SOME REFINEMENTS OF LYAPUNOV'S SECOND METHOD

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1. Lyapunov's second method is a well-known and powerful tool for studying the behaviour of solutions of a system of differential equations. One approach to the theory is the comparison method developed by Corduneanu **(4)**. This approach has the advantage that it also leads to other results on asymptotic behaviour which originally appeared to be unrelated to Lyapunov's method. Some of these results have been obtained by the author in **(2)**. The purpose of this paper is to make use of the comparison method to obtain some refinements of Lyapunov's theory. A classical theorem of Lyapunov states essentially that if the derivative of a suitably well-behaved Lyapunov function is negative definite, then the trivial solution of a system of differential equations is asymptotically stable. In this paper a result is obtained on asymptotic stability when the derivative of the Lyapunov function is negative everywhere but in a weaker sense. In another result, this derivative is allowed to be positive for some values of the independent variable, provided its average is sufficiently strongly negative. Easy proofs of asymptotic stability theorems due to Massera **(10)** and Persidskiĭ **(11)** can be obtained as very special cases of these results.

2. Let us begin by collecting some basic definitions and preliminary results. We are interested in a system of differential equations

$$x' = f(t, x), \tag{1}$$

where x and f are real n-dimensional vectors and t is a real non-negative scalar. A solution of this system will be denoted by $x(t)$. Sometimes we shall wish to emphasize the dependence of the solution on initial conditions, by using $x(t, t_0, x_0)$ to denote a solution which takes the value x_0 for $t = t_0$.

We shall always assume that f is continuous in (t, x) on $0 \leqslant t < \infty$, $|x| < \infty$. ($|\ |$ will denote any suitable norm for vectors.) We shall also assume $f(t, 0) = 0$ for $t \geqslant 0$, so that $x = 0$ is a solution of (1). Following the standard terminology **(1; 5)**, we say that the solution $x = 0$ is stable if for every $\epsilon > 0$ and $t_0 \geqslant 0$ there exists $\delta(\epsilon, t_0) > 0$ such that $|x_0| < \delta$ implies $|x(t, t_0, x_0)| < \epsilon$ for $t_0 \leqslant t < \infty$. If δ in this definition can be chosen independent of t_0, then we shall say that the solution $x = 0$ is uniformly stable. If $x = 0$ is stable and, in addition, $|x_0| < \delta$ implies that $\lim_{t\to\infty} x(t, t_0, x_0) = 0$, then we shall say

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that the solution $x = 0$ is asymptotically stable. If $x = 0$ is uniformly stable and if for every $\eta > 0$ there exists $\tau(\eta)$, independent of t_0, such that $|x(t, t_0, x_0)| < \eta$ for $t > t_0 + \tau(\eta)$ when $|x_0| < \delta$, where δ is independent of η, then we shall say that the solution $x = 0$ is uniformly asymptotically stable. The region of asymptotic stability is the set of initial values x_0 for which $x(t, t_0, x_0) \to 0$. It has been shown by Massera **(10)** that if f does not depend on t explicitly, then stability or asymptotic stability of $x = 0$ in (1) implies uniform stability or uniform asymptotic stability respectively of $x = 0$.

In discussing Lyapunov functions, we shall find it convenient to use the class of functions K introduced by Hahn **(5)**. A real-valued function $\Theta(r)$ belongs to the class K if it is defined for $r \geqslant 0$, if $\Theta(0) = 0$, and if $\Theta(r)$ is strictly increasing. We shall be interested in real-valued functions $V(t, x)$ of the scalar variable t and the vector variable x, defined for $t \geqslant t_0$ and x in some region $|x| \leqslant h$. A function V is said to be positive definite if there exists a function $\Theta(r)$ in the class K such that

$$V(t, x) \geqslant \Theta(|x|). \tag{2}$$

A function is said to be negative definite if its negative is positive definite. A function V is said to have an infinitesimal upper bound if there exists a function ψ in the class K such that

$$|V(t, x)| \leqslant \psi(|x|). \tag{3}$$

We shall always assume that the function V is continuous in (t, x) and has one-sided partial derivatives with respect to t and the components of x. We use V_t to denote a partial derivative of V with respect to t, V_x to denote a gradient vector of V with respect to x, and $\cdot$ to denote the usual scalar product of vectors. Any condition involving V_t or V_x will be understood to be required for all one-sided derivatives. We define

