

Since

$$[\xi(\mathbf{0}),I_{m-l}]\begin{pmatrix}\sigma_l g & \\ & 1_{m-l}\end{pmatrix} = s^+(g),$$

we get from (3.1) that

$$\Phi_g([\xi(\mathbf{0}),I_{m-l}]) = |\det(g)|^{n+1/2}\frac{\gamma(1,\psi)}{\gamma(\det(g)^{2n+1},\psi)}\Phi(s^+(g)).$$

We have shown Lemma 6.3.

This completes the proof of Proposition 6.1 (for the case we consider).

Proof of Theorem 1.1. We can assume that the measure is given as in the start of the section. Let the bijection of orbits ι be defined as in Proposition 4.4. Given $f \in C_c^\infty(G_n)$, we find $\Phi_f \in C_c^\infty(M_{2n+1,n})$ by Lemma 5.3. Lemma 5.5 associates to Φ_f a function $\tilde{f} \in \mathcal{S}(\tilde{G}_m)$. Let $\epsilon(f) = \tilde{f}$. From Proposition 6.1 and Lemmas 5.4 and 5.7, the identity (1.17) is satisfied for the pair $(f, \tilde{f})$. Since clearly for $g \in S_l(F)$ a rational element, $\prod_v \Delta_v(s^{\pm}(g)) = 1$ and $\prod_v \Delta_v(s(g)) = 1$, we have shown Theorem 1.1. □

7. Fundamental lemma

We prove Theorem 1.2 in this section. Fix F_v a p-adic field with odd residue characteristic. We will drop the reference to v in the notations. Assume ψ is of order 0. Choose the measure so that $\tilde{G}_m(\mathcal{O})$ and $G_n(\mathcal{O})$ have volume 1.

The proof of Theorem 1.2 uses the Howe duality, Proposition 6.1, and the more precise version of Lemmas 5.3 and 5.5.

We recall the Howe duality [How90, Ral82, Wal90]. The statement we will use is the following.

THEOREM 7.1. *If $f \in \mathcal{H}_n$, $\tilde{f} \in \tilde{\mathcal{H}}_m$, such that $\tilde{f} = \lambda(f)$ under the Hecke algebra homomorphism, let Φ_0 be the characteristic function on $M_{2n+1,m}$ of the lattice $M_{2n+1,m}(\mathcal{O})$, then as functions on $M_{2n+1,m}$,*

$$\int_{G_n} f(g^{-1})\omega_\psi(g, \tilde{1}_{2m})\Phi_0\, dg = \int_{\tilde{G}_m} \tilde{f}(h^{-1})\omega_\psi(1_{2n+1}, h)\Phi_0\, dh. \tag{7.1}$$

Theorem 7.1 is an immediate consequence of the first statement in § 6.1 of [Ral82].

Let f_0 be the characteristic function of $G_n(\mathcal{O})$ and $\tilde{f}_0$ be the genuine function which takes value 1 at $\tilde{g}$, $g \in G'_m(\mathcal{O})$ (note that $\tilde{G}_m$ splits over $G'_m(\mathcal{O})$), and 0 at any $(g, 1)$ with $g \notin G'_m(\mathcal{O})$. The two functions are the unit elements of the Hecke algebras $\mathcal{H}_n$ and $\tilde{\mathcal{H}}_m$. If $f \in \mathcal{H}_n$, then $f = f * f_0$ where $*$ is the convolution of functions. Similarly $\tilde{f} = \tilde{f} * \tilde{f}_0$ if $\tilde{f} \in \tilde{\mathcal{H}}_m$.

We now state the more precise version of Lemmas 5.3 and 5.5 applied to the case when $f = f_0$ and $\tilde{f} = \tilde{f}_0$.

LEMMA 7.2. *If $\tilde{f}_0$, Φ_0 are as above, then*

$$\int_{N'_m} \tilde{f}_0(\tilde{n}^{-1}g)\theta'(n)\, dn = \int_{R^0_{m+1,n}\backslash U_{m,n}} \omega_\psi(u, g)\Phi_0(I_m)\mu^{-1}(u)\, du. \tag{7.2}$$

Proof. Let $F_1(g)$ and $F_2(g)$ be the left- and right-hand sides of (7.2) respectively. Then we have

$$F_i(\tilde{n}g) = \theta'(n)F_i(g), \quad n \in N'_m,\ i = 1, 2.$$

The equivariance is proved in the proof of Lemma 5.5. It is also clear that $F_i(g\tilde{k}) = F_i(g)$ for $k \in G'_m(\mathcal{O})$ and $i = 1, 2$. By the Iwasawa decomposition, to show $F_1(g) = F_2(g)$, we only need to prove it for the case $g = (\mathbf{a}, 1)$ where $\mathbf{a}$ is a diagonal matrix $\mathrm{diag}[a_1, \ldots, a_m, a_m^{-1}, \ldots, a_1^{-1}]$.

