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TRANSITIVITY PROPERTIES OF FUCHSIAN GROUPS

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1. Introduction. A *Fuchsian group* G is a discrete group of fractional linear transforms each of which preserve a disc (or half plane). We consider only groups which preserve the unit disc $\Delta = \{z: |z| < 1\}$ and none of whose transforms, except the identity, fix infinity (any Fuchsian group is conjugate to such a group). In this case the elements of G are of the form

$$(1)\quad V(z) = \frac{az + \bar{c}}{cz + \bar{a}}, \quad |a|^2 - |c|^2 = 1, c \neq 0.$$

In this paper we will investigate the relationships between the various types of transitivity properties which a Fuchsian group may have. In this section we give the relevant definitions, a brief survey of the known results in the area and we will state our results. The remaining sections are devoted to the proofs of the theorems.

The *isometric circle* of the transform V which has the form (1) is the circle $\{z: |cz + \bar{a}| = 1\}$ being the set of points z for which $|V'(z)| = 1$. We note that the transform V is a rigid motion of the non-euclidean metric ρ in Δ defined by the differential

$$d\rho = \frac{2|dz|}{1 - |z|^2}, \quad z \in \Delta.$$

We denote by D the *Dirichlet fundamental region* for G centered at the origin —which is defined as the set of points z in Δ satisfying

$$\rho(z, 0) < \rho(z, V(0))$$

for all V in G except the identity. Note [**11**, p. 151], that D is also the *Ford fundamental region* for G (the set of points in Δ exterior to all isometric circles of transforms in G). We set $U = \partial\Delta$, $e = U \cap \partial D$, and $E = \bigcup_{V \in G} V(e)$.

The set of points on U at which G does not act discontinuously is the *limit set* $L(G)$. It is well known that $L(G)$ is the set of points of accumulation of centers of isometric circles belonging to transforms in G [**4**, p. 42]. If $L(G) = U$ then G is of the *first kind*, otherwise $L(G)$ is a nowhere dense subset of U and G is of the *second kind*.

The group G is said to be of *convergence type* if

$$\textstyle\sum_{V \in G}(1 - |V(z)|) < \infty \quad (z \in D)$$

otherwise of *divergence type*. We are now in a position to define the various transitivity properties with which we will be concerned.

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The group G is said to be *transitive on* U if it has the property that every measurable set of points S on U which is invariant under G is either of Lebesgue measure 0 or 2π. This notion can clearly be generalized to two dimensions and one is led to a much stronger property called metric transitivity. Consider the torus $R = U \times U$. We say a subset S of R is *G-invariant* if

$$S = \{(V(z_1), V(z_2)) | (z_1, z_2) \in S\}$$

for all $V \in G$. Then G is said to be *metric transitive* if each measurable G-invariant set S has two-dimensional measure 0 or $4\pi^2$. Generalizations to higher dimensions are useless as the analogous condition becomes so strong that it is not satisfied by any Fuchsian group [**23**, p. 541].

Let λ be a hyperbolic ray ab, $a \in U$, $b \in \Delta$. Let L be a hyperbolic line with end points α, β say. If there exists a sequence of transforms $\{V_n\} \subset G$ such that $V_n(a) \to \alpha$ and $V_n(b) \to \beta$ then we write $V_n(\lambda) \to L$. We say λ is *transitive under* G if, for any hyperbolic line L, there exists a sequence V_n such that $V_n(\lambda) \to L$. A point ξ of U is called *transitive* if every hyperbolic ray through ξ is transitive. We use T to denote the set of transitive points.

A *horocycle* is a euclidean circle which is internally tangent to U. A horocycle is said to be *transitive* if its G images approximate any horocycle. A point ξ of U is called *h-transitive* if every horocycle with point of tangency ξ is transitive. We use S to denote the set of h-transitive points.

Our last definition concerns the approximation to limit points by centers of isometric circles. A limit point ξ is said to be a *point of approximation* for the group G if there exists a sequence $\{V_n\} \subset G$ such that

$$|\xi - c(V_n)| = O(r(V_n)^2) \quad \text{as } n \to \infty$$

where $c(V_n)$ and $r(V_n)$ denote, respectively, the center and radius of the isometric circle of V_n. We use H to denote the set of points of approximation.