$$V'(t, x) = V_t(t, x) + V_x(t, x) \cdot f(t, x),$$

the total derivative of V with respect to the system of differential equations (1). The reason for this terminology is that if V has continuous first-order partial derivatives and if $x(t)$ is a solution of (1), then $V'(t, x(t)) = d[V(t, x(t))]/dt$. We shall assume that $V'(t, x)$ satisfies an inequality of the form

$$V'(t, x) \leqslant w[t, V(t, x)], \tag{4}$$

where $w(t, r)$ is continuous on $0 \leqslant t < \infty$, $r \geqslant 0$, and $w(t, 0) = 0$ for $t \geqslant 0$. We shall make use of the following result, proved in **(2; 4)**.

THEOREM 1. *Suppose that V is positive definite and satisfies* (4). *If the solution $r = 0$ of the scalar differential equation*

$$r' = w(t, r) \tag{5}$$

is stable, then the solution $x = 0$ of (1) *is stable. If the solution $r = 0$ of* (5) *is* *asymptotically stable, then the solution* $x = 0$ *of* (1) *is asymptotically stable. Moreover, if* V *has an infinitesimal upper bound, then uniform stability and uniform asymptotic stability for* $r = 0$ *in* (5) *imply uniform stability and uniform asymptotic stability respectively for* $x = 0$ *in* (1).

The last statement in Theorem 1 dealing with uniformity is not given explicitly in **(2)** but it is easy to see that the proof given there yields this part of the theorem. If the equation (5) is autonomous ($w(t, r)$ independent of t), then any stability or asymptotic stability of $r = 0$ in (5) must be uniform **(5**, pp. 62–64**)**.

Before proceeding to our main results, we give a useful preliminary lemma.

LEMMA 1. *If* V *is positive definite and has an infinitesimal upper bound, and if the total derivative of* V *with respect to* (1) *is negative definite, then there exists a function* χ *in the class* K *such that*

$$V'(t, x) \leqslant \chi[V(t, x)].$$

Proof. By definition, there exist functions ϕ, ψ in the class K such that

$$-V'(t, x) \geqslant \Theta(|x|), \qquad V(t, x) \leqslant \psi(|x|).$$

Since ψ is strictly increasing, it has an inverse ψ^{-1} which is also strictly increasing and belongs to the class K. Then $|x| \geqslant \psi^{-1}(V(t, x))$, and $-V'(t, x) \geqslant \Theta \circ \psi^{-1}[V(t, x)]$. It is easy to verify that the composite of two functions in the class K also belongs to K, and the lemma follows, with $\chi = \Theta \circ \psi^{-1}$.

3. The comparison theorem quoted in the previous section suggests that detailed study of a first-order differential equation may yield theorems on the stability of the zero solution of a system. To this end, we examine the scalar equation

$$r' = \lambda(t)\phi(r), \tag{6}$$

where λ is continuous on $0 \leqslant t < \infty$, and ϕ is continuous and non-negative on $0 \leqslant r < \infty$. This equation has been studied in **(3)** with the assumption that λ is also non-negative. It is easy to verify that the formula obtained for the solution in **(3)** remains valid when λ is allowed to take negative values, so long as precautions are taken to prevent the solution from becoming negative. We define

$$J(r) = \int_0^r \frac{du}{\phi(u)} \quad \text{if} \quad \int_0^r \frac{du}{\phi(u)} < \infty$$

(otherwise

$$J(r) = \int_\epsilon^r \frac{du}{\phi(u)}$$

for sufficiently small $\epsilon > 0$). If

$$\int_0^\infty \frac{du}{\phi(u)} = R \leqslant \infty,$$

then J is a monotone function mapping the interval $[0, \infty)$ homeomorphically onto the interval $(0, R)$. The solution $r(t)$ of (6) with $r(t_0) = r_0$ is given by

$$J(r(t)) = J(r_0) + \int_{t_0}^{t} \lambda(s)ds, \tag{7}$$

as long as

$$0 \leqslant J(r_0) + \int_{t_0}^{t} \lambda(s)ds < R,$$

or

$$\int_0^{r_0} \frac{du}{\phi(u)} \leqslant \int_{t_0}^{t} \lambda(s)ds < \int_{r_0}^{\infty} \frac{du}{\phi(u)}. \tag{8}$$