For $n\mathbf{a} \in G'_m(\mathcal{O})$, we have $\mathbf{a}, n$ both lie in $G'_m(\mathcal{O})$. Thus the integral $F_1(\mathbf{a})$ equals 1 when $|a_i| = 1$ for $1 \leqslant i \leqslant m$, and equals 0 otherwise.

To compute $F_2(\mathbf{a})$, we use the isomorphism $\rho : R^0_{m+1,n}\backslash U_{m,n} \cong N_{m+1}$. For $u \in U_{m,n}$, write

$$\rho(u) = n = \begin{pmatrix} 1 & {}^{\mathrm{t}}\mathbf{v} \\ & n' \end{pmatrix} \in N_{m+1}$$

with $n' \in N_m$. From (3.1), we see that $F_2(\mathbf{a})$ equals

$$\int_{\mathbf{v}\in F^m} \int_{n'\in N_m} \Phi_0([I_m n' + e_1\, {}^{\mathrm{t}}\mathbf{v}]\mathbf{a}')\theta^{-1}(n)\psi^{-1}(\mathbf{v}_1)\, dn'\, d\mathbf{v},$$

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where $\mathbf{a}' = \mathrm{diag}[a_1, \dots, a_m] \in GL_m$. The set

$$\{n \in N_{m+1} \mid [I_m n' + e_1 {}^{\mathrm{t}}\mathbf{v}]\mathbf{a}' \in M_{2n+1,m}(\mathcal{O})\}$$

can be described as

$$\{n \in N_{m+1} \mid n'\mathbf{a}'' \in GL_{m+1}(\mathcal{O})\}$$

with $\mathbf{a}'' = \mathrm{diag}[1, a_1, \dots, a_m]$. Thus the above integral is

$$\int_{n \in N_{m+1}, n\mathbf{a}'' \in GL_m(\mathcal{O})} \theta^{-1}(n)\, dn.$$

This integral is 0 unless $|a_i| = 1$ for $1 \leqslant i \leqslant m$, in which case it clearly equals 1. □

Similarly we have the following lemma.

Lemma 7.3. *Let f_0, Φ_0 be as above, then*

$$\int_{R_{m,n}} f_0(r^{-1}g)\chi^{-1}(r)\, dr = \int_{N_m} \Phi_0(g^{-1}[e_1, \dots, e_{m-1}, e_{n+1}]n)\theta^{-1}(n)\, dn. \tag{7.3}$$

Proof. Let $F_1(g)$, $F_2(g)$ be the left- and right-hand sides of the equation respectively. We have the equivariance condition under $R_{m,n}$ and invariance condition under $G_n(\mathcal{O})$:

$$F_i(rgk) = F_i(g)\chi(r), \quad r \in R_{m,n},\ k \in G_n(\mathcal{O}),\ i = 1, 2.$$

From the Iwasawa decomposition, we only need to check $F_1(\mathbf{a}) = F_2(\mathbf{a})$ where

$$\mathbf{a} = \mathrm{diag}[a_1, \dots, a_m, 1, \dots, 1, a_m^{-1}, \dots, a_1^{-1}].$$

A simple calculation as above gives $F_1(\mathbf{a}) = F_2(\mathbf{a}) = 1$ when $|a_i| = 1$, $1 \leqslant i \leqslant m$, and they equal 0 otherwise. □

Proof of Theorem 1.2. Let f be an element in the Hecke algebra and $\tilde{f} = \lambda(f)$. From Lemma 7.3 and the equation $f * f_0 = f$, we see in Lemma 5.3 that one can associate to f the function Φ_f defined as the left-hand side of (7.1). From Lemma 7.2 and the equation $\tilde{f} * \tilde{f}_0 = \tilde{f}$, we see that Lemma 5.5 holds for $\tilde{f}$ and Φ with Φ being the right-hand side of (7.1). By Theorem 7.1, Φ equals Φ_f. Lemmas 5.4 and 5.7 and Proposition 6.1 then imply the identity (1.18). □

Proof of Theorem 1.3. We choose the set of bad places S_0 to be the union of archimedean places, p-adic places with even residue characteristics, and places where ψ is not of order 0. Then the discussion in § 1.4 shows that Theorems 1.1 and 1.2 and Proposition 5.1 imply Theorem 1.3. □

8. Some remarks

We mention the relation between the relative trace identities in § 2 and those considered in some previous works.

