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On Locally Uniformly Rotund Renormings in $C(K)$ Spaces

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Abstract. A characterization of the Banach spaces of type $C(K)$ that admit an equivalent locally uniformly rotund norm is obtained, and a method to apply it to concrete spaces is developed. As an application the existence of such renorming is deduced when K is a Namioka–Phelps compact or for some particular class of Rosenthal compacta, results which were originally proved with *ad hoc* methods.

1 Introduction

The class of Banach spaces that admit an equivalent locally uniformly rotund norm (LUR) has been extensively studied and some characterizations of such spaces have already been obtained in terms of linear-topological conditions [2]. The LUR renorming techniques for a Banach space developed until now, which are free of martingale techniques, are based in two different approaches. In the first one, enough convex functions on the Banach space are constructed to apply Deville's lemma (see the decomposition method [1, Chapter 7, Lemma 1.1]), sometimes adding an iteration processes and Banach's contraction mapping theorem, to finally get an equivalent LUR norm. In the second one the existence of such norm is deduced from the existence of a σ-slicely isolated network of the norm topology in $C(K)$, introducing a countable family of equivalent pointwise lower semicontinuous norms such that, roughly speaking, the LUR condition on a fixed sequence (x_n) and a point x controls whether the segments $[x_n, x]$ live inside small slices, this gives the equivalent LUR norm [8, 10]. In this paper, taking ideas from both approaches, a characterization of the existence of LUR norms on $C(K)$ type spaces is presented and applied to the very recent cases obtained in [3, 6] by means of the first method. More applications of this method can be found in [7] where the cases when K is a Haydon tree [4] or a totally ordered compact space [5] are discussed.

Theorem 1.1 *Let K be a compact space. The Banach space $C(K)$ admits an equivalent pointwise lower semicontinuous and LUR norm if and only if there is a countable family of subsets $\{C_n : n \in \mathbb{N}\}$ in $C(K)$ such that for every $x \in C(K)$ and every $\varepsilon > 0$, there*

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exist $q \in \mathbb{N}$, a pointwise open half space H with $x \in H \cap C_q$, and a finite covering $\mathcal{L}$ of K, such that $|y(s) - y(t)| < \varepsilon$ whenever $s, t \in L$, $y \in H \cap C_q$, and $L \in \mathcal{L}$.

Some compacta K that are relevant in this field are not defined by *internal* topological properties but in terms of their immersion in a product of real lines $K \subset \mathbb{R}^\Gamma$, where the elements of Γ can be viewed as *coordinates* of the elements of K; see [6]. Therefore when we deal with this sort of compacta it is easier to apply Corollary 1.2, which may be understood as a version of Theorem 1.1 where sets of *controlling coordinates* play the role that coverings have in Theorem 1.1.

Let us recall that any compact Hausdorff space can be embedded in a cube $[0,1]^\Gamma$ for some Γ. So let $K \subset [0,1]^\Gamma$ and $x \in C(K)$. Since any continuous function on K is uniformly continuous, given $\varepsilon > 0$ there must exist a finite set $T \subset \Gamma$ and $\delta > 0$ such that

$$s, t \in K, \quad \sup_{\gamma \in T} |s(\gamma) - t(\gamma)| < \delta \Longrightarrow |x(s) - x(t)| < \varepsilon. \tag{1.1}$$

Following [8], we say that *T ε-controls x with δ* whenever (1.1) holds.

Corollary 1.2 *Let K be a compact space. The Banach space $C(K)$ admits an equivalent pointwise lower semicontinuous and LUR norm if and only if there is a countable family of subsets $\{C_n : n \in \mathbb{N}\}$ in $C(K)$ such that for every $x \in C(K)$ and every $\varepsilon > 0$ there exist $q \in \mathbb{N}$, a pointwise open half space H with $x \in H \cap C_q$, a finite set $T \subset \Gamma$, and $\delta > 0$ such that T ε-controls every $y \in H \cap C_q$ with δ.*

Therefore, roughly speaking, the existence of a LUR renorming in $C(K)$ is equivalent to describing *regularly* the members of a finite covering of K on which each $x \in C(K)$ has arbitrarily small oscillation, alternatively a finite set of coordinates that ε-controls it. The *regularity* of these *descriptions* is based, as in the case of [2], on the existence of half spaces with certain properties; this is the motivation for developing in Section 3 a method to obtain such half spaces. The characterization and the method together allow us to deduce in Section 4 a unified approach to reprove two new results in this field: on the one hand, the existence of a LUR norm in a Banach space X such that X^* admits a LUR dual norm, a result due to R. Haydon [3]; on the other hand, the existence of such renorming in $C(K)$ spaces where K belongs to a particular class of Rosenthal compacta [6].

As usual we denote by $(K, \mathcal{T})$ a compact Hausdorff topological space and by $C(K)$ the Banach space of real-valued continuous functions on K, endowed with the supremum norm $\|x\|_\infty = \sup\{|x(t)| : t \in K\}$. Let us remember that a norm $\|\cdot\|$ on a normed space X is said to be locally uniformly rotund (LUR) if $\lim_{n\to\infty} \|x - x_n\| = 0$ whenever $\lim_{n\to\infty}(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0$.

Let us recall that given a bounded set A of a normed space X, $\alpha(A)$, the *Kuratowski index of non-compactness* of A, is defined by $\alpha(A) = \inf_{n\in\mathbb{N}} \alpha(A, n)$, where

$$\alpha(A, n) = \inf\{\varepsilon > 0 : A \text{ can be covered by } n \text{ sets of diameter less than } \varepsilon\}.$$

We shall need the characterizations of Banach spaces admitting a LUR norm of the following theorem.

Theorem 1.3 ([2, 8, 10]) *Let X be a Banach space and let F be a norming linear subspace of X^*. The following assertions are equivalent:*

(i) *X admits an equivalent $\sigma(X,F)$-lower semicontinuous LUR norm;*

(ii) *there is a countable family of subsets $\{X_n : n \in \mathbb{N}\}$ in X such that given $\varepsilon > 0$ and $x \in X$ there exist $n \in \mathbb{N}$ with $x \in X_n$ and a $\sigma(X,F)$-open half space H containing x such that* $\operatorname{diam}(H \cap X_n) < \varepsilon$;

(iii) *there is a countable family of subsets $\{X_n : n \in \mathbb{N}\}$ in X such that given $\varepsilon > 0$ and $x \in X$ there exist $n \in \mathbb{N}$ with $x \in X_n$ and a $\sigma(X,F)$-open half space H containing x such that $\alpha(H \cap X_n) < \varepsilon$.*

2 A Characterization

Proof of Theorem 1.1 Assume that $C(K)$ admits a pointwise semicontinuous LUR norm. If $C(K) = \bigcup_{n\in\mathbb{N}} C_n$ is the decomposition of Theorem 1.3(ii), given $x \in C(K)$ and $\varepsilon > 0$, let n be a natural number and let H be a pointwise open half space such that $x \in C_n \cap H$ and

$$\operatorname{diam}(H \cap C_n) < \varepsilon/3. \tag{2.1}$$

Since x is continuous, by compactness we get a finite covering $\mathcal{L}$ of K such that the oscillation of x in every $L \in \mathcal{L}$ is

$$\operatorname{osc}(x, L) < \varepsilon/3. \tag{2.2}$$

From (2.1) and (2.2) it follows that $\operatorname{osc}(y, L) < \varepsilon$ for every $y \in H \cap C_n$ and every $L \in \mathcal{L}$.

Conversely let $\{C_n : n \in \mathbb{N}\}$ be the family of the statement. Then given $\varepsilon > 0$ and $x \in C(K)$ there are $n \in \mathbb{N}$ with $x \in C_n$, a pointwise open half space H containing x, and a finite covering $\mathcal{L}$ of K such that $\operatorname{osc}(y, L) < \varepsilon/9$ for every $y \in H \cap C_n$ and every $L \in \mathcal{L}$. If $C_{n,M} = \{x \in C_n : \|x\|_\infty \le M\}$, then $C_n = \bigcup_{M\in\mathbb{N}} C_{n,M}$.

Following [6, Proposition 5], take $x \in C_{n,M}$ and set $\{I_j\}_{j=1}^{\ell}$ a finite family of open real intervals of length less than $\varepsilon/9$ satisfying $[-M, M] \subset \bigcup_{j=1}^{\ell} I_j$. If $\mathcal{L} = \{L_i\}_{i=1}^m$ for some $m \in \mathbb{N}$, then for every $1 \le i \le m$ choose a point $s_i \in L_i$ and for every map $\pi : \{1, \dots, m\} \to \{1, \dots, \ell\}$ fix a function $x_\pi \in H \cap C_{n,M}$ satisfying $x_\pi(s_i) \in I_{\pi(i)}$ for all $1 \le i \le m$ (whenever this is possible). We claim that for these x_π we have $H \cap C_{n,M} \subset \bigcup_\pi B(x_\pi, \varepsilon/3)$. Indeed, if $y \in H \cap C_{n,M}$, then for every $1 \le i \le m$ there exists $1 \le k_i \le \ell$ such that $y(s_i) \in I_{k_i}$. Denote by π the map $i \mapsto k_i$. If $s \in K$, there exists $1 \le i \le m$ such that $s \in L_i$ so that

$$\begin{aligned}|y(s) - x_\pi(s)| &\le |y(s) - y(s_i)| + |y(s_i) - x_\pi(s_i)| + |x_\pi(s_i) - x_\pi(s)| \\ &< \operatorname{osc}(y, L_i) + \operatorname{length}(I_{\pi(i)}) + \operatorname{osc}(x_\pi, L_i) < \varepsilon/3.\end{aligned}$$

Since the Kuratowski index of non-compactness of $H \cap C_{n,M}$ is less than ε, from Theorem 1.3(iii) we conclude that $C(K)$ admits an equivalent pointwise semicontinuous LUR norm. ∎

Proof of Corollary 1.2 Assume that $C(K)$ admits a pointwise lower semicontinuous LUR norm. Take C_n, $x \in C(K)$, $\varepsilon > 0$, H, $x \in H \cap C_n$ as in the proof of Theorem 1.1 satisfying (2.1). If the finite set $T \subset \Gamma$ $\varepsilon/3$-controls x with $\delta > 0$, it is easy to check that T ε-controls every $y \in H \cap C_n$ with $\delta > 0$.

