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RESTRICTED WEAK UPPER SEMICONTINUOUS DIFFERENTIALS OF CONVEX FUNCTIONS

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We characterise restricted weak upper semicontinuity of the subdifferential of convex functions in terms of the Fenchel biconjugate mapping.

1. INTRODUCTION.

Given a convex lower semicontinuous function f defined on a real Banach space X, the *subdifferential* of f at $x \in X$ is defined by

$$\partial f(x) := \{x^* \in X^* : \langle y - x, x^* \rangle \leqslant f(y) - f(x), \forall y \in X\},$$

if $x \in \operatorname{dom}(f)$, while $\partial f(x) = \emptyset$ if $x \in X \setminus \operatorname{dom}(f)$.

A set-valued mapping Φ from a topological space (X, τ') into subsets of another topological space (Y, τ) is said to be $[\tau'$-$\tau]$-*upper semicontinuous* at $x \in X$ if given a τ-open subset W of Y such that $\Phi(x) \subset W$, there exists a τ'-neighbourhood U of x such that $\Phi(U) \subset W$. In this paper we shall always consider X a real Banach space endowed with the norm topology and $Y = X^*$ endowed with a topology τ. We shall write τ-upper semicontinuous instead of $[\|\cdot\|$-$\tau]$-upper semicontinuous.

Given a convex function f on an open subset D of a Banach space X and a point of continuity $x_0 \in D$ of f, it can be proved that $\partial f(x_0)$ is a nonempty, w^*-compact and convex subset of X^*, and the mapping $x \mapsto \partial f(x)$ is w^*-upper semicontinuous at x_0.

Gâteaux differentiability and Fréchet differentiability can be characterised in terms of the continuity of the subdifferential mapping: Given a continuous convex function f on an open subset D of a Banach space X and a point $x_0 \in D$, f is Gâteaux differentiable at $x_0 \in A$ if and only if $\partial f(x)$ is a singleton, and f is Fréchet differentiable at x_0 if and only if $\partial f(x)$ is a singleton and the subdifferential mapping $x \mapsto \partial f(x)$ is $\|\cdot\|$-upper semicontinuous at x_0 (for these and related concepts see, for example, [11]).

If the one sided limit in the definition of the derivative of f at a point x_0 is uniform in every direction, we get a weaker concept than Fréchet differentiability. More precisely,

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given a continuous function f defined on an open subset D of a Banach space X, we say (following [5, 8]) that f is *strongly subdifferentiable* at $x_0 \in D$ if

$$d^+ f_{x_0}(u) := \lim_{t \to 0+} \big(f(x_0 + tu) - f(x_0)\big)/t$$

is uniform in $\|u\| = 1$. This non-smooth extension of Fréchet differentiability has found several applications (see for example, [1, 5, 6, 9, 10]).

The following definition was introduced in [6]: A set-valued mapping Φ from a Banach space X into the subsets of X^* endowed with the topology τ is said to be *restricted τ-upper semicontinuous* at $x \in X$ if given a τ-neighbourhood W of 0 in X^* there exists an open neighbourhood U of x in X such that $\Phi(U) \subset \Phi(x) + W$.

In [9] it was proved that given a continuous convex function f defined on an open subset D of a Banach space X, f is strongly subdifferentiable at $x_0 \in D$ if and only if ∂f is restricted $\|\cdot\|$-upper semicontinuous at x_0.

In this note we provide, in the spirit of [6], a characterisation of restricted w-upper semicontinuity of the subdifferential mapping of a convex function by using the Fenchel biconjugate mapping. Notice that a partial characterisation was obtained in [6] for the duality mapping (that is, $x \mapsto \partial\|\cdot\|(x)$). For the use of the concept of restricted w-upper semicontinuity of the subdifferential mapping in questions related to the Asplundness and reflexivity of a Banach space we refer to [2, 4, 6, 7] and references therein.

