Oscillations of an equation relevant to an industrial problem

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The oscillatory behaviour of the solutions of a functional
differential equation, of more general form than an equation
arising from an industrial problem, is examined. The question
of whether one can maintain the oscillatory properties of this
equation by adding a forcing term is also answered. The
results obtained extend already known results on the subject and
complete the relevant literature.

1.   Introduction

The problem of analysing the dynamics of an overhead current
collection system for an electric locomotive has been the subject of
discussion in [9], where the analysis gave rise to a system of linear first
order ordinary differential equations, in which the argument of one of the
dependent variables is multiplied by a factor λ . Asymptotic and
numerical methods to study the properties and practical solutions of such
systems were used in [4], where, in particular, a simple physical
situation, which gives rise to the scalar equation

(1.1) $y'(t) = ay(\lambda t) + by(t)$,

was described.

Series and integral representations of the solutions to (1.1), where

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a and b are constants, were discussed in [4], while in [6] it was
shown, that the boundary-value problem associated with (1.1) and the
boundary condition $y(0) = 1$, is well-posed if $\lambda < 1$, but not if
$\lambda > 1$. Asymptotic properties of solutions of (1.1), as $t \to \infty$, were also
treated in [6], as well as in [1, 2 for $b = 0$], [3], and [5] and its
oscillatory behaviour (only for $b = 0$) was mentioned in [8]

The purpose of the present paper is to add information to the
literature, related to (1.1), extending and generalising notions contained
in it.

Thus, we examine first the oscillatory behaviour of the solutions of
the functional differential equation

$$(1.2) \qquad\qquad y'(t) = p(t)y\big(g(t)\big) + q(t)y(t) \ ,$$

which is of more general form than (1.1) — in fact, (1.1) can be derived
from (1.2) for $p(t) = a$, $g(t) = \lambda t$, $q(t) = b$ — and of course,
oscillatory properties concerning the solutions of (1.2), will also
concern those of (1.1), under proper modifications. Secondly we extend the
results obtained for (1.2) to the functional differential equation,
resulting from it by adding a forcing term, namely

$$(1.3) \qquad\qquad y'(t) = p(t)y\big(g(t)\big) + q(t)y(t) + r(t) \ .$$

In the sequel, a solution $y(t)$ of (1.2) or (1.3) is said to be
oscillatory, if it has arbitrarily large zeros, while it is said to be
nonoscillatory, if it is eventually of constant sign.

2.   Unforced oscillation

In this section, we give sufficient growth conditions on $p(t), g(t)$,
and $q(t)$, so that all solutions of (1.2) oscillate.

We shall need first the following lemma, which is an adaptation of a
corollary in [8, p. 222].

LEMMA 2.1. *Consider the functional differential equation*

$$(2.1) \qquad\qquad x'(t) + p(t)x\big(g(t)\big) = 0 \ ,$$

where

(C1) $p(t) \in C\big[[0, \infty), R^+\big]$, $g(t) \in C^1\big[[0, \infty), R^+\big]$, *and*

(C2) $g(t) < t$, $\lim_{t\to\infty} g(t) = \infty$, *and* $g'(t) \geq 0$.

If, in addition,

(2.2) $\lim\sup_{t\to\infty} \displaystyle\int_{g(t)}^{t} p(s)ds > 1$,

then every solution $x(t)$ of (2.1) *oscillates.*

Now, set $y(t)\exp\left[-\displaystyle\int q(t)dt\right] = z(t)$, to obtain (1.2) in the form

(2.3) $z'(t) + l(t)z\big(g(t)\big) = 0$,

– where $l(t) = -p(t)\exp\displaystyle\int_{t}^{g(t)} q(T)dT$ – which is the same as (2.1) and

for which Lemma 2.1 holds of course, under minor modifications. So,
observing that if $z(t)$ oscillates, so does $y(t)$, we apply Lemma 2.1 to
(2.3), to establish the following result for (1.2).

THEOREM 2.1. *Consider the functional differential equation* (1.2)
subject to the following conditions:

(i) $-p(t) \in C\big[[0, \infty), R^+\big]$;

(ii) $g(t) \in C^1\big[[0, \infty), R^+\big]$, $g(t) \leq t$, $\lim_{t\to\infty} g(t) = \infty$, *and*

$g'(t) \geq 0$;

(iii) $q(u)$ *is continuous for all $u \neq 0$.*

If, in addition,

(2.4) $\lim\sup_{t\to\infty} \displaystyle\int_{g(t)}^{t} -p(s)\left[\exp\displaystyle\int_{s}^{g(s)} q(T)dT\right]ds > 1$,

then every solution $y(t)$ of (1.2) *oscillates.*

REMARKS. It would not seem too easy for someone to derive Theorem
2.1 from usual oscillation theory techniques, if he had worked directly
from (1.2) (note especially condition (2.4)). This theorem, however, came
out naturally, as an application of the exposition in [8], after the
transformation of (1.2) to (2.3). In this fact lies the importance of that
transformation, which is a new concept in the relevant literature, as far

as we know.