Conversely let $\{C_n : n \in \mathbb{N}\}$ in $C(K)$ satisfy the assertion of the corollary. Given $x \in C(K)$ and $\varepsilon > 0$, choose $q \in \mathbb{N}$, a pointwise open half space H with $x \in H \cap C_q$, a finite set $T \subset \Gamma$, and $\delta > 0$ such that

$$T \ \varepsilon\text{-controls every } y \in H \cap C_q \text{ with } \delta > 0. \tag{2.3}$$

Given $s \in K$, let V_s be the open neighbourhood of s made up of all $t \in K$ such that $\sup_{\gamma \in T} |s(\gamma) - t(\gamma)| < \delta/2$. The compactness of K yields a finite covering $\mathcal{L}$ of K such that $\sup_{\gamma \in T} |s(\gamma) - t(\gamma)| < \delta$ for each $s, t \in L$, and each $L \in \mathcal{L}$. This and (2.3) show that $|y(s) - y(t)| < \varepsilon$ whenever $s, t \in L$, $y \in H \cap C_q$, and $L \in \mathcal{L}$. To finish the proof it is enough to apply Theorem 1.1. ■

Remark Recall that a subset A of X is said to be *radial* if for every $x \in X$ there exists $\rho > 0$ such that $\rho x \in A$. A linear subspace F of X^* is called *norming* whenever $|x| = \sup\{|f(x)| : f \in B_{X^*} \cap F\}$, $x \in X$, is an equivalent norm on X. Theorem 1.1 and Corollary 1.2 hold if we replace $C(K)$ by any linear subspace X of it, if we change the pointwise topology by $\sigma(X, F)$, where F is a norming subspace of X^*, and if the sets C_n are taken to be a covering of some radial subset of X and the condition holds for any $x \in \bigcup_{n=1}^{\infty} C_n$. This observation may be of some use in applying the above characterizations to spaces $C_0(L)$, where L is a locally compact space. Moreover, it shows that to apply the characterizations above, it is enough to decompose the unit ball $B_{C(K)} = \bigcup C_n$ instead of the whole space $C(K)$, as will be done below.

To apply Theorem 1.1 it is enough to show that for every $\varepsilon > 0$, there is a countable family of subsets $\{C_{n,\varepsilon} : n \in \mathbb{N}\}$ in $C(K)$ such that for every $x \in C(K)$ there are $q \in \mathbb{N}$ and a pointwise open half space H with $x \in H \cap C_{q,\varepsilon}$, together with a finite covering $\mathcal{L}$ of K such that $|y(s) - y(t)| < \varepsilon$ whenever $s, t \in L$, $y \in H \cap C_{q,\varepsilon}$, and $L \in \mathcal{L}$. Indeed, the (countable) family $\{C_{n,1/m} : n, m \in \mathbb{N}\}$ satisfies the requirements of Theorem 1.1. A similar remark holds for Corollary 1.2.

3 A Method for Constructing Half Spaces

To apply Theorem 1.1 and Corollary 1.2 to concrete compact spaces K, it is necessary to obtain a method to associate each $x \in C(K)$ and each $\varepsilon > 0$ with a finite covering $\mathcal{L}$ of K such that $\operatorname{osc}(x, L) < \varepsilon$ for every $L \in \mathcal{L}$ or, alternatively, a finite set of coordinates that ε-controls x. Often this method gives a decomposition of $C(K)$ that fulfills this requirement, where H is not a pointwise open half space, but a finite intersection of pointwise open half spaces. In general it is not possible to obtain a refinement of this decomposition for which the above characterization holds [4]. In the lemma below we develop a method to obtain necessary conditions to get such half spaces. Specialists will recognize in (3.3) a rigidity condition.

Lemma 3.1 *Let φ_k, $1 \le k \le n$, be convex and lower semicontinuous maps on a convex set B of a locally convex space X. Let $A_0 \subset B$ for which*

$$\operatorname{osc}(\varphi_k, A_0) \le 1 \quad \text{for all } 1 \le k \le n. \tag{3.1}$$

Let δ and θ be such that $0 < 4\delta^{1/n} \le \theta \le 1$. Fix $x \in A_0$ and for every $1 \le k \le n$ set $A_k = \{y \in A_{k-1} : \varphi_k(x) - \varphi_k(y) < \delta\}$. Suppose that for every $1 \le k \le n$ we have

$$\varphi_k(x) \ge \sup\{\varphi_k(y) : \ y \in A_{k-1}\} - \delta, \tag{3.2}$$

$$\{y \in A_{k-1} : \ \delta \le \varphi_k(x) - \varphi_k(y) < \theta\} = \varnothing. \tag{3.3}$$

Then there exists a continuous linear map f on X such that

$$\{y \in A_0 : f(x-y) < 1\} \subset A_n.$$

Proof Set $q = 4/\theta$ and $\varphi = \sum_{i=1}^{n} q^{n+1-i}\varphi_i$. Since φ is a convex and lower semicontinuous function on B, there must exist an ε-subdifferential at x for every $\varepsilon > 0$ [9, p. 48]. Then there exists a continuous linear map g on X such that $\varphi(x) - \varphi(y) < g(x-y) + \theta/6$ for every $y \in B$. Set $S = \{y \in A_0 : 6\,g(x-y) < \theta\}$. We will show by induction that $S \subset A_k$ for every $1 \le k \le n$. Clearly $S \subset A_0$. Assume that for some k, $1 \le k \le n$, we have $S \subset A_{k-1}$ and pick $y \in S$. Since $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{k-1} \supseteq S$, from (3.2) we get $\varphi_i(x) - \varphi_i(y) \ge -\delta$ for every $1 \le i \le k-1$. From this and (3.1) we have

$$\begin{aligned}
q^{n+1-k}(\varphi_k(x) - \varphi_k(y)) &= \varphi(x) - \varphi(y) - \sum_{1 \le i \le k-1} q^{n+1-i}(\varphi_i(x) - \varphi_i(y)) \\
&\qquad - \sum_{k+1 \le i \le n} q^{n+1-i}(\varphi_i(x) - \varphi_i(y)) \\
&< g(x-y) + \frac{\theta}{6} + \delta \sum_{1 \le i \le k-1} q^{n+1-i} + \sum_{k+1 \le i \le n} q^{n+1-i}.
\end{aligned}$$

Then

$$q^{n+1-k}(\varphi_k(x) - \varphi_k(y)) < \frac{\theta}{3} + \frac{\delta q^{n+1}}{q-1} + \frac{q^{n+1-k}}{q-1}.$$

Taking into account that $q = 4/\theta$, the above inequality yields

$$\varphi_k(x) - \varphi_k(y) < \frac{\theta}{3} + \theta\frac{\delta(4/\theta)^n + 1}{4-\theta} < \frac{\theta}{3} + \frac{2\theta}{4-\theta} \le \theta.$$

Since $y \in A_{k-1}$, from (3.3) we deduce that $\varphi_k(x) - \varphi_k(y) < \delta$, so $y \in A_k$. ∎

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4 Some Applications

4.1 Namioka–Phelps Compacta

In this section we will deduce from Theorem 1.1 and Lemma 3.1 the existence of an equivalent locally uniformly rotund norm on $C(K)$, when K is a *Namioka–Phelps* compact. This result was proved by R. Haydon [3], deducing that a Banach space X has an equivalent LUR norm whenever X^* has a LUR dual norm. This class of compacta was introduced by M. Raja [11], proving that (B_{X^*}, ω^*) belongs to this class whenever X^* is a dual Banach space with a LUR dual norm.

Let us recall that a family $\mathcal{H} = \{H_i : i \in I\}$ of subsets of a topological space $(X, \mathcal{T})$ is said to be $\mathcal{T}$-*isolated* if for every $i \in I$

$$H_i \cap \overline{\bigcup\{H_j : j \in I,\ j \neq i\}}^{\mathcal{T}} = \varnothing.$$

We will say that $\mathcal{H}$ is a $\mathcal{T}$-σ-*isolated* family if $\mathcal{H}$ is a countable union of $\mathcal{T}$-isolated families.

A collection $\mathcal{N}$ of subsets of a topological space $(X, \mathcal{T})$ is said to be a *network* for the topology $\mathcal{T}$ if for every $U \in \mathcal{T}$ and every $x \in U$ there exists $N \in \mathcal{N}$ such that $x \in N \subset U$.

Definition 4.1 ([11]) A compact Hausdorff space $(K, \mathcal{T})$ is said to be a *Namioka–Phelps compact* if there is a $\mathcal{T}$-lower semicontinuous metric ρ on K such that the metric topology induced by ρ has a network that is $\mathcal{T}$-σ-isolated.

In this section, using ideas of [3], we will deduce from our characterization and our method of constructing half spaces that $C(K)$ admits an equivalent pointwise lower semicontinuous LUR norm when K is a Namioka–Phelps compact. As in [3], we first show that there exists a σ-isolated covering of K with some special properties (see Theorem 4.3 below). Then we associate each $x \in C(K)$ and each $\varepsilon > 0$ with a finite covering $\mathcal{L}$ of K such that

$$\operatorname{osc}(x, L) < \varepsilon \quad \text{for every } L \in \mathcal{L}, \tag{4.1}$$

and using Lemma 3.1 we will deduce that the requirements of Theorem 1.1 are fulfilled.

Definition 4.2 Given a compact space $(K, \mathcal{T})$, a family $\mathcal{I}$ of subsets of K and a subset H of K, we say that $\mathcal{I}$ is *rigidly finite at H* when the family $\{I \in \mathcal{I} : I \cap H \neq \varnothing\}$ is finite, nonempty and $H \cap \overline{\bigcup\{I \in \mathcal{I} : I \cap H = \varnothing\}} = \varnothing$.

We start by proving the following result based on [3, Lemma 3.3]. It is essential to associate with each $x \in C(K)$ and each $\varepsilon > 0$ a finite covering $\mathcal{L}$ of K for which (4.1) holds; moreover it plays a key role in fulfilling the requirements of Lemma 3.1, see Proposition 4.5(iii)–(iv) and (4.3) below.

Theorem 4.3 *Let K be a compact space and let $\mathcal{I}$ be a σ-isolated covering of K. Then there exists another covering $\mathcal{J}$ of K such that $\mathcal{J} = \bigcup_{i \in \mathbb{N}} \mathcal{J}(i)$, where each family $\mathcal{J}(i)$ is isolated and the following hold:*

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(i) *for every nonempty closed subset H of K there exists $i \in \mathbb{N}$ such that $\mathcal{J}(i)$ is rigidly finite at H;*
(ii) *for every $J \in \mathcal{J}$ there is $I \in \mathcal{I}$ such that $J \subset \bar{I}$.*

As usual, given a family of sets $\mathcal{J}$, the symbol $\bigcup \mathcal{J}$ stands for the union of all the elements of $\mathcal{J}$.

Lemma 4.4 *Let K be a compact space, let H be a closed subset of K and $\mathcal{J}$ an isolated family in K. If $\mathcal{J}$ is not rigidly finite at H, then either $H \cap \overline{\bigcup \mathcal{J}} = \varnothing$ or $H \cap \overline{\bigcup \mathcal{J}} \setminus \bigcup \mathcal{J} \neq \varnothing$.*

Proof Set $\mathcal{M} = \{J \in \mathcal{J} : J \cap H \neq \varnothing\}$. If $\mathcal{M}$ is empty, then $H \cap \bigcup \mathcal{J} = \varnothing$ and $H \cap \overline{\bigcup \mathcal{J}} = H \cap \overline{\bigcup \mathcal{J}} \setminus \bigcup \mathcal{J}$.