The identity (2.34) is considered in [Fli93]; the special case when $n = 3$ is considered in [Mao92] and [Fli97]. The identity (2.9) is considered for the special case $m = 1$ in [MR99b]. The identity (2.14) is considered for the special case $m = 1$, $n = 2$ in [FM04] and [Zin98]. The identity (2.9) in the setting of § 2.3 is considered for the special case $m = 2, n = 3$ in [FJ96] (there they treat the similitude group case).

Jacquet's identity discussed in § 1.3 involves another generalization of identity (2.9) (in the case $n = m = 1$), i.e. the introduction of a quadratic character χ_τ. We now discuss how to introduce a quadratic character in the identity (2.9). Recall that there is a spinor norm defined on $SO(n+1, n)$, it is a homomorphism $\mathcal{N} : SO(n+1, n) \mapsto F^\times/F^{\times 2}$. Then $\chi_\tau \mathcal{N}$ is a character of $SO(n+1, n)$. We have the following proposition.

PROPOSITION 8.1. *For $\tau \in F^\times$, there is a relative trace identity (in the sense of (1.2))*

$$I_{G_n}(f : R_{m,n}, \chi^{-1}, U_{m,n}, \mu^{-1}) = I_{G_n}(f' : R_{m,n}, \chi^{-1}(\chi_\tau \mathcal{N}), U_{m,n}, \mu^{-1}). \tag{8.1}$$

Here both distributions are over $SO(n+1,n)$. There is a relative trace identity

$$I_{\tilde{G}_m}(\tilde{f} : N'_m, \theta', N'_m, \theta'^{-1}) = I_{\tilde{G}_m}(\tilde{f}' : N'_m, \theta'_\tau, N'_m, \theta'^{-1}_\tau). \tag{8.2}$$

Here both distributions are over $\widetilde{Sp}(m)$.

Proof. For the identity (8.1), let $f'(g) = \epsilon(f)(g) = f(g)\chi_\tau(\mathcal{N}(g))$. Then

$$\begin{aligned} I(f' &: R_{m,n}, \chi^{-1}(\chi_\tau \mathcal{N}), U_{m,n}, \mu^{-1}) \\ &= \iint \sum_\gamma f'(r^{-1}\gamma u)\chi^{-1}(r)\chi_\tau(\mathcal{N}(r))\mu^{-1}(u)\, du\, dr \\ &= \iint \sum_\gamma f(r^{-1}\gamma u)\chi^{-1}(r)\mu^{-1}(u)\, du\, dr, \end{aligned}$$

as $\mathcal{N}(u) \equiv 1$ for $u \in U_{m,n}$ and $\mathcal{N}(\gamma) \in F^\times$ for $\gamma \in SO(n+1,n)(F)$. Thus we get that the identity (8.1) holds for $f' = \epsilon(f)$ defined above. It is easy to check that ϵ is restricted to a Hecke algebra homomorphism λ_{G_n} at almost all non-archimedean places.

To show the identity (8.2), we consider $\widetilde{Sp}(m)$ as a subgroup of $\widetilde{GSp}(m)$, and define $\tilde{f}'(g) = \epsilon(\tilde{f})(g) = \tilde{f}(D_\tau g D_\tau^{-1})$ where $D_\tau \in \widetilde{GSp}(m)$ is given by $(\mathrm{diag}[\tau 1_m, 1_m], 1)$. One can check again that this definition of ϵ yields identity (8.2) and is restricted to a Hecke algebra homomorphism $\lambda_{\tilde{G}_m}$ at almost all non-archimedean places. □

From the above proposition and the identity (2.9) which we proved, one gets a relative trace identity:

$$I_{G_n}(f : R_{m,n}, \chi^{-1}(\chi_\tau \mathcal{N}), U_{m,n}, \mu^{-1}) = I_{\tilde{G}_m}(\tilde{f} : N'_m, \theta'_\tau, N'_m, \theta'^{-1}_\tau), \tag{8.3}$$

which is the generalization of Jacquet's identity described in § 1.3.

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References

Ber84 J. Bernstein, *p-invariant distributions on $GL(N)$ and the classification of unitary representations of $GL(N)$ (non-archimedean case)*, in *Lie group representations II*, Lecture Notes in Mathematics, vol. 1041 (Springer, Berlin, 1984), 50–102.