Given a continuous convex function f on an open convex subset D of a Banach space X, we can extend f to a lower semicontinuous convex function on X, denoted again by f, by defining

$$f(x) := \begin{cases} \liminf_{y \to x} f(y) & \text{for } x \in \overline{D}, \\ +\infty & \text{otherwise.} \end{cases}$$

Given a convex, proper, lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, the *Fenchel conjugate* of f is defined by

$$f^*(x^*) := \sup\{\langle x, x^*\rangle - f(x) : x \in X\}.$$

Now f^* is again convex, proper and lower semicontinuous (in fact, lower w^*-semicontinuous). Obviously $\langle x, x^*\rangle \leqslant f(x) + f^*(x^*)$ for all $x \in X$, $x^* \in X^*$ (and the inequality becomes equality if and only if $x^* \in \partial f(x)$). Moreover, if $\varepsilon \geqslant 0$, then $x^* \in \partial_\varepsilon f(x)$ if and only if $f(x) + f^*(x^*) \leqslant \langle x, x^*\rangle + \varepsilon$ (where $\partial_\varepsilon f$ denotes the ε-subdifferential). Also, $f^{**}|_X = f$ (see [3, 11]).

2. Preliminary results.

We shall need the following results:

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THEOREM 2.1. (Brøndsted-Rockafellar) *Suppose that f is a convex proper lower semicontinuous function on the Banach space X. Then given any point $x_0 \in \operatorname{dom}(f)$, $\varepsilon > 0$ and any $x_0^* \in \partial_\varepsilon f(x_0)$, there exists $x_\varepsilon \in \operatorname{dom}(f)$ and $x_\varepsilon^* \in X^*$ such that*

$$x_\varepsilon^* \in \partial f(x_\varepsilon), \quad \|x_\varepsilon - x_0\| \leqslant \sqrt{\varepsilon}, \quad \|x_\varepsilon^* - x_0^*\| \leqslant \sqrt{\varepsilon}.$$

The Brøndsted-Rockafellar Theorem, together with the local boundedness of the subdifferential mapping at a point of continuity x_0, allows us to interweave the ε-subdifferential at x_0 and the subdifferential at a neighbourhood of x_0. The precise relationship is formulated in the next result:

LEMMA 2.2. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let x_0 be a point of continuity of f. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\partial f\big[B(x_0;\delta)\big] \subset \partial_\varepsilon f(x_0) \subset \partial f\big[B(x_0;\sqrt{\varepsilon})\big] + \sqrt{\varepsilon}B_{X^*}.$$

PROOF: ∂f is locally bounded at x_0, that is, there exists $M > 0$ and $N(x_0)$, a neighbourhood of x_0, such that

$$\|x^*\| \leqslant M, \quad \forall x^* \in \partial f(x), \quad \forall x \in N(x_0).$$

Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$B(x_0;\delta) \subset N(x_0), \quad M\delta < \frac{\varepsilon}{2}, \quad \big|f(x) - f(x_0)\big| < \frac{\varepsilon}{2}, \ \forall x \in B(x_0;\delta).$$

Let $x^* \in \partial f\big[B(x_0;\delta)\big]$, say $x^* \in \partial f(x)$ for some $x \in B(x_0;\delta)$. Then

$$\begin{aligned}\langle y - x_0, x^*\rangle &= \langle y - x, x^*\rangle + \langle x - x_0, x^*\rangle \\ &\leqslant f(y) - f(x) + \|x^*\|\|x - x_0\| < f(y) - f(x_0) + \big|f(x_0) - f(x)\big| + M\delta \\ &< f(y) - f(x_0) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = f(y) - f(x_0) + \varepsilon,\end{aligned}$$

hence $x^* \in \partial_\varepsilon f(x_0)$.

The second inclusion is the Brøndsted-Rockafellar Theorem, and does not need the continuity of f at x_0. ∎

The following proposition can be found in [**11**]:

PROPOSITION 2.3. *Let $f : D \to \mathbb{R}$ be a convex function on D (a non-empty open and convex subset of X), continuous at $x_0 \in D$. Then, for all $y \in X$,*

$$d^+ f_{x_0}(y) = \sup\big\{\langle y, x^*\rangle : x^* \in \partial f(x_0)\big\}$$

and this supremum is attained at some point $x^* \in \partial f(x_0)$.