Points of approximation were first studied by Hedlund [**5**] in his investigation of h-transitive points. He proved that for any Fuchsian group $H \subset S$ and that for a finitely generated group of the first kind H comprises the whole of U with the exception of the parabolic fixed points. The result was extended by Lehner [**11**, p. 181] who showed that for any finitely generated group $L(G)$ comprises H and the set of parabolic fixed points. This result has been generalized to the Kleinian case by Beardon and Maskit [**3**] who also give several equivalent definitions for points of approximation. A paper of the author's gives some further information and applications of points of approximation [**14**]. The original question as to the nature of the set of h-transitive points has been answered by the author who proved [**15**] that for any group $S = U \backslash E$. We summarize this information as follows:

THEOREM A. *For any Fuchsian group, $H \subset S$ and $S = U\backslash E$. The group is finitely generated if and only if $L(G)$ comprises H and the set of parabolic fixed points.*

The existence of transitive points for groups of the first kind (it is clear that there are no transitive points if G is of the second kind) was investigated by Artin [**2**], Myrberg [**12**] and Koebe [**10**]. In 1931 Myrberg [**13**] proved that if G is of the first kind and finitely generated then $m(T) = 2\pi$. Using results of Tsuji [**23**, p. 530] and Yujobo [**24**], Shimada [**21**] generalized this result and proved that if G is a group of divergence type, then $m(T) = 2\pi$. The relation between T and H is clearly seen from the author's paper [**14**] where it is proved that $T \subset H$ and that this is a strict inclusion (since any hyperbolic fixed point belongs to H but not T). We remark that for a group of convergence type $m(H) = 0$ (see [**3**, p. 4] and [**23**, p. 530] for the proof). Summarizing this information we have:

THEOREM B. *For any Fuchsian group, $T \subset H \subset S$ where T is empty if G is of the second kind. The inclusion $T \subset H$ is strict. If G is of divergence type then $m(T) = 2\pi$, while G of convergence type implies $m(H) = 0$.*

Concerning transitivity on U, it was proved by Seidel [**20**] that a group G is transitive on U if and only if every bounded harmonic function in Δ which is automorphic with respect to G is identically constant. Thus G is transitive on U if and only if the quotient surface Δ/G belongs to the class O_{HB}. It is well known [**23**, p. 522] that a group G is of divergence type if and only if Δ/G belongs to the class O_G (Δ/G does not have a Green's function). Since $O_G \subset O_{HB}$ we have:

THEOREM C. *A Fuchsian group G is transitive on U if and only if $\Delta/G \in O_{HB}$. A group of divergence type is transitive on U.*

Our first result shows that the converse of the last statement of Theorem C is false.

THEOREM 1. *There exists a Fuchsian group G of convergence type which is transitive on U.*

Tsuji has proved [**23**, p. 514] that any group for which $m(E) > 0$ is of convergence type. A very easy construction shows the stronger result that $m(E) > 0$ implies G is intransitive on U. The converse is false; in [**18**] Pommerenke gives an explicit construction of a group with $m(E) = 0$ and $\Delta/G \notin O_{HB}$ so this group is intransitive.

We have the following:

COROLLARY 1. *The inclusion $H \subset S$ is, in general, strict. In fact there exists a group G for which $m(H) = 0$ and $m(S) = 2\pi$.*

To prove the corollary we let G be the group of Theorem 1; then G is of convergence type, and so $m(H) = 0$ (Theorem B). G is transitive on U so by our remarks above $m(E) = 0$ then by Theorem A, $m(S) = 2\pi$.

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Finally we consider metric transitivity. This is a very strong property—in fact if G is metric transitive then G is transitive on U and $m(T) = 2\pi$ [**23**, p. 542]. Hopf [**7**] proved the transitivity property for any group of the first kind which is finitely generated. Other proofs were subsequently given by Hedlund [**6**] and Tsuji [**23**, p. 537]. In the other direction Tsuji [**22**] has shown that if G is metric transitive then it is of divergence type. Our main result shows that the converse is true.

THEOREM 2. *A Fuchsian group G is metric transitive if and only if it is of divergence type.*

We also have an analogue of Seidel's result.