If

$$-\int_0^{r_0} \frac{du}{\phi(u)} = \int_{t_0}^{t_1} \lambda(s)ds \qquad \text{for some } t_1 \ (t_0 < t_1 < \infty),$$

then $r(t_1) = 0$; while if

$$-\int_0^{r_0} \frac{du}{\phi(u)} = \int_{t_0}^{\infty} \lambda(s)ds,$$

then $\lim_{t\to\infty} r(t) = 0$. If

$$\int_{t_0}^{t_2} \lambda(s)ds = \int_{r_0}^{\infty} \frac{du}{\phi(u)} \qquad \text{for some } t_2 \ (t_0 < t_2 \leqslant \infty),$$

then $\lim_{t\to t_2} r(t) = \infty$. From (7) and the fact that J is a homeomorphism it is easy to see that: (i) $r = 0$ is stable in (6) if (8) holds for each t_0 and $t_0 \leqslant t < \infty$; (ii) $r = 0$ is uniformly stable in (6) if (8) holds and either $\int_{r_0}^{\infty} du/\phi(u) < \infty$ or (*a*) $\int_{r_0}^{\infty} du/\phi(u) = \infty$ and (*b*) $\int_{t_0}^{t} \lambda(s)ds$ is bounded above for every t $(t_0 \leqslant t < \infty)$ uniformly in t_0; (iii) $r = 0$ is asymptotically stable in (6) if there exists T $(t_0 \leqslant T \leqslant \infty)$ such that

$$\int_{t_0}^{T} \lambda(s)ds = -\int_0^{r_0} \frac{du}{\phi(u)}.$$

In this case the region of asymptotic stability includes $0 \leqslant r(t_0) \leqslant r_0$.

Combination of this analysis with Theorem 1 yields the following result:

THEOREM 2. *Let V be a positive definite Lyapunov function whose total derivative with respect to* (1) *satisfies*

$$V'(t, x) \leqslant \lambda(t)\phi(V(t, x)), \tag{9}$$

where λ is continuous on $t_0 \leqslant t < \infty$, and ϕ is continuous and non-negative on $0 \leqslant r < \infty$. Then $x = 0$ is stable in (1) *if* (8) *holds for some $r_0 > 0$, every $t_0 \geqslant 0$, and $t_0 \leqslant t < \infty$. Also, $x = 0$ is asymptotically stable in* (1) *if there exists T $(t_0 \leqslant T \leqslant \infty)$ such that*

$$\int_{t_0}^{T} \lambda(s)ds = -\int_0^{r_0} \frac{du}{\phi(u)}.$$

Finally, $x = 0$ *is uniformly stable if* V *has an infinitesimal upper bound, if* (8) *holds, and if either* $\int_{r_0}^{\infty} du/\phi(u) < \infty$ *or* (*a*) $\int_{r_0}^{\infty} du/\phi(u) = \infty$ *and* (*b*) $\int_{t_0}^{t} \lambda(s)ds$ *is bounded above for every* t $(t_0 \leqslant t < \infty)$ *uniformly in* t_0. *In the case of asymptotic stability, the region of asymptotic stability includes initial values* x_0 *with* $V(t_0, x_0) \leqslant r_0$.

In Theorem 2, λ is allowed to take both positive and negative values. The easier special case in which λ is always negative is of sufficient interest to merit explicit description. We consider

$$r' = -\mu(t)\phi(r), \tag{10}$$

where μ is continuous and non-negative on $0 \leqslant t < \infty$ and ϕ is continuous and non-negative on $0 \leqslant r < \infty$, and where $\phi(u) > 0$ for $u > 0$. The study of (10) is simpler than the study of (6) mainly because all solutions of (10) are monotone decreasing.

LEMMA 2. *If*

$$\int_{t_0}^{\infty} \mu(s)ds \geqslant \int_0^{r_0} \frac{du}{\phi(u)}, \tag{11}$$

for some $r_0 > 0$, *then the zero solution of* (10) *is asymptotically stable, and the region of asymptotic stability contains* $0 \leqslant r(t_0) \leqslant r_0$.