PROPOSITION 2.4. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex, proper and lower semicontinuous function. Then $\operatorname{epi}(f^{**}) = \overline{\operatorname{epi}(f)}^{w^*}$.*

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PROOF: First, assume $f \geqslant 0$. The inclusion $\overline{\text{epi}(f)}^{w^*} \subset \text{epi}(f^{**})$ follows from $\text{epi}(f) \subset \text{epi}(f^{**})$ and the lower w^*-semicontinuity of f^{**}. Let $(x_0^{**}, \lambda_0) \in \text{epi}(f^{**})$. Suppose that $(x_0^{**}, \lambda_0) \notin \overline{\text{epi}(f)}^{w^*}$. By the Hahn-Banach Theorem, there are $x_0^* \in X^*$, k, α, $\beta \in \mathbb{R}$ such that:

$$\langle x_0^{**}, x_0^* \rangle + k\lambda_0 < \alpha < \beta < \langle x^{**}, x_0^* \rangle + k\lambda, \quad \forall\, (x^{**}, \lambda) \in \overline{\text{epi}(f)}^{w^*}. \tag{1}$$

From (1) we get $k \geqslant 0$ (if $k < 0$, it is enough to take $x \in \text{dom}(f)$ and $\lambda \to +\infty$ in order to obtain a contradiction). In particular, from (1), we get $\langle x, x_0^* \rangle + kf(x) > \beta$, for all $x \in \text{dom}(f)$. Take $\varepsilon > 0$. Since $f \geqslant 0$, we get

$$\left\langle x, -\frac{x_0^*}{k+\varepsilon} \right\rangle - f(x) < -\frac{\beta}{k+\varepsilon}, \qquad \forall\, x \in \text{dom}(f),$$

hence $f^*\big(-x_0^*/(k+\varepsilon)\big) \leqslant -\beta/(k+\varepsilon)$. Then

$$\begin{aligned} f^{**}(x_0^{**}) &\geqslant \left\langle x_0^{**}, -\frac{x_0^*}{k+\varepsilon} \right\rangle - f^*\left(-\frac{x_0^*}{k+\varepsilon}\right) \\ &\geqslant \left\langle x_0^{**}, -\frac{x_0^*}{k+\varepsilon} \right\rangle + \frac{\beta}{k+\varepsilon} = \frac{1}{k+\varepsilon}\left[\beta - \langle x_0^{**}, x_0^* \rangle\right] > \frac{\beta - \alpha + k\lambda_0}{k+\varepsilon}. \end{aligned}$$

If $k = 0$, then $f^{**}(x_0^{**}) > (\beta - \alpha)/\varepsilon$. As $\varepsilon > 0$ was arbitrary, we get $x_0^{**} \notin \text{dom}(f^{**})$, a contradiction. If $k \neq 0$, since $\varepsilon > 0$ was arbitrary, we get $f^{**}(x_0^{**}) \geqslant (\beta - \alpha + k\lambda_0)/k > \lambda_0$. This contradicts $(x_0^{**}, \lambda_0) \in \text{epi}(f^{**})$.

Now, if $f : X \to \mathbb{R} \cup \{+\infty\}$ is an arbitrary proper semicontinuous convex function, choose $x_0^* \in \text{dom}(f^*)$. Consider $g : X \to \mathbb{R} \cup \{+\infty\}$ given by $g(x) = f(x) + f^*(x_0^*) - \langle x, x_0^* \rangle$. This function, obviously, is proper, lower semicontinuous and convex. Moreover $\text{dom}(f) = \text{dom}(g)$ and $g \geqslant 0$. Now, a simple calculation shows $g^{**}(x^{**}) = f^{**}(x^{**}) + f^*(x_0^*) - \langle x^{**}, x_0^* \rangle$ for all $x^{**} \in X^{**}$. By the first part of the proof, the proposition holds for g, and hence for f. ∎

REMARKS.

1. Note that Goldstine's Theorem is a particular case of the former proposition: It is enough to take as f the indicator function δ_{B_X} of the closed unit ball of X (that is, $\delta_{B_X}(x) = 0$ if $\|x\| \leqslant 1$, $\delta_{B_X}(x) = +\infty$ if $\|x\| > 1$), a proper lower semicontinuous convex function. Obviously, f^* is the dual norm. Let $x^{**} \in B_{X^{**}}$. As

$$f^{**}(x^{**}) = \sup\big\{\langle x^{**}, x^* \rangle - \|x^*\| : x^* \in X^*\big\} \leqslant 0 < +\infty,$$

we get $x^{**} \in \text{dom}(f^{**})$. By Proposition 2.4, $\text{dom}(f^{**}) = \overline{\text{dom}(f)}^{w^*} = \overline{B_X}^{w^*}$.