THEOREM 3. *A Fuchsian group G is metric transitive if and only if any function $u(z, w)$ which is harmonic in each variable in $\Delta \times \Delta$, bounded and invariant under G reduces to a constant.*

Our application of the transitivity theorems is to uniform distribution questions of the following type. Let G be a Fuchsian group with fundamental polygon D and suppose A is a measurable subset of D. Since the images of D cover Δ without overlapping, we might expect that the images of A would cover a subregion of Δ whose size would be proportional to the size of A. In order to make these ideas more precise we need some notation.

For $\xi \in U$ and r, $0 < r < 1$, we denote by $L(\xi, r)$ the ray joining 0 to $r\xi$. If σ denotes the non-euclidean area measure in Δ then we have the following result of Tsuji [**23**, p. 547]:

THEOREM D. *Let G be a Fuchsian group with $\sigma(D) < \infty$ and let $M \subset D$ be measurable with $M^* = \bigcup_{V \in G} V(M)$. Then*

$$\lim_{r \to 1} \frac{\rho\{M^* \cap L(\xi, r)\}}{\rho\{L(\xi, r)\}} = \frac{\sigma(M)}{\sigma(D)}$$

for almost all $\xi \in U$.

We obtain the following extension of this result:

THEOREM 4. *Let G be a Fuchsian group with $\sigma(D) = \infty$ and let M be a disc contained in D. Then, with $M^* = \bigcup_{V \in G} V(M)$,*

$$\lim_{r \to 1} \frac{\rho\{M^* \cap L(\xi, r)\}}{\rho\{L(\xi, r)\}} = 0$$

for almost all $\xi \in U$.

We have been considering the covering of radii—the corresponding results for discs and circles seem to lie deeper. In what follows M will denote a small disc centered at 0 and contained in D. $C(r, M)$ will denote the non-euclidean linear measure of that part of the circle $\{|z| = r\}$ which lies in

$M^* \{= \bigcup_{V \in G} V(M)\}$. $A(r, M)$ will denote the non-euclidean area measure of that part of the disc $\{|z| < r\}$ which lies in M^*. We introduce also the *orbital counting function*—for $a \in D$, $n(r, a)$ denotes the number of transforms $V \in G$ such that $|V(a)| < r$. We have the following result:

THEOREM 5. *If a Fuchsian group G has one of the following properties it has all three of them.*

(i) $\limsup_{r \to 1} (1 - r)\, n(r, 0) > 0$

(ii) $\limsup_{r \to 1} (1 - r)\, C(r, M) > 0$

(iii) $\limsup_{r \to 1} (1 - r)\, A(r, M) > 0$

Property (i) of Theorem 5 is of independent interest. Tsuji proved [**23**, p. 518] that if G is a group with $\sigma(D) < \infty$ then G has property (i). Very recently S. J. Patterson [**16**] obtained an asymptotic estimate for $(1 - r)\, n(r, 0)$ in this case and in another paper [**17**] has shown that no group with $\sigma(D) = \infty$ has property (i) of Theorem 5.

THEOREM E (Patterson). *If G is a Fuchsian group and*

(i) *if* $\sigma(D) < \infty$ *then*

$$n(r, 0) \sim \frac{2\pi}{\sigma(D)} \cdot \frac{1}{1 - r} \quad \text{as } r \to 1.$$

(ii) *if* $\sigma(D) = \infty$ *then*

$$(1 - r)\, n(r, 0) \to 0 \quad \text{as } r \to 1.$$

2. Proof of Theorem 1. Theorem 1 is an easy consequence of the fact that for Riemann surfaces the inclusion $O_G \subset O_{HB}$ is strict [**1** and **19**, p. 235, p. 304]. Let R be a surface in O_{HB} but not in O_G. The universal covering surface of R is Δ with a discrete group Γ of covering transforms. It is well known that Δ/Γ is conformally equivalent to R and is thus in O_{HB} but not O_G. By Theorem C, Γ is transitive on U and from Tsuji's result [**23**, p. 522], Γ is of convergence type.

3. Proofs of Theorems 2 and 4. To prove Theorem 2 we consider a geodesic flow in Δ and appeal to some results of Hopf [**8**, **9**] which show that the flow is ergodic in certain instances. The fact that the only measurable sets preserved by an ergodic flow on a space have either measure zero or the measure of the whole space leads to the conclusion of Theorem 2.