Assume now that $\mathcal{M} \neq \varnothing$. If every neighborhood of H meets infinitely many members of $\mathcal{J}$, then by compactness of H some point $p \in H$ has the property that every neighborhood of p meets infinitely many members of $\mathcal{J}$. Since $\mathcal{J}$ is isolated, p does not belong to any member of $\mathcal{J}$, so $p \in \overline{\bigcup \mathcal{J}} \setminus \bigcup \mathcal{J}$. ∎

Proof of Theorem 4.3 Let $\mathcal{I}$ be a σ-isolated covering of K. Then $K = \bigcup\{I : I \in \mathcal{I}\}$ and for every $i \in \mathbb{N}$ there is an isolated family $\mathcal{I}(i)$ such that $\mathcal{I}$ is the family of all sets that belong to some $\mathcal{I}(i)$ for some $i \in \mathbb{N}$. The proof is divided into three steps.

Step 1. We can assume that $\overline{\bigcup \mathcal{I}(i)} \setminus \bigcup \mathcal{I}(i)$ is closed for every $i \in \mathbb{N}$. Indeed, for every $i \in \mathbb{N}$ we define the family

$$\widetilde{\mathcal{I}}(i) := \Big\{ \bar{I} \setminus \overline{\bigcup \mathcal{I}(i) \setminus I} : I \in \mathcal{I}(i) \Big\}.$$

It is clear that $\widetilde{\mathcal{I}}(i)$ is an isolated family and that each $\overline{\bigcup \widetilde{\mathcal{I}}(i)} \setminus \bigcup \widetilde{\mathcal{I}}(i)$ is just the set of all points t in K such that each neighbourhood of t meets at least two members of $\widetilde{\mathcal{I}}(i)$. Then $\overline{\bigcup \widetilde{\mathcal{I}}(i)} \setminus \bigcup \widetilde{\mathcal{I}}(i)$ is closed. Since each $\mathcal{I}(i)$ is isolated, we have $I \subset \bar{I} \setminus \overline{\bigcup \mathcal{I}(i) \setminus I}$ for every $I \in \mathcal{I}(i)$, therefore the family of all sets that belong to some $\widetilde{\mathcal{I}}(i)$ for some $i \in \mathbb{N}$ is a σ-isolated covering of K. From now on we will write $\mathcal{I}(i)$ instead of $\widetilde{\mathcal{I}}(i)$ for $i \in \mathbb{N}$.

Step 2. The construction of $\mathcal{J}$. Following [3] we define recursively isolated families $\mathcal{I}(i_1, \dots, i_n)$ for $(i_1, \dots, i_n) \in \mathbb{N}^{<\omega}$, the family of all finite sequences of natural numbers. If $\mathcal{I}(i_1, \dots, i_n)$ has been defined, set

$$I(i_1, \dots, i_n) = \bigcup \mathcal{I}(i_1, \dots, i_n) \quad \text{and} \quad J(i_1, \dots, i_n) = \overline{I(i_1, \dots, i_n)} \setminus I(i_1, \dots, i_n).$$

Given $j \in \mathbb{N}$, let $\mathcal{I}(i_1, \dots, i_n, j)$ be the (isolated) family

$$\mathcal{I}(i_1, \dots, i_n, j) := \{J(i_1, \dots, i_n) \cap I : I \in \mathcal{I}(j)\}.$$

From Step 1 it follows that each $J(i_1, \dots, i_n)$ is closed and $I(i_1, \dots, i_n) = \varnothing$ if $(i_1, \dots, i_n)$ has repeated terms. Then set $\mathcal{J}$ as the family of all sets in any $\mathcal{I}(i_1, \dots, i_n)$ for some $(i_1, \dots, i_n) \in \mathbb{N}^{<\omega}$. From the choice of $\mathcal{J}$ we have that (ii) holds. Let us show condition (i).

Step 3. For every nonempty closed subset H of K there is $(i_1,\dots,i_n)\in\mathbb{N}^{<\omega}$ such that $\mathcal{I}(i_1,\dots,i_n)$ is rigidly finite at H. Indeed, otherwise from Lemma 4.4 there exists a nonempty closed set H such that

$$H\cap\overline{I(i_1,\dots,i_n)}=\varnothing \text{ or } H\cap J(i_1,\dots,i_n)\neq\varnothing \quad \text{for any } (i_1,\dots,i_n)\in\mathbb{N}^{<\omega}. \tag{4.2}$$

Since $\mathcal{I}$ is a covering of K, there is $i\in\mathbb{N}$ such that $H\cap I(i)\neq\varnothing$. If we write $i_1=\min\{i\in\mathbb{N}: H\cap\overline{I(i)}\neq\varnothing\}$, then $H\cap\overline{I(i_1)}\neq\varnothing$ and from (4.2) it follows that $H\cap J(i_1)\neq\varnothing$. Thus, there is $j\in\mathbb{N}$ such that $\varnothing\neq H\cap J(i_1)\cap I(j)=H\cap I(i_1,j)$ and we can set $i_2=\min\{i\in\mathbb{N}: H\cap\overline{I(i_1,i)}\neq\varnothing\}$. Note that $i_2\neq i_1$, since otherwise $I(i_1,i_2)$ is empty. Proceeding recursively we can obtain a sequence of pairwise distinct natural numbers $(i_n)_{n\geq 1}$ satisfying the following:

(a) $H\cap\overline{I(\ell)}=\varnothing$ if $\ell<i_1$ but $H\cap\overline{I(i_1)}\neq\varnothing$;
(b) $H\cap\overline{I(i_1,\dots,i_{n-1},\ell)}=\varnothing$ if $\ell<i_n$ and $n\geq 2$ and $H\cap\overline{I(i_1,\dots,i_n)}\neq\varnothing$.

We claim that $(i_n)_{n\geq1}$ is strictly increasing. Indeed, from $H\cap\overline{I(i_1,i_2)}\subset H\cap\overline{I(i_2)}$ and (a) we get $i_2>i_1$. Let $n\geq 2$. Since each set $J(i_1,\dots,i_n)$ is closed, we have

$$\begin{aligned} I(i_1,\dots,i_{n+1}) &= J(i_1,\dots,i_n)\cap I(i_{n+1})\subset\overline{I(i_1,\dots,i_n)}\cap I(i_{n+1})\\ &\subset J(i_1,\dots,i_{n-1})\cap I(i_{n+1})=I(i_1,\dots,i_{n-1},i_{n+1}),\end{aligned}$$

which implies $H\cap\overline{I(i_1,\dots,i_{n+1})}\subset H\cap\overline{I(i_1,\dots,i_{n-1},i_{n+1})}$. From (b) we deduce that $i_{n+1}>i_n$.

Finally, by compactness there exists a point $t\in\bigcap_{n=1}^{+\infty}H\cap J(i_1,\dots,i_n)$ and there is $i\in\mathbb{N}$ such that $t\in I(i)$. For every $n\in\mathbb{N}$ it follows that $t\in H\cap J(i_1,\dots,i_n)\cap I(i)=H\cap I(i_1,\dots,i_n,i)$ and by minimality we get $i\geq i_{n+1}$ for all $n\in\mathbb{N}$, which is a contradiction. ■

We denote by K a (Namioka–Phelps) compact space, by $\mathcal{T}$ its topology, and by ρ a $\mathcal{T}$-lower semicontinuous metric on K such that the metric topology induced by ρ has a network $\mathcal{D}$ that is $\mathcal{T}$-σ-isolated. From now on and unless otherwise stated, all the closures are taken with respect to $\mathcal{T}$. Theorem 4.3 applied to the $\mathcal{T}$-σ-isolated family $\mathcal{D}^\ell=\{A\in\mathcal{D}:\rho\text{-}\operatorname{diam}(A)\leq 1/\ell\}$ yields a covering $\mathcal{I}^\ell$ of K made up by all the sets that belong to any $\mathcal{I}^\ell(i)$ for $i\in\mathbb{N}$, where each family $\mathcal{I}^\ell(i)$ is $\mathcal{T}$-isolated, satisfying the following:

(a) each set of $\mathcal{I}^\ell$ is included in the closure of some set of ρ-diameter at most $1/\ell$;
(b) for every nonempty closed subset H of K there is $i\in\mathbb{N}$ such that $\mathcal{I}^\ell(i)$ is rigidly finite at H.

Following [3], for every $x\in C(K)$ and every $\varepsilon>0$, we are going to describe a method to split up every closed subset L of K, when the oscillation of x on L is bigger than ε, in such a way that Theorem 1.1 gives a pointwise lower semicontinuous renorming.

Given $\mathcal{M}\subset\mathcal{I}^\ell(i)$ such that the cardinal of $\mathcal{M}$ is $\#\mathcal{M}=m$, let

$$\Phi(x,L,\mathcal{M})=\frac{1}{m}\sum_{M\in\mathcal{M}}\sup x\restriction_{L\cap\overline{M}} \quad\text{and}\quad \Psi(x,L,\mathcal{M})=\frac{1}{m}\sum_{M\in\mathcal{M}}\inf x\restriction_{L\cap\overline{M}}.$$

Proposition 4.5 ([3]) *Let $x \in C(K)$ and $\varepsilon > 0$. Then there exists $\ell \in \mathbb{N}$ such that if L is a closed subset of K with $\operatorname{osc}(x, L) \geq \varepsilon$, then there are $m, n, i, j \in \mathbb{N}$ and a pair $(\mathcal{M}, \mathcal{N})$ satisfying*

(i) $\mathcal{M} \subset \mathcal{I}^\ell(i)$, $\mathcal{N} \subset \mathcal{I}^\ell(j)$, $\#\mathcal{M} = m$, $\#\mathcal{N} = n$, *and* $A \cap L \neq \varnothing$ *for every* $A \in \mathcal{M} \cup \mathcal{N}$;
(ii) $\overline{\bigcup \mathcal{M}} \bigcap \overline{\bigcup \mathcal{N}} = \varnothing$;
(iii) $\Phi(x, L, \mathcal{M}) > (1 - 1/m) \sup x\restriction_L + 1/m \sup x\restriction_{L \cap \overline{\bigcup(\mathcal{I}^\ell(i) \setminus \mathcal{M})}}$;
(iv) $\Psi(x, L, \mathcal{N}) < (1 - 1/n) \inf x\restriction_L + 1/n \ \inf x\restriction_{L \cap \overline{\bigcup(\mathcal{I}^\ell(j) \setminus \mathcal{N})}}$.