2. This proposition gives a description of f^{**}, sometimes simpler than the original one.

Corollary 2.5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then, given $x_0 \in X$,*

1. $\partial f(x_0) = \partial f^{**}(x_0) \cap X^*$.
2. *If f is continuous at x_0, f^{**} is also continuous at x_0.*

Proof: (1) is a consequence of two well-known facts (see [**11**]): f^{**} induces f on X, and $x_0^* \in \partial f(x_0)$ if and only if $\langle x_0, x_0^* \rangle = f(x_0) + f^*(x_0^*)$.

To prove (2), assume f (but not f^{**}) is continuous at x_0. Let $\mathcal{N}$ be a basis of w^*-open neighbourhoods of 0 in X^{**}. Then there exists $\varepsilon > 0$ and $x_N^{**} \in x_0 + N$, $N \in \mathcal{N}$, such that $\left|f^{**}(x_N^{**}) - f(x_0)\right| \geqslant \varepsilon$. As $\overline{\operatorname{epi}(f)}^{w^*} = \operatorname{epi}(f^{**})$ and f^{**} is lower semicontinuous, it is possible to choose $x_N \in (x_0 + N) \cap X$ such that $f^{**}(x_N^{**}) \leqslant f(x_N) < f^{**}(x_N^{**}) + \varepsilon/2$, $N \in \mathcal{N}$. It follows that $x_N \xrightarrow{w} x_0$ and $\left|f(x_N) - f(x_0)\right| \geqslant \varepsilon/2$, a contradiction. ∎

Corollary 2.6. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then, given $x_0^{**} \in \operatorname{dom}(f^{**})$,*

$$f^{**}(x_0^{**}) = \inf\Big\{\liminf_i f(x_i) : x_i \subset \operatorname{dom}(f), x_i \xrightarrow{w^*} x_0^{**}\Big\}.$$

Proof: By Proposition 2.4 it is obvious that $\operatorname{dom}(f^{**}) = \overline{\operatorname{dom}(f)}^{w^*}$. Now, given a net $(x_i)_{i\in I} \subset \operatorname{dom}(f)$, $x_i \xrightarrow{w^*} x_0^{**}$, by the w^*-lower semicontinuity of f^{**} we get $f^{**}(x_0^{**}) \leqslant \liminf_i f(x_i)$. On the other hand, again by Proposition 2.4, given $\varepsilon > 0$ we can find a net $(x_i)_{i\in I} \subset \operatorname{dom}(f)$ and $\lambda_i \in \mathbb{R}$ such that $x_i \xrightarrow{w^*} x_0^{**}$, $(x_i, \lambda_i) \in \operatorname{epi}(f)$ and $\lambda_i < f^{**}(x_0^{**}) + \varepsilon$. As $f(x_i) \leqslant \lambda_i$, $i \in I$, we get the conclusion. ∎

Corollary 2.7. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then, given $x_0 \in \operatorname{dom}(f)$ and $\varepsilon > 0$,*

$$\partial_\varepsilon f^{**}(x_0) \subset \overline{\partial_{\varepsilon+k} f(x_0)}^{X^{***}[w^*]}, \quad \forall k > 0.$$

Proof: Let $x^{***} \in \partial_\varepsilon f^{**}(x_0)$. Then $f^{**}(x_0) + f^{***}(x^{***}) \leqslant \langle x_0, x^{***} \rangle + \varepsilon$. It follows that

$$f^{***}(x^{***}) \leqslant \langle x_0, x^{***} \rangle - f^{**}(x_0) + \varepsilon < \langle x_0, x^{***} \rangle - f^{**}(x_0) + \varepsilon + k/2.$$