Proof. The solution $r(t)$ of (10) with $r(t) = r_0$ satisfies

$$\int_{r(t)}^{r_0} \frac{du}{\phi(u)} = \int_{t_0}^{t} \mu(s)ds.$$

If $\int_0^{r_0} du/\phi(u)$ diverges, then (11) implies the divergence of $\int_{t_0}^{\infty} \mu(s)ds$. Thus

$$\int_{r(t)}^{r_0} \frac{du}{\phi(u)} \to \infty \qquad \text{as } t \to \infty,$$

which can happen only if $r(t) \to 0$. If $\int_0^{r_0} du/\phi(u)$ converges and

$$\int_{t_0}^{\infty} \mu(s)ds > \int_0^{r_0} \frac{du}{\phi(u)},$$

then there exists $T > t_0$ such that

$$\int_{t_0}^{T} \mu(s)ds = \int_0^{r_0} \frac{du}{\varphi(u)}.$$

This implies that $r(T) = 0$, and then $r(t) = 0$ for $t \geqslant T$. If

$$\int_{t_0}^{\infty} \mu(s)ds = \int_0^{r_0} \frac{du}{\phi(u)} < \infty,$$

then it is clear that $r(t) \to 0$ as $t \to \infty$.

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Combination of Lemma 2 and Theorem 1 yields the following result.

THEOREM 3. *Let V be a positive definite Lyapunov function whose total derivative with respect to* (1) *satisfies*

$$V'(t, x) \leqslant -\mu(t)\phi(V(t, x)), \tag{12}$$

where μ is continuous and non-negative on $t_0 \leqslant t < \infty$ and ϕ is continuous and non-negative on $0 \leqslant r < \infty$, with $\phi(u) > 0$ for $r > 0$. If (11) *is satisfied for some $r_0 > 0$, then the zero solution of* (1) *is asymptotically stable and the region of asymptotic stability includes initial values x_0 with $V(t_0, x_0) \leqslant r_0$.*

COROLLARY 1 (Massera **10**). *If $V'(t, x) \leqslant -\phi(V(t, x))$, where V is positive definite and ϕ is as in Theorem* 3, *then the zero solution of* (1) *is asymptotically stable. If V also has an infinitesimal upper bound, then the asymptotic stability is uniform.*

Proof. If $\mu(t) \equiv 1$, then (11) is automatically satisfied for every $r_0 > 0$, so that Theorem 3 is applicable. Also, the comparison equation (10) is autonomous, and therefore the asymptotic stability of $r = 0$ for (10) is uniform. If V has an infinitesimal upper bound, then the asymptotic stability of $x = 0$ for (1) is uniform by Theorem 1.

COROLLARY 2 (Persidskiĭ **11**). *If there exists a positive definite Lyapunov function, with an infinitesimal upper bound, whose total derivative with respect to* (1) *is negative definite, then $x = 0$ is uniformly asymptotically stable for* (1).

Proof. By Lemma 1, there exists a function ϕ in the class K such that $V'(t, x) \leqslant -\phi(V(t, x))$, and the result follows from Corollary 1.

The part of Theorem 3 dealing with asymptotic stability has been obtained by Krasovskiĭ (**6**), using a slightly different method. The role of Lemma 1 is to show that when V has an infinitesimal upper bound, the results of Massera and Persidskiĭ are equivalent.

To conclude this section, let us apply Theorem 2 to a system of the form

$$x' = A(t)x + g(t, x), \tag{13}$$

where $A(t)$ is a continuous $n \times n$ matrix and $q(t, x)$ is an n-dimensional vector We assume that there exists a function σ such that the eigenvalues of the symmetric matrix $\frac{1}{2}[A(t) + A^T(t)]$ (where A^T denotes the transpose of A) are no greater than $\sigma(t)$. We also assume that there exists a non-negative function γ such that

$$||g(t, x)|| \leqslant \gamma(t)||x||,$$

where $||x||$ denotes the Euclidean norm. We use $V(t, x) = x^T x = ||x||^2$, and then $V'(t, x) \leqslant 2[\sigma(t) + \gamma(t)]V(t, x)$. We are led to the comparison equation $r' = 2[\sigma(t) + \gamma(t)]r$. Theorem 2 shows that the zero solution of (13) is stable if

$$\int_{t_0}^{t} [\sigma(s) + \gamma(s)]ds < \infty \qquad \text{for } t_0 \leqslant t < \infty,$$

uniformly stable if

$$\int_{t_0}^{t} [\sigma(s) + \gamma(s)]ds \qquad \text{is bounded above for all } t \text{ uniformly in } t_0,$$

and asymptotically stable if

$$\int_{t_0}^{T} [\sigma(s) + \gamma(s)]ds = -\infty \qquad \text{for some } T\ (t_0 \leqslant T \leqslant \infty).$$

Note that σ is allowed to take both positive and negative values, while γ is non-negative. Thus a sufficient condition for asymptotic stability is the existence of T $(t_0 \leqslant T \leqslant \infty)$, such that

$$\int_{t_0}^{T} \sigma(s)ds = -\infty \quad \text{and} \quad \int_{t_0}^{T} \gamma(s)ds < \infty.$$

4. The assumption that the Lyapunov function V has an infinitesimal upper bound can often be replaced by an assumption of boundedness on the function f on the right side of (1). An example is the following theorem due to Marachkov **(8)**, which has been generalized by Levin **(7)**.