Proof It is easy to see that x is ρ-uniformly continuous. Thus, there exists $\ell \in \mathbb{N}$ such that $|x(s) - x(t)| < \varepsilon/3$ whenever $\rho(s, t) \leq 1/\ell$ with $s, t \in K$. Suppose that $\operatorname{osc}(x, L) \geq \varepsilon$ for some closed subset L of K and let $H_1 = \{t \in L : x(t) = \sup x\restriction_L\}$. According to the properties of $\mathcal{I}^\ell$ there exists $i \in \mathbb{N}$ such that the family $\mathcal{M} := \{I \in \mathcal{I}^\ell(i) : I \cap H_1 \neq \varnothing\}$ is finite and nonempty, say $\#\mathcal{M} = m$ for some $m \in \mathbb{N}$, and $H_1 \cap \overline{\bigcup(\mathcal{I}^\ell(i) \setminus \mathcal{M})} = \varnothing$. This clearly implies

$$\sup x\restriction_{L \cap \overline{\bigcup(\mathcal{I}^\ell(i) \setminus \mathcal{M})}} < \sup x\restriction_L. \tag{4.3}$$

Note that $\Phi(x, L, \mathcal{M}) = \sup x\restriction_L$ since $\sup x\restriction_L = \sup x\restriction_{L} \cap \overline{M}$ for all $M \in \mathcal{M}$. A similar argument with $H_2 = \{t \in L : x(t) = \inf x\restriction_L\}$ gives a $j \in \mathbb{N}$ such that the set $\mathcal{N} := \{I \in \mathcal{I}^\ell(j) : I \cap H_2 \neq \varnothing\}$ is finite and nonempty, say $\#\mathcal{N} = n$ for some $n \in \mathbb{N}$, and $\Psi(x, L, \mathcal{N}) = \inf x\restriction_L < \inf x\restriction_{L \cap \overline{\bigcup(\mathcal{I}^\ell(j) \setminus \mathcal{N})}}$. Hence,(i), (iii), and (iv) hold.

To prove (ii) observe that given $M \in \mathcal{M}$, there exists $A \in \mathcal{D}$ such that $M \subset \overline{A}$ and ρ- $\operatorname{diam}(A) \leq 1/\ell$. By the choice of ℓ, it follows that $\operatorname{osc}(x, \overline{M}) \leq \varepsilon/3$ and since $M \cap H_1 \neq \varnothing$, we get $x(t) \geq \sup x\restriction_L - \varepsilon/3$ for every $t \in \overline{M}$. Similarly, $x(t) \leq \inf x\restriction_L + \varepsilon/3$ for every $N \in \mathcal{N}$ and every $t \in \overline{N}$. Condition (ii) follows from the fact that $\operatorname{osc}(x, L) \geq \varepsilon$. ■

We say that a pair $(\mathcal{M}, \mathcal{N})$ satisfying Proposition 4.5(i)–(iv) is a *good choice* of x of type (ℓ, m, n, i, j) on L. From condition (ii), for every good choice $(\mathcal{M}, \mathcal{N})$ we fix a pair of closed sets, $X(\mathcal{M}, \mathcal{N})$ and $Y(\mathcal{M}, \mathcal{N})$, such that $K = X(\mathcal{M}, \mathcal{N}) \cup Y(\mathcal{M}, \mathcal{N})$ and $\bigcup \mathcal{N} \cap X(\mathcal{M}, \mathcal{N}) = \bigcup \mathcal{M} \cap Y(\mathcal{M}, \mathcal{N}) = \varnothing$.

Observe that given $x \in C(K)$ and a closed subset L of K such that $\operatorname{osc}(x, L) \geq \varepsilon$ for some $\varepsilon > 0$, there exists a good choice $(\mathcal{M}, \mathcal{N})$ of x on L of some type and we can split L up into $L \cap X(\mathcal{M}, \mathcal{N})$ and $L \cap Y(\mathcal{M}, \mathcal{N})$. Proposition 4.5(iii)–(iv) enable us to prove the next lemma, which shows that the good choice of x on L is unique if we fix its type and that a *suitable rigidity condition* holds. From this a *rule* will be deduced to decompose every closed set on which a continuous function has an oscillation not less than ε. Set

$$\mathcal{B}(L, \ell, m, i) = \{\mathcal{M} \subset \mathcal{I}^\ell(i) : \#\mathcal{M} = m, \ M \cap L \neq \varnothing \text{ for all } M \in \mathcal{M}\}.$$

Lemma 4.6 ([3]) *If $(\mathcal{M}, \mathcal{N})$ is a good choice of $x \in C(K)$ on a closed subset L of K of type (ℓ, m, n, i, j), then*

(i) $\sup\{\Phi(x, L, \mathcal{M}') : \mathcal{M}' \in \mathcal{B}(L, \ell, m, i), \mathcal{M}' \neq \mathcal{M}\} < \Phi(x, L, \mathcal{M})$;
(ii) $\inf\{\Psi(x, L, \mathcal{N}') : \mathcal{N}' \in \mathcal{B}(L, \ell, n, j), \mathcal{N}' \neq \mathcal{N}\} > \Psi(x, L, \mathcal{N})$.

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Proof If $\mathcal{M}' \in \mathcal{B}(L, \ell, m, i)$ and $\mathcal{M}' \neq \mathcal{M}$, then there exists $M_0 \in \mathcal{M}' \setminus \mathcal{M} \subset \mathcal{I}^\ell(i) \setminus \mathcal{M}$ so that

$$\begin{aligned}\Phi(x, L, \mathcal{M}') &= \frac{1}{m}\Big(\sum_{M \in \mathcal{M}' \setminus \{M_0\}} \sup x\restriction_{L \cap \overline{M}} \Big) + \frac{1}{m} \sup x\restriction_{L \cap \overline{M_0}} \\ &\leq \Big(1 - \frac{1}{m}\Big) \sup x\restriction_L + \frac{1}{m} \sup x\restriction_{L \cap \overline{\bigcup(\mathcal{I}^\ell(i) \setminus \mathcal{M})}}.\end{aligned}$$

The proof of (ii) is similar. ∎

Given $x \in C(K)$ with $\operatorname{osc}(x, K) \geq \varepsilon$, we are going to iterate the above decomposition to get a covering $\mathcal{L}$ fulfilling the requirements of Theorem 1.1. Such a covering should be finite, so this iterative process will be defined in such a way that it finishes after a finite number of steps.

In order to cope with this requirement, fix a map τ from the nonnegative integers into $\mathbb{N}^5$ with the property that for every $(\ell, m, n, i, j) \in \mathbb{N}^5$ the set $\tau^{-1}(\ell, m, n, i, j)$ is infinite. Let $\mathcal{S} = \{-1, 0, 1\}^{<\omega}$ be the set of all finite sequences of integers $s = (i_1, \dots, i_n)$, where $i_k \in \{-1, 0, 1\}$ for $1 \leq k \leq n$; this n is called the *length* $|s|$ of s. We agree that the *empty* sequence $s = (\cdot)$ belongs to $\mathcal{S}$ and has length zero. If $s \in \mathcal{S}$ and $i \in \{-1, 0, 1\}$, we write (s, i) for the element of $\mathcal{S}$ that extends s and has i in its last place. And $\mathcal{F}(K)$ will stand for the set of all closed subsets of K.

Proposition 4.7 ([3]) *Given $x \in C(K)$ and $\varepsilon > 0$, there exists a finite subset $\Omega \subset \mathcal{F}(K) \times \mathcal{S}$ and a tree order on Ω with the following properties:*

(i) *The unique minimal element of Ω is $(K, (\,\cdot\,))$.*
(ii) *An element (L, s) is maximal in Ω if and only if $\operatorname{osc}(x, L) < \varepsilon$.*
(iii) *If (L, s) is not maximal in Ω and $|s| = n$ with $n \geq 0$, then there are two possibilities:*
   (a) *There exists a good choice $(\mathcal{M}, \mathcal{N})$ of x of type $\tau(n)$ on L. Then the immediate successors of (L, s) in Ω are $(L \cap X(\mathcal{M}, \mathcal{N}), (s, 0))$ and $(L \cap Y(\mathcal{M}, \mathcal{N}), (s, 1))$.*
   (b) *No good choice of x of type $\tau(n)$ exists on L. In this case, the unique immediate successor of (L, s) in Ω is $(L, (s, -1))$.*

Moreover, the family $\mathcal{L} = \{L \in \mathcal{F}(K) : (L, s)$ is a maximal node of Ω for some $s \in \mathcal{S}\}$ is a finite covering of K satisfying $\operatorname{osc}(x, L) < \varepsilon$ for all $L \in \mathcal{L}$.

Proof Conditions (i)–(iii) define a tree Ω; we claim that it has no infinite branches. Indeed, otherwise there exists $\sigma \in \{-1, 0, 1\}^\omega$ and there is a sequence $\{(L_k, \sigma\restriction_k)\}_{k \geq 0}$ in Ω such that $(L_{k+1}, \sigma\restriction_{k+1})$ is an immediate successor of $(L_k, \sigma\restriction_k)$ for all $k \geq 0$. Since $(L_k)_k$ is a decreasing sequence of closed sets with the property that $\operatorname{osc}(x, L_k) \geq \varepsilon$ for all $k \geq 0$, it follows that $L = \bigcap_{k \geq 0} L_k$ is a nonempty closed set satisfying $\operatorname{osc}(x, L) \geq \varepsilon$. From Proposition 4.5 there exists a good choice $(\mathcal{M}, \mathcal{N})$ of x of type (ℓ, m, n, i, j) on L for some $\ell, m, n, i, j \in \mathbb{N}$. By compactness there must exist k_0 such that $(\mathcal{M}, \mathcal{N})$ is a good choice on L_k for all $k \geq k_0$. As $\tau^{-1}(\ell, m, n, i, j)$ is infinite, there is some $k \geq k_0$ such that $\tau(k) = (\ell, m, n, i, j)$. By construction, either $L_{k+1} = L_k \cap X(\mathcal{M}, \mathcal{N})$ or $L_{k+1} = L_k \cap Y(\mathcal{M}, \mathcal{N})$. Therefore, either $\bigcup \mathcal{M} \cap L_{k+1} = \varnothing$ or $\bigcup \mathcal{N} \cap L_{k+1} = \varnothing$, contradicting that $A \cap L \neq \varnothing$ for every $A \in \mathcal{M} \cup \mathcal{N}$.

According to König's lemma, Ω is a finite tree. Consequently, if $\mathcal{L}$ is the family of all sets L for which (L, s) is maximal in Ω for some $s \in \mathcal{S}$, we have that $\mathcal{L}$ is finite. Moreover, from the choice of Ω it follows that $\mathcal{L}$ is a finite covering of K such that for every $L \in \mathcal{L}$ we have $\operatorname{osc}(x, L) < \varepsilon$. ■

Now we are ready to prove the main result about Namioka–Phelps compacta.