By the previous corollary, there exists a net $(x_i^*)_{i\in I}$ in X^* such that $x_i^* \to x^{***}$ in $X^{***}[w^*]$, $f^*(x_i^*) < \langle x_0, x^{***} \rangle - f^{**}(x_0) + \varepsilon + k/2$, $\forall i \in I$ and $\left|\langle x_0, x_i^* - x^{***} \rangle\right| < k/2$. We get

$$f(x_0) + f^*(x_i^*) < \langle x_0, x^{***} \rangle + \varepsilon + \frac{k}{2} < \langle x_0, x_i^* \rangle + \varepsilon + k.$$

Then, $x_i^* \in \partial_{\varepsilon+k} f(x_0)$, $\forall i \in I$. The conclusion follows. ∎

3. A characterisation of the restricted w-upper semicontinuity.

If $x \in X$ and $\delta > 0$, we shall denote by $B^{**}(x_0; \delta)$ the open ball in X^{**} of radius δ and centred at x_0.

Now we are ready to prove the main result in this note:

Theorem 3.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let x_0 be a point of continuity of f. Then the following assertions are equivalent:*

1. *∂f is restricted w-upper semicontinuous at x_0.*
2. *For all N, a w-neighbourhood of 0 in X^*, there is $\varepsilon > 0$ such that $\partial_\varepsilon f(x_0) \subset \partial f(x_0) + N$.*
3. *$\partial f(x_0)$ is dense in $\partial f^{**}(x_0)$ in $X^{***}[w^*]$.*
4. *$d^+ f^{**}_{x_0}(\cdot) = \sup\{\langle \cdot, x^* \rangle : x^* \in \partial f(x_0)\}$.*

Proof: (1) $\Rightarrow$ (2): Let N be a convex w-neighbourhood of 0 in X^*. By hypothesis there is $\delta > 0$ such that $\partial f\left[B(x_0;\delta)\right] \subset \partial f(x_0) + N/2$, $\delta B_{X^*} \subset N/2$. Now, by Lemma 2.2,

$$\partial_{\delta^2} f(x_0) \subset \partial f\left[B(x_0;\delta)\right] + \delta B_{X^*} \subset \partial f(x_0) + \frac{1}{2}N + \frac{1}{2}N \subset \partial f(x_0) + N.$$

It is enough to choose $\varepsilon = \delta^2$.

(2) $\Rightarrow$ (3): Given a closed neighbourhood N^{**} of 0 in $X^{***}[w^*]$, let $\varepsilon > 0$ be as in (2). Then, using Corollary 2.5, Corollary 2.7 and the fact that $\overline{\partial f(x_0)}^{X^{***}[w^*]}$ is compact and $N := N^{**} \cup X^*$ is closed in $X^{***}[w^*]$,

$$\begin{aligned}\partial f(x_0) \subset \partial f^{**}(x_0) \subset \partial_{\varepsilon/2} f^{**}(x_0) \subset \overline{\partial_\varepsilon f(x_0)}^{X^{***}[w^*]} \\ \subset \overline{\partial f(x_0) + N}^{X^{***}[w^*]} \subset \overline{\partial f(x_0)}^{X^{***}[w^*]} + N^{**}.\end{aligned}$$

This proves (3).

(3) $\Rightarrow$ (1): Let N a w-neighbourhood of 0 in X^*. Take a convex w^*-neighbourhood of 0 in X^{***}, N^*, such that $N^* \cap X^* \subset N$. By Corollary 2.5, f^{**} is continuous at x, so ∂f^{**} is upper w^*-semicontinuous at x. Hence there exists $\delta > 0$ such that

$$\partial f^{**}\left(B^{**}(x;\delta)\right) \subset \partial f^{**}(x) + N^*/2.$$

By hypothesis, $\partial f^{**}(x) \subset \partial f(x) + N^*/2$. It follows that

$$\partial f^{**}\left(B^{**}(x;\delta)\right) \subset \partial f^{**}(x) + N^*/2 \subset \partial f(x) + N^*/2 + N^*/2 \subset \partial f(x) + N^*.$$

It is now enough to use Corollary 2.5 to get $\partial f\left(B(x;\delta)\right) \subset \partial f(x) + N$.