Let B be the subset of R^3 defined by:

$$B = \{(x, y, \theta): x^2 + y^2 < 1, 0 \leqq \theta < 2\pi\}.$$

The point (x, y, θ) of B is to be regarded as a line element in Δ with carrier point (x, y) and direction parallel to the line segment joining the origin to $e^{i\theta}$. We introduce a metric in the line element space B by:

$$s((x_1, y_1, \theta_1), (x_2, y_2, \theta_2)) = \rho((x_1, y_1), (x_2, y_2)) + \alpha$$

where α is the least positive angle between the directions θ_1 and θ_2. Clearly s is a Riemannian metric on B and is invariant under bilinear transforms preserving Δ. The invariant element of volume induced by ds in B is found to be

$$dm = \frac{4dxdyd\alpha}{(1 - |z|^2)^2}, \quad z = x + iy.$$

We now define the *geodesic flow*. For t real and $b \in B$, $T^t(b)$ is the line element obtained by moving b, along the geodesic which it defines, through an s-distance $|t|$. If $t > 0$, we move in the direction determined by b; if $t < 0$, we move in the opposite direction.

Now if G is a Fuchsian group let Σ be the quotient space Δ/G. Directed line elements P on Σ are defined by identification of congruent line elements in B. The distance between two such elements P, P^1 is defined by:

$$s(P, P^1) = \inf_{b_i \in P, b_i{}^1 \in P^1} s(b_i, b_i{}^1)$$

The space of such elements P is denoted by Ω. A set in Ω is said to have *m-measure zero* if the set of all representative points in B has this property. We define the *m-measure* of a general measurable set on Ω as the measure, $\int dm$, of the intersection of the set of all representatives with those elements of B carried by points in D. Measure zero defined this way clearly agrees with measure 0 defined above, due to the countability of G.

The geodesic flow $T^t(P)$ is unambiguously defined on Ω since

$$V\,T^t(b) = T^t\,V(b)$$

holds for any transform V preserving Δ.

Let $P \in \Omega$ be a line element with carrier point $p \in \Sigma$. The geodesic flow $T^t(P)$ is said to be *divergent on* Ω if

$$s(T^t(P), P) \to \infty \quad \text{as } t \to \infty.$$

In this case the geodesic defined by P is said to be *divergent* (see Hopf [**9**, p. 869] for more details). Following Hopf we make a definition:

Definition. The surface Σ is of the *first class* if the divergent geodesics issuing from a fixed point p of Σ form a set of directions at p of angular measure zero. Σ is said to be of the *second class* if it is not of the first class.

We recall that a geodesic flow T defined on Ω is ergodic if and only if for every measurable set λ invariant under T either $m(\lambda) = 0$ or $m(\Omega \backslash \lambda) = 0$. We have the following result of Hopf [**9**, p. 871].

LEMMA 1. *For a surface Σ the geodesic flow on Ω is ergodic if and only if Σ is of the first class.*

Let G be a group such that $\Sigma = \Delta/G$ is of the first class and let S be a measurable G invariant subset of $U \times U$. We define a subset λ of Ω in the following way: $P \in \Omega$ is a point of λ provided every point $b \in B$ which is a projection of P determines a geodesic whose end points η_1 and η_2 give a point (η_1, η_2) of S. This definition of λ makes sense because S is G invariant.

Clearly $P \in \lambda$ if and only if $T^t(P) \in \lambda$ for all real t. Thus λ is a measurable, T invariant subset of Ω and as Σ is of the first class, it follows from Lemma 1 that either $m(\lambda) = 0$ or $m(\Omega \backslash \lambda) = 0$.