Theorem 4.8 ([3]) *Let K be a Namioka–Phelps compact space. Then $C(K)$ admits an equivalent pointwise lower semicontinuous LUR norm.*

Proof We divide the proof of this theorem into three steps. Given $\varepsilon > 0$ we begin by decomposing the unit ball of $C(K)$ into countably many sets $\{C_n : n \in \mathbb{N}\}$ in such a way that for every $n \in \mathbb{N}$ the set C_n codifies the *countable* information relative to the tree Ω_x associated with each ε and $x \in C_n$ according to Proposition 4.7. In the second step, for every $n \in \mathbb{N}$ and every $x \in C_n$ we define a family of maps $\Phi(x)$ associated with x and we prove that $\Phi(x)$ fulfils the hypothesis of Lemma 3.1. Finally, we deduce that for every $n \in \mathbb{N}$ and every $x \in C_n$ there is a pointwise open half space H containing x such that $\Omega_y = \Omega_x$ for every $y \in H \cap C_n$. According to Proposition 4.7, the statement follows from Theorem 1.1.

Let $\varepsilon > 0$. We write $T(\mathcal{S})$ for the countable family of all finite trees $(\Omega, \sqsubseteq)$ in $\mathcal{S}$, where $\sqsubseteq$ is the end-extension order, with the property that every $\Omega \in T(\mathcal{S})$ has one minimal element and the set s^+ of the immediate successors of each s of Ω has at most two elements. Given $x \in B_{C(K)}$, let Ω_x be the tree associated with x and ε by Proposition 4.7. We denote by $P(\Omega_x)$ the tree made up by all $s \in \mathcal{S}$ for which there exists L such that $(L, s) \in \Omega_x$. If $\Omega \in T(\mathcal{S})$, let $C_\Omega = \{x \in B_{C(K)} : P(\Omega_x) = \Omega\}$.

Fix $\Omega \in T(\mathcal{S})$. We assume that $\#\Omega > 1$, otherwise $\operatorname{osc}(x, K) < \varepsilon$ for every $x \in C_\Omega$. For every $s \in \Omega$ and every $x \in C_\Omega$ we write $L_{x,s}$ for the closed subset of K such that $(L_{x,s}, s) \in \Omega_x$. If $\Omega_2 = \{s \in \Omega : \#s^+ = 2\}$, then for every $s \in \Omega_2$ and every $x \in C_\Omega$ there exists a good choice $(\mathcal{M}_{x,s}, \mathcal{N}_{x,s})$ of x on $L_{x,s}$ of type $\tau(|s|)$ such that

$$L_{x,(s,0)} = L_{x,s} \cap X(\mathcal{M}_{x,s}, \mathcal{N}_{x,s}) \quad \text{and} \quad L_{x,(s,1)} = L_{x,s} \cap Y(\mathcal{M}_{x,s}, \mathcal{N}_{x,s}). \tag{4.4}$$

If $\tau(|s|) = (\ell_s, m_s, n_s, i_s, j_s)$ we let

$$\alpha_s(x) = \sup\{\Phi(x, L_{x,s}, \mathcal{M}) : \mathcal{M} \in \mathcal{B}(L_{x,s}, \ell_s, m_s, i_s), \mathcal{M} \neq \mathcal{M}_{x,s}\};$$

$$\beta_s(x) = \inf\{\Psi(x, L_{x,s}, \mathcal{N}) : \mathcal{N} \in \mathcal{B}(L_{x,s}, \ell_s, n_s, j_s), \mathcal{N} \neq \mathcal{N}_{x,s}\}.$$

From Lemma 4.6 it follows that $\Phi(x, L_{x,s}, \mathcal{M}_{x,s}) > \alpha_s(x)$ and $\Psi(x, L_{x,s}, \mathcal{N}_{x,s}) < \beta_s(x)$ for every $s \in \Omega_2$ and every $x \in C_\Omega$. Hence, for every $r \in \mathbb{N}$ and every family $\mathcal{U} = \{(U_s, V_s) : s \in \Omega_2\}$ of pairs (U_s, V_s) of open real intervals with rational end points and length equal to $(1/12r)^{2\#\Omega_2}$, let $C_{\Omega,r,\mathcal{U}}$ be the set of all $x \in C_\Omega$ such that $\Phi(x, L_{x,s}, \mathcal{M}_{x,s}) \in U_s$ and $\Psi(x, L_{x,s}, \mathcal{N}_{x,s}) \in V_s$ for every $s \in \Omega_2$ and

$$r^{-1} \leq \min_{s \in \Omega_2}\{\Phi(x, L_{x,s}, \mathcal{M}_{x,s}) - \alpha_s(x), \beta_s(x) - \Psi(x, L_{x,s}, \mathcal{N}_{x,s})\}.$$

It is clear that $B_{C(K)}$ is the (countable) union of the sets $C_{\Omega,r,\mathcal{U}}$.

Fix $\Omega, r, \mathcal{U}$ and $x \in C_{\Omega,r,\mathcal{U}}$. For every $s \in \Omega_2$ and every $i \in \{0,1\}$ we define the map $\varphi_s^i : B_{C(K)} \to \mathbb{R}$ by

$$\varphi_s^i(y) = \begin{cases} \Phi(y, L_{x,s}, \mathcal{M}_{x,s})/2 & \text{if } i = 0, \\ -\Psi(y, L_{x,s}, \mathcal{N}_{x,s})/2 & \text{if } i = 1, \end{cases}$$

and fix the values $\theta = 1/3r$ and $\delta = (1/12r)^{2\#\Omega_2}$. If we write $\Phi(x)$ for the collection $\{\varphi_s^i : s \in \Omega_2,\ i = 0,1\}$, then $\Phi(x)$ is a family of convex and pointwise lower semi-continuous maps satisfying $\operatorname{osc}(\varphi, B_{C(K)}) \le 1$ for every $\varphi \in \Phi(x)$. The following result yields information about setting a suitable order on $\Phi(x)$ to apply Lemma 3.1.

Lemma 4.9 *Let $y \in C_{\Omega,r,\mathcal{U}}$ be such that $L_{y,s} = L_{x,s}$ for some $s \in \Omega_2$. If $i \in \{0,1\}$, then*

$$\varphi_s^i(x) > \varphi_s^i(y) - \delta \quad \textit{and} \quad \varphi_s^i(x) - \varphi_s^i(y) \notin [\delta, \theta). \tag{4.5}$$

Moreover,

$$\textit{if } \max_{i=0,1}(\varphi_s^i(x) - \varphi_s^i(y)) < \delta, \textit{ then } L_{y,(s,0)} = L_{x,(s,0)} \textit{ and } L_{y,(s,1)} = L_{x,(s,1)}. \tag{4.6}$$

Proof Since $L_{y,s} = L_{x,s}$, we have

$$\mathcal{M}_{x,s} \in \mathcal{B}(L_{y,s}, \ell_s, m_s, i_s) \quad \text{and} \quad \mathcal{N}_{x,s} \in \mathcal{B}(L_{y,s}, \ell_s, n_s, j_s).$$

Moreover, if

$$\Phi(x, L_{x,s}, \mathcal{M}_{x,s}), \Phi(y, L_{y,s}, \mathcal{M}_{y,s}) \in U_s, \text{ and } \Psi(x, L_{x,s}, \mathcal{N}_{x,s}), \Psi(y, L_{y,s}, \mathcal{N}_{y,s}) \in V_s,$$

then

$$\begin{aligned} 2\varphi_s^0(y) &= \Phi(y, L_{y,s}, \mathcal{M}_{x,s}) \le \Phi(y, L_{y,s}, \mathcal{M}_{y,s}) \\ &< \Phi(x, L_{x,s}, \mathcal{M}_{x,s}) + \operatorname{length}(U_s) = 2\varphi_s^0(x) + \delta. \end{aligned}$$

Similarly we get $-2\varphi_s^1(y) > -2\varphi_s^1(x) - \delta$. Hence for each $i \in \{0,1\}$ we have $\varphi_s^i(x) > \varphi_s^i(y) - \delta$, and the first part of (4.5) follows.

To show the second part of (4.5) suppose that $\varphi_s^i(x) - \varphi_s^i(y) \in [\delta, \theta)$. Since $\varphi_s^0(x) - \varphi_s^0(y) < \theta$, we get

$$\begin{aligned} \Phi(y, L_{y,s}, \mathcal{M}_{x,s}) &> \Phi(x, L_{x,s}, \mathcal{M}_{x,s}) - 2\theta > \Phi(y, L_{y,s}, \mathcal{M}_{y,s}) - \operatorname{length}(U_s) - 2\theta \\ &> \Phi(y, L_{y,s}, \mathcal{M}_{y,s}) - 3\theta \ge \alpha_s(y), \end{aligned}$$

which implies $\mathcal{M}_{x,s} = \mathcal{M}_{y,s}$ so that $\Phi(y, L_{y,s}, \mathcal{M}_{x,s}) \in U_s$. Similarly we get

$$\Psi(y, L_{y,s}, \mathcal{N}_{x,s}) < \Psi(y, L_{y,s}, \mathcal{N}_{y,s}) + 3\theta \le \beta_s(y) \quad \text{and} \quad \mathcal{N}_{x,s} = \mathcal{N}_{y,s},$$

so that $\Psi(y, L_{y,s}, \mathcal{N}_{x,s}) \in V_s$. Nevertheless the inequalities $\varphi_s^i(x) - \varphi_s^i(y) \ge \delta$ for $i = 0,1$ imply $\Phi(y, L_{y,s}, \mathcal{M}_{x,s}) \le \Phi(x, L_{x,s}, \mathcal{M}_{x,s}) - 2\delta < \inf U_s$ and $\Psi(y, L_{y,s}, \mathcal{N}_{x,s}) \ge \Psi(x, L_{x,s}, \mathcal{N}_{x,s}) + 2\delta > \sup V_s$, a contradiction that establishes the second part of (4.5).