BM04 E. M. Baruch and Z. Mao, *Central values of automorphic L-functions*, Preprint (2004).

Böc86 S. Böcherer, *Bemerkungen über die Dirichletreihen von Koecher und Maass*, Preprint Math. Gottingensis, Heft 68 (1986).

BZ86 J. Bernstein and Z. V. Zelevinsky, *Representations of the group $GL(n,F)$ where F is a non-archimedean local field*, Russian Math. Surveys **31** (1976), 1–68.

Cas89 W. Casselman, *Introduction to the Schwartz space of* $\Gamma\backslash G$, Canad. J. Math. **41** (1989), 285–320.

FJ96 S. Friedberg and H. Jacquet, *The fundamental lemma for the Shalika subgroup of $GL(4)$*, Mem. Amer. Math. Soc., no. 594 (American Mathematical Society, Providence, RI, 1996).

Fli93 Y. Flicker, *A Fourier summation formula for the symmetric space $GL(n)/GL(n-1)$*, Compositio Math. **88** (1993), 39–117.

Fli97 Y. Flicker, *Cyclic automorphic forms on a unitary group*, J. Math. Kyoto. Univ. **37** (1997), 367–439.

FM04 Y. Flicker and J. G. M. Mars, *Cusp forms on $GSp(4)$ with $SO(4)$ periods*, Preprint (2004).

FS99 M. Furusawa and J. Shalika, *The fundamental lemma for the Bessel and Novodvorsky subgroups of $GSp(4)$*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 105–110.

Fur95 M. Furusawa, *On the theta lift from SL_{2n+1} to $\tilde{G}$*, J. reine angew. Math. **466** (1995), 87–110.

Gin90 D. Ginzburg, *L-functions for $SO_n \times GL_k$*, J. reine angew. Math. **405** (1990), 156–180.

Guo96 J. Guo, *On the positivity of the central critical values of automorphic L-functions for $GL(2)$*, Duke Math. J. **83** (1996), 157–190.

How90 R. Howe, *Another look at the local θ-correspondence for an unramified dual pair*, Israel Math. Conf. Proc. **2** (1990), 93–124.

Jac87 H. Jacquet, *On the nonvanishing of some L-functions*, Proc. Indian Acad. Sci. **97** (1987), 117–155.

Jac95 H. Jacquet, *The continuous spectrum of the relative trace formula for $GL(3)$ over a quadratic extension*, Israel J. Math. **89** (1995), 1–59.

JR92 H. Jacquet and S. Rallis, *Kloosterman integrals for skew symmetric matrices*, Pacific J. Math. **154** (1992), 265–283.

Koh85 W. Kohnen, *Fourier coefficients of modular forms of half-integral weight*, Math. Ann. **271** (1985), 237–268.

KZ81 W. Kohnen and D. Zagier, *Values of L-series of modular forms at the center of the critical strip*, Invent. Math. **64** (1981), 175–198.

Mao92 Z. Mao, *Relative Kloosterman integrals for the unitary group*, C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), 381–386.

Mao93 Z. Mao, *Relative Salie sums*, C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), 1257–1262.

MR99a Z. Mao and S. Rallis, *The induced Weil representation and the Shalika period*, Pacific J. Math. **101** (1999), 313-327.

MR99b Z. Mao and S. Rallis, *Howe duality and the trace formula*, Pacific J. Math. **101** (1999), 329–346.

MR99c Z. Mao and S. Rallis, *On a cubic lifting*, Israel J. Math. **113** (1999), 95–124.

Ral82 S. Rallis, *Langlands' functoriality and the Weil representation*, Amer. J. Math. **104** (1982), 469–515.

Ral84 S. Rallis, *On the Howe duality conjecture*, Compositio Math. **51** (1984), 333–399.

Sou93 D. Soudry, *Rankin–Selberg convolutions for $SO_{2\ell+1} \times GL_m$: local theory*, Mem. Amer. Math. Soc., no. 500 (American Mathematical Society, Providence, RI, 1993).

Wal90 J. L. Waldspurger, *Démonstration d'une conjecture de dualité de Howe dans le cas p-adique, $p \neq 2$*, Israel Math. Conf. Proc. **2** (1990), 267–324.

Wei64 A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143–211.

Zin98 D. Zinoviev, *Relation of orbital integrals on $SO(5)$ and $PGL(2)$*, Israel J. Math. **106** (1998), 29–77.

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