Now if $P \in \Omega$ and $b \in B$ is a projection of P, let $b = (z, \phi)$ then $z(= x + iy)$ lies on the geodesic joining η_1 to η_2 and is a non-euclidean distance r, say, from the mid-point of this geodesic. We have [**23**, p. 545]

$$dm = \frac{4dxdyd\phi}{(1 - |z|^2)^2} = \frac{2|d\eta_1|\,|d\eta_2|\,dr}{|\eta_1 - \eta_2|^2}.$$

It follows easily that $m(\lambda) = 0$ implies the two-dimensional measure of S is zero and $m(\Omega\backslash\lambda) = 0$ implies the two-dimensional measure of $(U \times U)\backslash S$ is zero. We have shown that if G is a group such that $\Sigma = \Delta/G$ is of the first class, then G is metric transitive. The proof of Theorem 2 is complete with the following:

LEMMA 2. *G is a Fuchsian group of divergence type if and only if Δ/G is of the first class.*

To prove Lemma 2 we note [**3**, p. 4] that ξ is a point of approximation for G if and only if there exists a sequence of points $\{z_n\}$ approaching ξ radially and a sequence $\{V_n\} \subset G$ such that $V_n(z_n)$ lies in a compact subset of Δ for all n. So the geodesic flow along a radius to ξ is divergent if and only if ξ is not a point of approximation. Thus G is of the first class if and only if almost every point of U is a point of approximation and this is the case if and only if G is of divergence type [Theorem B].

We now prove Theorem 4. If $\xi \notin H$ we note from the proof of Lemma 2 that only finitely many images of M will meet the radius to ξ. It follows that $\rho\{M^* \cap L(\xi, r)\}$ is bounded as $r \to 1$. If G is of convergence type then $m(H) = 0$ and Theorem 4 is proved in this case.

Now suppose G is of divergence type and note the following easy consequence of ergodicity (it follows from [**9**, p. 871]). If T is ergodic and $m(\Omega) = \infty$ then for a bounded measurable function f on Ω,

$$(2) \quad \lim_{x\to\infty} \frac{1}{x}\int_0^x f(T^t(P))\,dt = 0$$

for almost all $P \in \Omega$.

Now suppose, for real numbers $r_1, \ldots, r_4$, the subset ψ of C defined by:

$$\psi = \{x + iy: r_1 \leqq x \leqq r_2, r_3 \leqq y \leqq r_4\}$$

is a rectangle in D, the fundamental polygon of the group G under consideration. If $r_1 \ldots r_4$ are rationals, it is called a *rational rectangle*. We define the function $f(P)$ on Ω as follows: if P has a representative which is carried by a point in ψ, set $f(P) = 1$; otherwise $f(P) = 0$. Now G is of divergence type so by Theorem 3, T is ergodic; since $\sigma(D) = \infty$ it follows that $m(\Omega) = \infty$ so we apply (2) above to the function f,

$$(3) \quad \lim_{x \to \infty} \frac{1}{x} \int_0^x f(T^t(P))dt = 0.$$

which is valid provided P does not belong to a certain null set $N(r_1, \ldots, r_4)$ in Ω. Now we define N, a null set in Ω, by

$$N = \bigcup_{r_1, \ldots, r_4} N(r_1, \ldots, r_4)$$

where the union is over all rationals. If $P \notin N$ then (3) holds for any rational rectangle ψ in D. Let M be a disc in D; then for any $\epsilon > 0$ we may choose two sets M_1 and M_2 each consisting of a finite number of non-overlapping rational rectangles in D, such that

$$M_1 \subset M \subset M_2 \quad \text{and} \quad \sigma(M_2 \backslash M_1) < \epsilon.$$

We have equation (3) for M_1 and M_2 and thus also for M.

For almost all $\xi = e^{i\theta}$ we may find a point P of Ω which has a representative (x, y, θ) such that $P \notin N$ and the line segment joining 0 to (x, y) is in the direction $e^{i\theta}$. For such a point P

$$(4) \quad \frac{1}{x} \int_0^x f(T^t(P))dt = \frac{\rho\{M^* \cap L(\xi, r)\}}{\rho(L(\xi, r))}$$

where $f(Q)$ is the function defined on Ω by: $f(Q) = 1$ if Q has a representative carried by a point of M, $f(Q) = 0$ otherwise. The number r in (4) is the number such that $\rho(L(\xi, r)) = x$.

The conclusion of Theorem 4 follows from (3) and (4).