Finally, suppose that $\varphi_s^i(x) - \varphi_s^i(y) < \delta$ for every $i \in \{0,1\}$. Then $\varphi_s^i(x) - \varphi_s^i(y) < \theta$ and we get $(\mathcal{M}_{y,s}, \mathcal{N}_{y,s}) = (\mathcal{M}_{x,s}, \mathcal{N}_{x,s})$ as above. Hence $X(\mathcal{M}_{y,s}, \mathcal{N}_{y,s}) = X(\mathcal{M}_{x,s}, \mathcal{N}_{x,s})$ and $Y(\mathcal{M}_{y,s}, \mathcal{N}_{y,s}) = Y(\mathcal{M}_{x,s}, \mathcal{N}_{x,s})$, so (4.6) follows from (4.4). ∎

Let us turn to the proof of Theorem 4.8. To enumerate the family $\Phi(x)$, we introduce an order $\prec$ as follows. Given two distinct maps $\varphi_s^i, \varphi_t^j \in \Phi(x)$, we write $\varphi_s^i \prec \varphi_t^j$ if and only if either $|s| < |t|$, or $|s| = |t|$ and $s <_{\text{lex}} t$, where $<_{\text{lex}}$ is the lexicographic order, or $s = t$ and $i < j$. Then we write $\Phi(x)$ as $\{\varphi_k : 1 \le k \le 2\#\Omega_2\}$, where $\varphi_k \prec \varphi_\ell$ if and only if $1 \le k < \ell \le 2\,\#\Omega_2$. Note that for every $y \in C_{\Omega,r,\mathcal{U}}$ we have $L_{y,(\cdot)} = L_{x,(\cdot)} = K$ and if $L_{y,s} = L_{x,s}$ for some $s \in \Omega$ with $\#s^+ = 1$, then it follows that $L_{y,(s,-1)} = L_{y,s} = L_{x,s} = L_{x,(s,-1)}$. If for some $n \le 2\,\#\Omega_2$ there is $y \in C_{\Omega,r,\mathcal{U}}$ such that $\varphi_j(x) - \varphi_j(y) < \delta$ for all $j < n$, then according to Lemma 4.9 we get $\varphi_n(x) > \varphi_n(y) - \delta$ and $\varphi_n(x) - \varphi_n(y) \notin [\delta, \theta)$. Since $0 < 4\delta^{1/(2\#\Omega_2)} \le \theta < 1$, we can apply Lemma 3.1 to the family $\Phi(x)$ with $A_0 = C_{\Omega,r,\mathcal{U}}$ and $B = B_{C(K)}$. Then there exists a pointwise open half space H containing x such that $\Omega_y = \Omega_x$ for every $y \in H \cap C_{\Omega,r,\mathcal{U}}$. From Proposition 4.7 we have that the family $\mathcal{L}$ of the projections of all maximal elements of Ω_x into $\mathcal{F}(K)$ is a finite covering of K satisfying $\operatorname{osc}(y, L) < \varepsilon$ for every $y \in H \cap C_{\Omega,r,\mathcal{U}}$ and every $L \in \mathcal{L}$. According to Theorem 1.1, $C(K)$ admits an equivalent pointwise lower semicontinuous LUR norm. ∎

Remark A compact Hausdorff space is said to be *descriptive* if its topology has a σ-isolated network. If $(K, \mathcal{T})$ is descriptive, then there exists a metric ρ_K on K such that the metric topology induced by ρ_K has a $\mathcal{T}$-σ-isolated network. Thus, a proof similar to that of Theorem 4.8 shows that for any descriptive compact space K the linear subspace of all continuous functions on K that are ρ_K-uniformly continuous admits an equivalent pointwise lower semicontinuous LUR norm [3].

4.2 A Class of Rosenthal Compacta

In what follows we denote by Γ a Polish space, *i.e.,* a separable complete metric space. Let K be a separable and pointwise compact set of functions on Γ, assume further that each function $s \in K$ has only countably many discontinuities. It is clear that every s in K is a Baire-1 function, so K is a Rosenthal compact [12]. For such subclass of Rosenthal compacta K it has been proved that $C(K)$ admits a pointwise lower semicontinuous LUR equivalent norm [6]. Using some ideas of [6], we are going to deduce the existence of such renorming from Corollary 1.2 describing the controlling coordinates of the functions in $C(K)$, unlike Section 4.1 where Theorem 1.1 played the key role.

In what follows Q stands for a countable dense subset of Γ. As in Section 1 we assume that $K \subset [0,1]^\Gamma$. If $m \in \mathbb{N}$, R is a subset of Q, and S is a subset of $\Gamma \setminus Q$, let

$$I(R,S,m) = \{(s,t) \in K \times K : \|(s-t)\restriction_R\|_\infty \le (4m)^{-1}, \|(s-t)\restriction_S\|_\infty \le m^{-1}\}.$$

Fix $\varepsilon > 0$. The uniform continuity of every $x \in C(K)$ yields $m(x) \in \mathbb{N}$ and a finite subset F of Γ such that

$$|x(s) - x(t)| < \varepsilon \text{ whenever } s,t \in K \text{ and } \sup\{|s(\gamma) - t(\gamma)| : \gamma \in F\} \le m(x)^{-1}.$$

If $S = F \cap (\Gamma \setminus Q)$, it is clear that $|x(s) - x(t)| < \varepsilon$ whenever $(s,t) \in I(Q, S, m(x))$. Furthermore, we can associate x with a finite subset $S(x)$ of $\Gamma \setminus Q$ of minimal cardinality satisfying

$$(s,t) \in I(Q, S(x), m(x)) \Longrightarrow |x(s) - x(t)| < \varepsilon. \tag{4.7}$$

Since $I(Q, S(x), m(x))$ is pointwise compact, there must exist $p(x) \in \mathbb{N}$ such that $|x(s) - x(t)| \leq \varepsilon - p(x)^{-1}$ whenever $(s,t) \in I(Q, S(x), m(x))$. We claim that there exists a finite subset $R(x)$ of Q such that

$$(s,t) \in I(R(x), S(x), m(x)) \Longrightarrow |x(s) - x(t)| < \varepsilon. \tag{4.8}$$

Indeed, otherwise for every finite subset R of Q there is $(s_R, t_R) \in I(R, S(x), m(x))$ such that $|x(s_R) - x(t_R)| \geq \varepsilon$. By compactness, we can choose a cluster point (s,t) of the net $\{(s_R, t_R)\}_{R \in [Q]^{<\omega}}$ in $K \times K$. It is easy to check that $(s,t) \in I(Q, S(x), m(x))$ but applying continuity, we get $|x(s) - x(t)| \geq \varepsilon$, a contradiction to (4.7) that proves our claim.

From (4.8) it follows that $R(x) \cup S(x)$ ε-controls x with $1/4m(x)$, so in order to apply Corollary 1.2, we will split $C(K)$ up into countably many subsets and, fixing one of these subsets, we will describe the set $S(x)$ for each x in it. Given $s \in K$ and $\delta > 0$, let $J(s,\delta) = \{\gamma \in \Gamma : \operatorname{osc}(s, U) > \delta$ whenever U is open and $\gamma \in U\}$. Each $J(s,\delta)$ is a countable closed subset of Γ, hence a scattered topological space. By means of arguments of descriptive set theory, in [6, Theorem 3] it was proved that there exists a countable ordinal Ω such that for all $s \in K$ and all $\delta > 0$ the Ω-th derived set $J(s,\delta)^{(\Omega)}$ is empty; fix such Ω. Given $s,t \in K$ and $m \in \mathbb{N}$, we write $J(s,t,m) = J(s, 1/4m) \cup J(t, 1/4m)$. It is easily checked that $J(s,t,m)^{(\Omega)} = \varnothing$ for all $s,t \in K$ and all $m \in \mathbb{N}$. The proof of the lemma below can be found in [6]; we include it here for the sake of completeness.

Lemma 4.10 ([6, Lemma 5]) *For every $x \in C(K)$ and every proper subset F of $S(x)$ the set $U(x,F) = \{(s,t) \in I(Q, F, m(x)) : |x(s) - x(t)| > \varepsilon - p(x)^{-1}\}$ is nonempty. Moreover, there exists an ordinal $\xi(x,F) < \Omega$ such that*

(i) *$S(x) \cap J(s,t,m(x))^{(\xi(x,F))} \setminus F \neq \varnothing$ for all $(s,t) \in U(x,F)$;*
(ii) *there is $(s,t) \in U(x,F)$ such that $S(x) \cap J(s,t,m(x))^{(\xi(x,F)+1)} \setminus F = \varnothing$.*

Proof Since $S(x)$ is minimal, the set $U(x,F)$ must be nonempty. For simplicity we write m instead of $m(x)$. By the choice of $S(x)$, given $(s,t) \in U(x,F)$, there is $\gamma \in S(x) \setminus F$ such that $|s(\gamma) - t(\gamma)| > m^{-1}$. We claim that any such γ belongs to $J(s,t,m)$. Indeed, if $\gamma \notin J(s,t,m)$, then there must exist an open set $U \subset \Gamma$ containing γ such that $|s(\alpha) - s(\gamma)| \leq (4m)^{-1}$ and $|t(\alpha) - t(\gamma)| \leq (4m)^{-1}$ for every $\alpha \in U$. By density of Q we can choose some $\alpha \in Q \cap U$. Since $(s,t) \in I(Q,F,m)$, it follows that $|s(\alpha) - t(\alpha)| \leq (4m)^{-1}$ and applying the triangle inequality we get $|s(\gamma) - t(\gamma)| \leq 3/4m$, a contradiction. Hence, for every $(s,t) \in U(x,F)$ the set $S(x) \cap J(s,t,m) \setminus F$ is nonempty. Since $S(x)$ is finite, for every $(s,t) \in U(x,F)$ there is a unique ordinal $\xi(s,t) < \Omega$ such that $S(x) \cap J(s,t,m)^{(\xi(s,t))} \setminus F \neq \varnothing$ and $S(x) \cap J(s,t,m)^{(\xi(s,t)+1)} \setminus F = \varnothing$. If $\xi(x,F) = \min\{\xi(s,t) : (s,t) \in U(x,F)\}$, then (i) and (ii) hold. ∎

Lemma 4.10 gives some information to describe the coordinates of $S(x)$. In fact, it shows that some of them belong to $J(s,t,m)^{(\xi)}$ for some $(s,t) \in K \times K$, $m \in$

$\mathbb{N}$ and $\xi < \Omega$. From now on we will *codify* the *new countable information* about the controlling coordinates of the functions by taking countable decompositions of $C(K)$. Indeed, for $R \in [Q]^{<\omega}$, m, $p \in \mathbb{N}$, and $n \geq 0$ let $C^R_{m,n,p}$ be the set of all $x \in B_{C(K)}$ for which $p(x) = p$, $\#S(x) = n$, and (4.8) holds with $R(x) = R$ and $m(x) = m$. Given $n \geq 1$, $x \in C^R_{m,n,p}$ and a proper subset F of $S(x)$, let $\xi(x, F)$, or ξ for simplicity, as in Lemma 4.10. We will focus on the coordinates of $S(x) \setminus F$ that are in $J(s, t, m)^{(\xi)}$ for a pair (s, t) such that $J(s, t, m)^{(\xi+1)} \cap S(x) \setminus F = \varnothing$; so we introduce the set $I(x, F)$ below. To fix the minimum number of coordinates we may find there, consider $j(x, F)$ below. Therefore set

$$\begin{aligned} I(x, F) &= \{(s, t) \in U(x, F) : S(x) \cap J(s, t, m)^{(\xi+1)} \setminus F = \varnothing\}; \\ j(x, F) &= \min\{\# \left(S(x) \cap J(s, t, m)^{(\xi)} \setminus F\right) : (s, t) \in I(x, F)\}; \\ V(x, F) &= \{(s, t) \in I(x, F) : \# \left(S(x) \cap J(s, t, m)^{(\xi)} \setminus F\right) = j(x, F)\}; \\ \mathcal{H}(x, F) &= \{S(x) \cap J(s, t, m)^{(\xi)} \setminus F : (s, t) \in V(x, F)\}. \end{aligned}$$

According to Lemma 4.10, we have $I(x, F) \neq \varnothing$ and $j(x, F) \geq 1$, so $\mathcal{H}(x, F) \neq \varnothing$. For every $H \in \mathcal{H}(x, F)$ let

$$\alpha(x, F, H) = \sup\left\{ |x(s) - x(t)| : (s, t) \in V(x, F),\ H = S(x) \cap J(s, t, m)^{(\xi)} \setminus F \right\}.$$

Since $|x(s) - x(t)| > \varepsilon - p^{-1}$ whenever $(s, t) \in V(x, F)$, it follows that

$$\alpha(x, F, H) > \varepsilon - p^{-1} \text{ for every } H \in \mathcal{H}(x, F). \tag{4.9}$$

Then if we let

$$D(x) = \{\alpha(x, F, H) : F \text{ is a proper subset of } S(x), H \in \mathcal{H}(x, F)\} \cup \{\varepsilon - p^{-1}\},$$

there exists $i(x) \in \mathbb{N}$ such that

$$\min\{|a - b| : a, b \in D(x), a \neq b\} > i(x)^{-1}. \tag{4.10}$$

For $i \in \mathbb{N}$ and $n \geq 1$ let $C^R_{m,n,p,i}$ be the set of all $x \in C^R_{m,n,p}$ such that $i(x) = i$. It is clear that $B_{C(K)}$ is the (countable) union of the sets $C^R_{m,0,p}$ and $C^R_{m,n,p,i}$.