On the other hand, Theorem 2.1, as a "qualitative result", completes the theory described in [6, paragraphs 4, 5, and 6] and one can now easily apply that theorem to (1.1), under proper modifications, to extend results obtained for it, in [8, p. 222], for $b = 0$; we leave the details to the reader.

3.   Forced oscillation

Proceeding further, we give sufficient growth conditions on $p(t)$, $g(t)$, $q(t)$, and $r(t)$, so that all solutions of (1.3) oscillate. To this end, in addition to the transformations established in Section 2, set $r(t)\exp\left[-\int q(t)dt\right] = m(t)$, to obtain

$$(3.1) \qquad\qquad z'(t) + l(t)z\big(g(t)\big) = m(t) \ .$$

But for (3.1), the following lemma holds, which is a modification of a theorem in [7, p. 274], using Lemma 2.1 given previously.

LEMMA   **3.1.** *Consider the functional differential equation* (3.1) *subject to the following hypotheses:*

(h1) $l(t), m(t) \in C\big[[0, \infty), R\big]$, $l(t) \geq 0$;

(h2) $g(t) \in C^1\big[[0, \infty), R\big]$, $g(t) \leq t$, $\lim_{t\to\infty} g(t) = \infty$,

$\qquad\quad g'(t) \geq 0$;

(h3) $\lim\sup_{t\to\infty} \int_{g(t)}^{t} l(s)ds > 1$;

(h4) $q(u)$ *is continuous for all* $u \neq 0$;

(h5)   *there exists a function* $Q(t) \in C^1\big[[0, \infty), R\big]$, *such that* $Q'(t) = m(t)$, $t \geq 0$, *and either*

(1) $\lim_{t\to\infty} Q(t) = 0$, *or*

(2)   *there exist constants* q_1, q_2 *and sequences* $\{t'_m\}$, $\{t''_m\}$, *such that* $\lim_{m\to\infty} t'_m = \lim_{m\to\infty} t''_m = \infty$ *and*

$$Q\big(t_m'\big) = q_1 \ , \quad Q\big(t_m''\big) = q_2 \ , \quad q_1 \le Q(t) \le q_2 \ , \quad t \ge 0 \ .$$

Then, if (1) *holds, every solution* $z(t)$ *of* (3.1) *oscillates or* $\lim_{t \to \infty} z(t) = 0$ *, while if* (2) *holds, every solution* $z(t)$ *of* (3.1) *oscillates.*

Now, using the above lemma and the previously given transformations, we establish the following result for (1.3).

THEOREM 3.1. *Consider the functional differential equation* (1.3) *subject to the following hypotheses:*

(H1) $p(t)$, $r(t) \in C\big[[0, \infty), R\big]$, $-p(t) \ge 0$;

(H2) $g(t) \in C^1\big[[0, \infty), R\big]$, $g(t) \le t$, $\lim_{t \to \infty} g(t) = \infty$,

   $g'(t) \ge 0$;

(H3) $\limsup_{t \to \infty} \displaystyle\int_{g(t)}^{t} -p(s)\exp\bigg[\int_s^{g(s)} q(T)dT\bigg]ds > 1$;

(H4) $q(u)$ *is continuous for all* $u \ne 0$;

(H5) *there exists a function* $Q(t) \in C^1\big[[0, \infty), R\big]$ *, such that*
   $Q'(t) = r(t)\exp\bigg[-\int q(t)dt\bigg]$ *,* $t \ge 0$ *and either*

   (I) $\lim_{t \to \infty} Q(t) = 0$ *, or*

   (II) *there exist constants* q_1, q_2 *and sequences*
      $\{t_m'\}$, $\{t_m''\}$ *, such that* $\lim_{m \to \infty} t_m' = \lim_{m \to \infty} t_m'' = \infty$

      and

   $$Q\big(t_m'\big) = q_1 \ , \quad Q\big(t_m''\big) = q_2 \ , \quad q_1 \le Q(t) \le q_2 \ , \quad t \ge 0 \ .$$

Then, if (I) *holds, every solution* $y(t)$ *of* (1.3) *oscillates or* $\lim_{t \to \infty} y(t) = 0$ *, while if* (II) *holds, every solution* $y(t)$ *of* (1.3) *oscillates.*

The above result accomplishes the purpose of this paper.

A QUESTION. We have seen so far, that one can deal directly with the

oscillatory character of the solutions of (1.2) and (1.3), by transforming them properly into equations which have already been examined.   Could this procedure - under suitable transformations - be applied to the higher-order retarded differential equations

$$y^{(n)}(t) = p(t)y\big(g(t)\big) + q(t)y(t)$$

and

$$y^{(n)}(t) = p(t)y\big(g(t)\big) + q(t)y(t) + r(t) \ ,$$

respectively, for $n > 1$?   This question remains open.

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