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WEAK IDEAL INVARIANCE AND ORDERS IN ARTINIAN RINGS: CORRIGENDUM

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Let R be an associative ring with identity element which has Krull dimension on the left (see [3] for the relevant definitions). For any ordinal α, and any module ${}_RX$, let $\tau_\alpha(X)$ denote the sum in X of all cyclic submodules of Krull dimension less than α. (The Krull dimension of a module X will be denoted by $|X|$.) It is clear that if X' is a submodule of X, then $\tau_\alpha(X') = X' \cap \tau_\alpha(X)$ and that $f(\tau_\alpha(X)) \subseteq \tau_\alpha(Y)$ for any R-homomorphism $f : X \to Y$. Thus τ_α defines a torsion preradical.

It has been pointed out to the author by Professor Ann Boyle that in [1] the torsion preradical τ_α was incorrectly assumed to always determine a torsion radical. Difficulties may arise if α is a limit ordinal and R is not left Noetherian. The main result of [1], Theorem 7, is valid as stated. The necessary corrections to the proof can be made easily by using the proposition proved in this note. The statements of Theorem 4 and Theorem 5 of [1] can be corrected by making the additional assumption that τ_α is a torsion radical. A revised version of Theorem 6 is given below.

To show that τ_α is a torsion radical, it is necessary to show that $\tau_\alpha(X/\tau_\alpha(X)) = (0)$ for all modules ${}_RX$. It is sufficient to show that if A and B are left ideals with $A \subseteq B$, $\tau_\alpha(B/A) = B/A$ and $|R/B| < \alpha$, then $|R/A| < \alpha$. This follows immediately if R is left Noetherian. If α is not a limit ordinal, then $|B/A| < \alpha$ by Corollary 4.2 of [3], and so

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$|R/A| < \alpha$.

Definitions and notation are those of [1], and the following notation will be fixed throughout. The prime radical of R will be denoted by N, and the Krull dimension of R will be denoted by α . The intersection of all prime ideals P of R such that $|R/P| = \alpha$ will be denoted by N_α .

PROPOSITION. *(a) If N is left weakly ideal invariant, then τ_α is a torsion radical.*

(b) N is left weakly ideal invariant if and only if N_α is left weakly ideal invariant.

(c) If R/N is K-homogeneous and $C(N) \subseteq C(0)$, then N is left weakly ideal invariant.

Proof. *(a)* Let $C = \{c \in R \mid |R/Rc| < \alpha\}$. By Corollary 5 of [4], if D is a left ideal of R with $|R/D| < \alpha$, then $D \cap C \neq \emptyset$. The proof that C is a left Ore set is similar to the first part of the proof of Theorem 8 of [4]. (Given $c \in C$ and $a \in R$, let $A = \{r \in R \mid ra \in Rc\}$. Then $|R/A| < \alpha$, and so there exists $c' \in A \cap C$, which implies that $c'a = a'c$ for some $a' \in R$. Furthermore, if $c, d \in C$, then $cd \in C$ since $|R/Rcd| = \sup\{|R/Rd|, |Rd/Rcd|\}$ and $|Rd/Rcd| \leq |R/Rc|$.) Thus the filter of left ideals $\{D \subseteq R \mid |R/D| < \alpha\}$ which defines τ_α has as a basis the set $\{Rc \mid c \in C\}$, and since C is a left Ore set, it follows immediately that τ_α is a torsion radical.

(b) Let I be the intersection of all minimal prime ideals P of R with $|R/P| < \alpha$ (if this intersection is empty, then of course $N = N_\alpha$) and let D be a left ideal of R with $|R/D| < \alpha$. Then $|R/I| < \alpha$ and $IN_\alpha \subseteq I \cap N_\alpha = N$, so N_α/N and $N_\alpha D/ND$ are left R/I-modules, which forces $|N_\alpha/N| < \alpha$ and $|N_\alpha D/ND| < \alpha$. Since

$$|N/ND| \leq |N_\alpha/ND| = \sup\{|N_\alpha D/ND|, |N_\alpha/N_\alpha D|\}$$

and

$$|N_\alpha/N_\alpha D| \leq |N_\alpha/ND| = \sup\{|N_\alpha/N|, |N/ND|\} ,$$

it follows that $|N/ND| < \alpha$ if and only if $|N_\alpha/N_\alpha D| < \alpha$.

(c) Let D be a left ideal of R with $|R/D| < \alpha$. Then $\mathrm{Hom}(R/D, E(R/N)) = (0)$ since R/N is K-homogeneous, and so there exists $d \in D \cap C(N)$. If $|R/Rd| = \beta$, then $\beta < \alpha$ since by assumption $C(N) \subseteq C(0)$ and hence d is a regular element of R . For any element $x \in N \cap D$, if $rx = ad$ for $r, a \in R$, then $ad \in N$ implies that $a \in N$ since $d \in C(N)$, and so $rx \in Nd$. Thus

$$\{r \in R \mid rx \in Nd\} = \{r \in R \mid rx \in Rd\} ,$$

and so $|R/A| \leq |R/Rd| = \beta$ for the left ideal $A = \{r \in R \mid rx \in Nd\}$. This shows that each cyclic submodule of $(N \cap D)/ND$ has Krull dimension less than or equal to β , which implies that $|(N \cap D)/ND| \leq \beta < \alpha$. Thus $|N/ND| < \alpha$ since $|N/ND| = \sup\{|N/(N \cap D)|, |(N \cap D)/ND|\}$ and $|N/(N \cap D)| = |(N+D)/D| \leq |R/D| < \alpha$.

THEOREM 6'. *Let* R *be a ring with Krull dimension* α *on the left, and let* N_α *be the intersection of all minimal prime ideals* P *of* R *such that* $|R/P| = \alpha$. *Then the prime radical* N *of* R *is left weakly ideal invariant if and only if* $C(N_\alpha)$ *is a left Ore set and* τ_α *is a torsion radical.*

Proof. If N is left weakly ideal invariant, then by part *(a)* of the preceding proposition, τ_α is a torsion radical. The proof of Theorem 6 of [1] then shows that $C(N_\alpha)$ is a left Ore set, since by part *(b)* of the preceding proposition, N is left weakly ideal invariant if and only if N_α is left weakly ideal invariant.

The converse follows from the preceding proposition and the proof of Theorem 6 of [1].

References

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