In the proposition below, after some new countable decompositions, we will define some families of functions that will allow us to apply Lemma 3.1. Conditions (i) and (iid) below will allow us to describe $S(x)$ inductively, and (iia)–(11c) below will allow us to apply Lemma 3.1.

Proposition 4.11 *Given $m, n, p, i \in \mathbb{N}$, and $R \in [Q]^{<\omega}$, let $\delta = (40i)^{-2^n}$ and $\theta = (10i)^{-1}$. Then for every $k = 0, 1, \ldots, n$, there is a decomposition $C^R_{m,n,p,i} = \bigcup_{\ell=1}^{+\infty} B^k_\ell$ such that the following hold:*

(i) *For every $\ell \geq 1$ and every $x \in B^k_\ell$ there is a subset $F^\ell_k(x)$ of $S(x)$ such that $\#F^\ell_k(x) \geq k$.*

(ii) *If $k \geq 1$, for every $\ell \geq 1$ and every $x \in B^k_\ell$ there are $\ell', r \geq 1$, and a family $\{\varphi_j\}_{j=1}^r$ of convex and pointwise lower semicontinuous maps, $\varphi_j \colon B_{C(K)} \to [0, +\infty)$, $1 \leq j \leq r$, such that $B^k_\ell \subseteq B^{k-1}_{\ell'}$ and the following hold:*

(a) $\operatorname{osc}(\varphi_j, B_{C(K)}) \leq 1$ *for all* $1 \leq j \leq r$*;*

(b) *if $y \in B^k_\ell$ and $F^{\ell'}_{k-1}(y) = F^{\ell'}_{k-1}(x)$, then $\varphi_j(x) > \varphi_j(y) - \delta$ for all $1 \leq j \leq r$;*

(c) $\{y \in B^k_\ell : F^{\ell'}_{k-1}(y) = F^{\ell'}_{k-1}(x),\ \delta \leq \varphi_j(x) - \varphi_j(y) < \theta\} = \varnothing$ *for all* $1 \leq j \leq r$*;*

(d) *if $y \in B^k_\ell$, $F^{\ell'}_{k-1}(y) = F^{\ell'}_{k-1}(x)$, and $\varphi_j(x) - \varphi_j(y) < \delta$ for all $1 \leq j \leq r$, then $F^\ell_k(y) = F^\ell_k(x)$.*

Proof We proceed by induction on $k \geq 0$. For $k = 0$ set $B^0_\ell = C^R_{m,n,p,i}$ and $F^\ell_0(x) = \varnothing$ for every $\ell \geq 1$ and every $x \in C^R_{m,n,p,i}$. Suppose that for some $k \geq 0$ there is a decomposition $C^R_{m,n,p,i} = \bigcup_{\ell=1}^{+\infty} B^k_\ell$ satisfying (i) and (ii). To complete the induction it suffices to split up $B^k_\ell = \bigcup_{s=1}^{+\infty} B^{k+1}_s$ for every $\ell \geq 1$ in such a way that (i) and (ii) hold for every $s \geq 1$ and every $x \in B^{k+1}_s$. To this end fix $\ell \geq 1$, and for every $x \in B^k_\ell$ write $F_k(x)$ instead of $F^\ell_k(x)$ for short. Let $B_0 = \{x \in B^k_\ell : F_k(x) = S(x)\}$, and suppose $B^k_\ell \setminus B_0 \neq \varnothing$. Given $x \in B^k_\ell \setminus B_0$, $F_k(x)$ is a proper subset of $S(x)$, so choose $\xi\,(x, F_k(x))$ satisfying (i) and (ii) from Lemma 4.10. For every ordinal $\xi < \Omega$ and every $j \in \mathbb{N}$ let $B_{\xi,j}$ be the set of all $x \in B^k_\ell \setminus B_0$ such that $\xi(x, F_k(x)) = \xi$ and $j(x, F_k(x)) = j$. It is clear that $B^k_\ell \setminus B_0$ is the union of the sets $B_{\xi,j}$. We need the following lemma to codify the new countable information.

Lemma 4.12 *Given $\xi < \Omega$, $j \in \mathbb{N}$, and $x \in B_{\xi,j}$ let $\mathcal{I}(\xi, j, x)$ be the family of all $H \in \mathcal{H}(x, F_k(x))$ such that $\alpha(x, F_k(x), H) = \sup\{|x(s) - x(t)| : (s,t) \in V(x, F_k(x))\}$. Then there is a pair of open real intervals (L, M) with rational end points such that*

$$L \subset M, \quad \sup L = \sup M, \quad \inf M > \varepsilon - p^{-1}, \qquad \text{length}\, L = \delta, \quad \text{length}\, M \geq (2i)^{-1}; \tag{4.11}$$

$$\alpha(x, F_k(x), H) \in L \textit{ whenever } H \in \mathcal{I}(\xi, j, x), \qquad \alpha(x, F_k(x), H) < \inf M \textit{ when } H \in \mathcal{H}(x, F_k(x)) \setminus \mathcal{I}(\xi, j, x). \tag{4.12}$$

Moreover, if $|x(s) - x(t)| \in M$ for some $(s,t) \in V(x, F_k(x))$, then there exists $(s', t') \in V(x, F_k(x))$ such that $|x(s') - x(t')| \in L$ and

$$S(x) \cap J(s', t', m)^{(\xi)} \setminus F_k(x) = S(x) \cap J(s, t, m)^{(\xi)} \setminus F_k(x).$$

Proof Let $\{M_H : H \in \mathcal{H}(x, F_k(x))\}$ be a family of pairwise disjoint open real intervals with rational end points such that $\alpha(x, F_k(x), H) \in M_H$ for every $H \in \mathcal{H}(x, F_k(x))$ and with the property that $M_H = M_{H'}$ in case $H, H' \in \mathcal{H}(x, F_k(x))$ and $\alpha(x, F_k(x), H) = \alpha(x, F_k(x), H')$. From (4.9) and (4.10) it follows that for every $H \in \mathcal{H}(x, F_k(x))$ we can assume that $\inf M_H > \varepsilon - p^{-1}$ and that $\text{length}\, M_H \geq (2i)^{-1}$; moreover we can take L_H to be an open real interval with rational end points satisfying $L_H \subset M_H$, $\sup L_H = \sup M_H$, $\text{length}\, L_H = \delta$, and $\alpha(x, F_k(x), H) \in L_H$. If (L, M) is the pair (L_H, M_H) corresponding to any $H \in \mathcal{I}(\xi, j, x)$ then (4.11),(4.12) are fulfilled.

Suppose that $|x(s) - x(t)| \in M$ for some $(s,t) \in V(x, F_k(x))$ and let $H = S(x) \cap J(s,t,m)^{(\xi)} \setminus F_k(x)$. Since $H \in \mathcal{H}(x, F_k(x))$, by the choice of M we have $\inf M_H \leq \inf M < |x(s) - x(t)| \leq \alpha(x, F_k(x), H) < \sup M_H$ so $M = M_H$. Then $\alpha(x, F_k(x), H) \in L$ and there must exist $(s', t') \in V(x, F_k(x))$ such that $|x(s') - x(t')| \in L$ and $H = S(x) \cap J(s', t', m)^{(\xi)} \setminus F_k(x)$. ∎

Let us turn to the proof of Proposition 4.11. Given $\xi < \Omega$, $j \in \mathbb{N}$, a real interval I, and $x \in B_{\xi,j}$, let $\mathcal{H}(x, F_k(x), I) = \{H \in \mathcal{H}(x, F_k(x)) : \alpha(x, F_k(x), H) \in I\}$. For every pair of open real intervals (L, M) with rational end points satisfying (4.11) and for every $r \in \mathbb{N}$ we write $B_{\xi,j,L,M,r}$ for the set of all $x \in B_{\xi,j}$ with $\#\mathcal{H}(x, F_k(x), L) = r$ such that (4.12) holds for x, L, and M. It is clear that each $B_{\xi,j}$ is the union of the sets $B_{\xi,j,L,M,r}$.

Given ξ, j, L, M, and r as above, let $\mathcal{B}$ be a countable basis for the topology of Γ. From the choice of $B_{\xi,j,L,M,r}$ and $\mathcal{H}(x, F_k(x), L)$ we have that for every $x \in B_{\xi,j,L,M,r}$ and every $H \in \mathcal{H}(x, F_k(x), L)$ there exists $(s,t) \in V(x, F_k(x))$ such that $|x(s) - x(t)| \in L$ and $H = S(x) \cap J(s,t,m)^{(\xi)} \setminus F_k(x)$. Since $J(s,t,m)^{(\xi+1)}$ is closed and $S(x) \cap J(s,t,m)^{(\xi+1)} \setminus F_k(x) = \varnothing$, for every $\gamma \in S(x) \setminus F_k(x)$ we can choose $U_\gamma \in \mathcal{B}$ with $\gamma \in U_\gamma$ such that if $\mathcal{U} = \{U_\gamma\}_{\gamma \in S(x) \setminus F_k(x)}$, then

$$J(s,t,m)^{(\xi+1)} \cap \bigcup \mathcal{U} = \varnothing \quad \text{and} \quad H = J(s,t,m)^{(\xi)} \cap \bigcup \mathcal{U}. \tag{4.13}$$

The second equality above shows that $\mathcal{U}$, a finite subset of the countable set $\mathcal{B}$, codifies which are the controlling coordinates of $S(x) \setminus F_k(x)$ lying in $J(s,t,m)^{(\xi)}$. Then for every $\mathcal{U} \in [\mathcal{B}]^{<\omega}$ we write $B^{\mathcal{U}}_{\xi,j,L,M,r}$ for the set of all $x \in B_{\xi,j,L,M,r}$ for which $S(x) \setminus F_k(x) \subset \bigcup_{U \in \mathcal{U}} U$ and with the property that for each $H \in \mathcal{H}(x, F_k(x), L)$ there exists $(s,t) \in I(Q, F_k(x), m)$ such that the equalities of (4.13) hold and $|x(s) - x(t)| \in L$. It is clear that $B_{\xi,j,L,M,r}$ is the union of the sets $B^{\mathcal{U}}_{\xi,j,L,M,r}$.