4. Proof of Theorem 3. Suppose G is metric transitive and let $u(z, w)$ be a function harmonic in each variable, bounded and invariant under G. Then for almost all $(e^{i\theta}, e^{i\phi}) \in U \times U$.

$$\lim_{z \to e^{i\theta},\, w \to e^{i\phi}} u(z, w) = u(e^{i\theta}, e^{i\phi})$$

exists uniformly when $z \to e^{i\theta}$, $w \to e^{i\phi}$ from the inside of fixed Stolz domains with vertices at $e^{i\theta}$ and $e^{i\phi}$ [**23**, p. 142]. For $a < b$ real, let $S(a, b)$ be the set of points $(e^{i\theta}, e^{i\phi})$ such that $a < u(e^{i\theta}, e^{i\phi}) \leqq b$. Since $u(z, w)$ is invariant by G it follows that $S(a, b)$ is also G-invariant. Since $S(a, b)$ is clearly measurable and G is metric transitive it follows that $S(a, b)$ has two dimensional measure 0 or $4\pi^2$. Thus there exists K such that $u(e^{i\phi}, e^{i\phi}) = K$ for almost all $(e^{i\theta}, e^{i\phi})$.

We know [**23**, p. 142] that $u(z, w)$ can be expressed by

$$u(z, w) = \frac{1}{4\pi^2}$$

$$\times \int\int_{U\times U} \frac{u(e^{i\theta^1}, e^{i\phi^1})}{(1 - 2r \operatorname{Cos}(\theta^1 - \theta) + r^2)} \frac{(1 - r^2)(1 - \rho^2)}{(1 - 2\rho \operatorname{Cos}(\phi^1 - \phi) + \rho^2)} d\theta^1 d\phi^1$$

where $z = re^{i\theta}$, $w = \rho e^{i\phi}$. Thus $u(z, w) = K$ identically in $\Delta \times \Delta$.

To prove the converse we suppose G is not metric transitive—so there exists a measurable set S on $U \times U$ which is invariant by G and whose two-dimensional measure lies between 0 and $4\pi^2$. Let $f(e^{i\theta}, e^{i\phi})$ be the characteristic function of S and set

$$u(z, w) = \frac{1}{4\pi^2} \int\int_{U\times U} f(e^{i\theta}, e^{i\phi}) \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \cdot \frac{1 - |w|^2}{|w - e^{i\phi}|^2} d\theta d\phi.$$

Then $u(z, w)$ is harmonic in each variable and is bounded and invariant under G [**23**, p. 537). Clearly u is not constant.

5. Proof of Theorem 5. The fact that (i) implies (ii) is a result of Tsuji [**22**, p. 267]. We assume that $\limsup_{r\to 1} (1 - r)\, C(r, M) > 0$ so there exists a sequence $\{r_n\}$ converging to 1 on which $(1 - r_n)\, C(r_n, M) \geqq a \geqq 0$. Each image of M has hyperbolic diameter δ, say, and hyperbolic area b, say. Thus for each n the number of images of M which intersect the circle $\{|z| = r_n\}$ is at least $C(r_n, M)/\delta$ which is at least $a/(1 - r_n)\delta$.

Set $R_n = (r_n + \epsilon)/(1 + \epsilon r_n)$, $(M = \{|z| < \epsilon\})$ and note [**23**, p. 511] that if an image of M meets $\{|z| = r_n\}$ then this image lies in $\{|z| < R_n\}$. Thus we see that

$$A(R_n, M) \geqslant \frac{a}{(1 - r_n)\delta} \cdot b$$
$$= \frac{ab(1 - \epsilon)}{\delta(1 + r_n\epsilon)(1 - R_n)}$$

So $\limsup_{r\to 1} (1 - r)A(r, M)$ is positive and (ii) implies (iii).

We assume that $\lim_{r\to 1} (1 - r)n(r, 0) = 0$ and let $a \in M$. Clearly if $|V(a)| < r$ then $|V(0)| < (r + \epsilon)/(1 + r\epsilon)$ $(=R)$. Thus

$$(1 - r)n(r, a) < \frac{1 + r\epsilon}{1 - \epsilon}(1 - R)n(R, 0)$$

and we see that $\lim_{r\to 1} (1 - r)\, n(r, a) = 0$ uniformly for all $a \in M$. Thus from the relation

$$A(r, M) = \int_{a\in M} n(r, a) d\sigma(a)$$

it follows that $\lim_{r\to 1} (1 - r)A(r, M) = 0$. Thus (iii) implies (i) and the proof is complete.

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