(3) $\Leftrightarrow$ (4): By Proposition 2.3 and Corollary 2.5,

$$d^+ f^{**}_{x_0}(\cdot) = \sup\{\langle \cdot, x^{***} \rangle : x^{***} \in \partial f^{***}(x_0)\}.$$

Now, using the Hahn-Banach Theorem, the equivalence is obvious. ∎

Note that the equivalence (1) $\Leftrightarrow$ (2) is valid not only for restricted w-upper semicontinuity, but also for τ-upper semicontinuity, τ a Hausdorff topology weaker than the norm-topology. More precisely:

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THEOREM 3.2. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let $x_0 \in X$ be a point of continuity of f. If τ is a topology on X^* weaker than the norm topology, then the following assertions are equivalent:*

1. *∂f is restricted τ-upper semicontinuous at x_0.*
2. *For every τ-neighbourhood N of 0 in X^*, there is $\varepsilon > 0$ such that $\partial_\varepsilon f(x_0) \subset \partial f(x_0) + N$.*

PROOF: For (1) $\Rightarrow$ (2) the same proof used in the previous theorem, (1) $\Rightarrow$ (2), works. To prove (2) $\Rightarrow$ (1), use Lemma 2.2. ∎

This characterisation can be considered as the analogue of the Šmulyan Test.

It is well known that if the dual norm of X^* is locally uniformly rotund, then the norm of X is Fréchet differentiable. The next proposition, that uses the previous theorem, extends this result. Note that the Fenchel conjugate of the norm of a Banach space X is the indicator function of B_{X^*}.

PROPOSITION 3.3. *Let $f : D \to \mathbb{R}$ be a convex, continuous function defined on D, a non-empty open subset of X. Let $x_0 \in D$. If τ is a Hausdorff topology on X^* weaker than the norm topology, then the following assertions are equivalent:*

1. *f is Gâteaux differentiable at x_0 and ∂f is restricted τ-upper semicontinuous at x_0.*
2. *For every τ-neighbourhood N of 0 in X^* and $x^* \in \partial f(x_0)$, there exists $\delta = \delta(x^*, N)$ such that*

$$f(x_0) + \frac{1}{2}\big(f^*(x^*) + f^*(y^*)\big) - \delta < \frac{1}{2}\langle x_0, x^* + y^*\rangle \Rightarrow y^* \in x^* + N.$$

PROOF: (1) $\Rightarrow$ (2). Let N be a τ-neighbourhood of 0 in X^* and $\{x^*\} = \partial f(x_0)$. By the previous theorem there exists $\varepsilon > 0$ such that $\partial_\varepsilon f(x_0) \subset x^* + N$. Take $y^* \in X^*$ such that

$$f(x_0) + \frac{1}{2}\big(f^*(x^*) + f^*(y^*)\big) - \frac{\varepsilon}{2} < \frac{1}{2}\langle x_0, x^* + y^*\rangle.$$

A simple calculation shows that $f(x_0) + f^*(y^*) < \langle x_0, y^*\rangle + \varepsilon$. It follows that $y^* \in \partial_\varepsilon f(x_0) \subset x^* + N$.

(2) $\Rightarrow$ (1). First, we shall prove that f is Gâteaux differentiable at x_0. If not, there would exist $x_1 \neq x_2$ in $\partial f(x_0)$. Choose a τ-neighbourhood N of 0 in X^* such that $(x_1^* + N) \cap (x_2^* + N) = \emptyset$. Let $\delta_i = \delta_i(x_i^*, N)$ be as in (2) $(i = 1, 2)$ and let $\delta := \min\{\delta_1, \delta_2\}$. Take $y^* \in \partial_\delta f(x_0)$. A simple calculation shows that

$$f(x_0) + \frac{1}{2}\big(f^*(x_i^*) + f^*(y^*)\big) - \delta_i < \frac{1}{2}\langle x_0, x_i^* + y^*\rangle,$$

for $i = 1, 2$. By hypothesis, $y^* \in (x_1^* + N) \cap (x_2^* + N)$, a contradiction.

Now, let N be a τ-neighbourhood of 0 in X^*. Since f is Gâteaux differentiable at x_0, $\partial f(x_0) = \{x_0^*\}$. Given N and x_0^*, we get $\delta = \delta(x_0^*, N)$ as in (2). It is easy to prove that $\partial_\delta f(x_0) \subset x_0^* + N = \partial f(x_0) + N$. By Theorem 3.2, ∂f is restricted τ-upper semicontinuous at x_0. ∎

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