Summarizing, we have written B^k_ℓ as the countable union of B_0 and the sets $B^{\mathcal{U}}_{\xi,j,L,M,r}$; the rest of the proof is devoted to showing that this decomposition satisfies the requirements of Proposition 4.11. Indeed if $x \in B_0$, it is enough to take $F_{k+1}(x) = S(x)$ because (i) and (ii) trivially hold associating with x the zero function on $B_{C(K)}$. On the other hand, fix ξ, j, L, M, r, and $\mathcal{U}$ as above. If $x \in B^{\mathcal{U}}_{\xi,j,L,M,r}$ take $F_{k+1}(x)$ as the union of $F_k(x)$ and $\bigcup\{H : H \in \mathcal{H}(x, F_k(x), L)\}$, it is clear that (i) holds. To show (ii), for every $H \in \mathcal{H}(x, F_k(x), L)$ we write $P(x, H)$ for the set of all $(s,t) \in I(Q, F_k(x), m)$ such that (4.13) hold. Take $\varphi_H \colon B_{C(K)} \to [0, \infty)$ defined by $\varphi_H(y) = (1/4) \sup\{|y(s) - y(t)| : (s,t) \in P(x, H)\}$. Condition (iia) is clearly fulfilled by any enumeration of the family $\{\varphi_H : H \in \mathcal{H}(x, F_k(x), L)\}$. To obtain one for which (iib)–(iid) hold, we will prove the following.

Claim Given $y \in B^{\mathcal{U}}_{\xi,j,L,M,r}$ and $H \in \mathcal{H}(x, F_k(x), L)$, if $F_k(y) = F_k(x)$, we have $4\,\varphi_H(y) < \sup L$.

Proof Pick $(s,t) \in P(x, H)$. Since $F_k(y) = F_k(x)$, we have $(s,t) \in I(Q, F_k(y), m)$. From this and (4.11) we get $|y(s) - y(t)| < \inf L$ whenever $|y(s) - y(t)| \leq \varepsilon - p^{-1}$; so we can suppose that $|y(s) - y(t)| > \varepsilon - p^{-1}$. Then $(s,t) \in U(y, F_k(y))$ and

$$\varnothing \neq (S(y) \setminus F_k(y)) \cap J(s,t,m)^{(\xi)} \subset \bigcup_{U \in \mathcal{U}} U \cap J(s,t,m)^{(\xi)} = H.$$

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Moreover, from the choice of $P(x,H)$ we have $\bigcup_{U\in\mathcal{U}} U \cap J(s,t,m)^{(\xi+1)} = \varnothing$. Since $S(y)\setminus F_k(y) \subset \bigcup_{U\in\mathcal{U}} U$, it follows that $(S(y)\setminus F_k(y))\cap J(s,t,m)^{(\xi+1)} = \varnothing$. Furthermore, since $j \le \#(S(y)\setminus F_k(y))\cap J(s,t,m)^{(\xi)} \le \#H = j$, we have $(s,t)\in V(y,F_k(y))$ and $H = (S(y)\setminus F_k(y))\cap J(s,t,m)^{(\xi)} \in \mathcal{H}(y,F_k(y))$. Hence, $|y(s)-y(t)| \le \alpha(y,F_k(y),H)$ and the claim follows from (4.12). ∎

According to the above claim and the choice of $B^{\mathcal{U}}_{\xi,j,L,M,r}$, we have $4\,\varphi_H(x)\in L$ for every $H\in\mathcal{H}(x,F_k(x),L)$. Given $y\in B^{\mathcal{U}}_{\xi,j,L,M,r}$ such that $F_k(y)=F_k(x)$, from the claim it follows that $4\varphi_H(y) < \sup L$, so $4\varphi_H(x) > \sup L - \operatorname{length} L > 4\varphi_H(y)-\delta$ and (iib) follows.

To show (iic), suppose there are $H\in\mathcal{H}(x,F_k(x),L)$ and $y\in B^{\mathcal{U}}_{\xi,j,L,M,r}$ with $F_k(y)=F_k(x)$ such that $\varphi_H(x)-\varphi_H(y)\in[\delta,\theta)$. Since $\varphi_H(x)-\varphi_H(y)<\theta$, we get

$$
\begin{aligned}
4\varphi_H(y) &> 4\varphi_H(x)-4\theta > \sup L - \operatorname{length} L - 4\theta = \sup M - \delta - 4\theta \\
&> \sup M - 5\theta = \sup M - (2i)^{-1} \ge \sup M - \operatorname{length} M = \inf M.
\end{aligned}
\tag{4.14}
$$

The inequality above, (4.11), and the claim imply $4\varphi_H(y)\in M$. Then there is $(s,t)\in P(x,H)$ such that $|y(s)-y(t)|\in M$. From this and (4.11) we get $|y(s)-y(t)| > \varepsilon - p^{-1}$. Bearing in mind this inequality, the proof of the claim yields $(s,t)\in V(y,F_k(y))$ and $H\in\mathcal{H}(y,F_k(y))$, where $H=(S(y)\setminus F_k(y))\cap J(s,t,m)^{(\xi)}$. According to Lemma 4.12, there exists $(s',t')\in V(y,F_k(y))$ such that $H=(S(y)\setminus F_k(y))\cap J(s',t',m)^{(\xi)}$ and $|y(s')-y(t')|\in L$. Therefore, we have $\inf L < |y(s')-y(t')| \le \alpha(y,F_k(y),H)$ that together with (4.12) imply $H\in\mathcal{H}(y,F_k(y),L)$. From the choice of $B^{\mathcal{U}}_{\xi,j,L,M,r}$ we deduce that there is $(s'',t'')\in P(y,H)$ such that $|y(s'')-y(t'')|\in L$. So keeping in mind that $P(x,H)=P(y,H)$, we get $4\varphi_H(y)\in L$. However, the inequality $\varphi_H(x)-\varphi_H(y)\ge\delta$ implies $4\varphi_H(y)\le 4\varphi_H(x)-4\delta < \inf L$, a contradiction that proves(iic).

To show (iid) let $y\in B^{\mathcal{U}}_{\xi,j,L,M,r}$ with $F_k(y)=F_k(x)$ such that $\varphi_H(x)-\varphi_H(y)<\delta$ for every $H\in\mathcal{H}(x,F_k(x),L)$. Arguing as in (4.14), we get that $\mathcal{H}(x,F_k(x),L)$ is included in $\mathcal{H}(y,F_k(y),L)$. Since both sets have the same cardinality r, it follows that $\mathcal{H}(x,F_k(x),L)=\mathcal{H}(y,F_k(y),L)$. Hence, $F_{k+1}(y)=F_{k+1}(x)$ and (iid) follows. The proof of Proposition 4.11 is now complete. ∎

Theorem 4.13 ([6, Theorem 1]) *Let Γ be a Polish space and let K be a separable and pointwise compact subset of functions on Γ with the property that each $s\in K$ has at most countably many discontinuities. Then $C(K)$ admits an equivalent pointwise lower semicontinuous LUR norm.*

Proof The ball $B_{C(K)}$ has already been decomposed as the union of the sets $C^R_{m,0,p}$ and $C^R_{m,n,p,i}$ for $R\in[Q]^{<\omega}$ and $m,n,p,i\in\mathbb{N}$. From the choice of $C^R_{m,0,p}$ it follows that R ε-controls every $y\in C^R_{m,0,p}$ with $(4m)^{-1}$. On the other hand, given $R\in[Q]^{<\omega}$ and $m,n,p,i\in\mathbb{N}$, let B^k_ℓ and $\{\varphi_j\}_{j=1}^r$ as in Proposition 4.11. For every $\ell\ge 1$ and $k=1,2,\dots,n$, let $\mathcal{F}_{k,\ell}(x)=\{\varphi_j\}_{j=1}^r$. For every $\ell=(\ell_1,\dots,\ell_n)\in\mathbb{N}^n$ set

$$
C^{R,\ell}_{m,n,p,i} = \Bigl\{x\in\bigcap_{k=1}^n B^k_{\ell_k} : B^k_{\ell_k}\subset C^R_{m,n,p,i} \text{ for all } 1\le k\le n\Bigr\}.
$$

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Then $C^R_{m,n,p,i} = \bigcup_{\ell\in\mathbb{N}^n} C^{R,\ell}_{m,n,p,i}$. We shall see that each $C^{R,\ell}_{m,n,p,i}$ satisfies the requirement of Corollary 1.2.

Indeed, given $\ell \in \mathbb{N}^n$ and $x \in C^{R,\ell}_{m,n,p,i}$, let $\mathcal{F}(x) = \bigcup_{k=1}^n \mathcal{F}_{k,\ell_k}(x)$. To enumerate $\mathcal{F}(x)$ we introduce an order $\prec$ as follows. Given $\varphi_j \in \mathcal{F}_{k,\ell_k}(x)$, $\varphi_{j'} \in \mathcal{F}_{k',\ell_{k'}}(x)$ with $\varphi_j \neq \varphi_{j'}$, we write $\varphi_j \prec \varphi_{j'}$ if and only if $(k, j) <_{\text{lex}} (k', j')$, where $<_{\text{lex}}$ is the lexicographic order. If $N = \#\mathcal{F}(x)$, then we can write $\mathcal{F}(x)$ as $\{\varphi_k\}_{k=1}^N$, where $\varphi_k \prec \varphi_{k'}$ if and only if $k < k'$. If for some $k < N$ there is $y \in C^{R,\ell}_{m,n,p,i}$ such that $\varphi_j(x) - \varphi_j(y) < \delta$ for all $j < k$, Proposition 4.11 shows that $\varphi_k(x) > \varphi_k(y) - \delta$ and $\varphi_k(x) - \varphi_k(y) \notin [\delta, \theta)$. By the choice of δ and θ we have $0 < 4\delta^{1/N} \leq \theta < 1$, so applying Lemma 3.1 to $\mathcal{F}(x)$ with $A_0 = C^{R,\ell}_{m,n,p,i}$ and $B = B_{C(K)}$ we get a pointwise open half space H containing x such that

$$H \cap C^{R,\ell}_{m,n,p,i} \subset \{y \in C^R_{m,n,p,i} : S(y) = S(x)\}.$$

From this and (4.8) we deduce that $R \cup S(x)$ ε-controls every $y \in H \cap C^{R,\ell}_{m,n,p,i}$ with $(4m)^{-1}$. According to Corollary 1.2 we conclude that $C(K)$ admits an equivalent pointwise lower semicontinuous LUR norm. ∎

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