THEOREM 4. *Let V be a positive definite Lyapunov function whose total derivative with respect to* (1) *is negative definite, and if f is bounded uniformly in t whenever x remains in a compact set, then the solution $x = 0$ of* (1) *is asymptotically stable.*

The proof in **(7)** is obtained by observing that there exists a function ϕ in the class K such that $V'(t, x) \leqslant -\phi(|x|)$, or

$$V(t, x(t)) - V(t_0, x_0) \leqslant -\int_{t_0}^{t} \phi(|x(s)|)ds,$$

where $x(t)$ is a solution of (1) with $x(t_0) = x_0$. This yields

$$\int_{t_0}^{t} \phi(|x(s)|)ds \leqslant V(t_0, x_0) \qquad \text{for every } t\ (t_0 \leqslant t < \infty),$$

which implies that

$$\int_{t_0}^{\infty} \phi(|x(s)|)ds \leqslant V(t_0, x_0). \tag{14}$$

The stability of $x = 0$ (a consequence of Theorem 1), the boundedness of f, and $x'(t) = f(t, x(t))$ imply that $|x(t)|$ and $|x'(t)|$ are uniformly bounded when $|x_0|$ is small enough. This, together with (14), yields the asymptotic stability of $x = 0$.

If neither boundedness of f nor the existence of an infinitesimal upper bound for V is assumed, the solution $x = 0$ of (1) is not necessarily asymptotically stable. This can be shown by an example due to Massera **(9)**. It has been shown, however, that if V is positive definite and V' is negative definite, then for every solution $x(t)$ of (1) with $|x(t_0)|$ sufficiently small, $\liminf_{t\to\infty} |x(t)| = 0$ **(1)**. We can give the following generalization of this result.

THEOREM 5. *Let V be a positive definite Lyapunov function whose total derivative with respect to* (1) *satisfies*

$$V'(t, x) \leqslant -\lambda(t)\phi(|x|), \tag{15}$$

where ϕ *is a function in the class* K *and* λ *is a continuous non-negative function on* $0 \leqslant t < \infty$ *with* $\int_0^\infty \lambda(t)dt = \infty$. *Then for every solution* $x(t)$ *of* (1) *with* $|x(t_0)|$ *sufficiently small,* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Inequality (15) yields

$$V(t, x(t)) - V(t_0, x(t_0)) \leqslant -\int_{t_0}^{t} \lambda(s)\phi(|x(s)|)ds,$$

which leads to

$$\int_{t_0}^{t} \lambda(s)\phi(|x(s)|)ds \leqslant V(t_0, x(t_0)).$$

Since this is true for all t,

$$\int_{t_0}^{\infty} \lambda(s)\phi(|x(s)|)ds \leqslant V(t_0, x(t_0)).$$

If $\liminf_{t\to\infty} |x(t)| > 0$, then for every $\epsilon > 0$ there exists $T = T(\epsilon)$ such that $|x(t)| > \epsilon$ for $t > T$. This implies that

$$V(t_0, x(t_0)) \geqslant \int_{t_0}^{\infty} \lambda(s)\phi(|s(s)|)ds \geqslant \int_{T}^{\infty} \lambda(s)\phi(|x(s)|)ds$$

$$\geqslant \phi(\epsilon)\int_{T}^{\infty} \lambda(s)ds,$$

which contradicts $\int_0^\infty \lambda(s)ds = \infty$. Thus $\liminf_{t\to\infty} |x(t)| = 0$, provided that $|x(t_0)|$ is small enough for $V(t_0, x(t_0))$ to be defined. Of course, if $V(t, x)$ is defined everywhere, this result holds for all solutions of (1).

It is not known whether the condition $\int_0^\infty \lambda(s)ds = \infty$ in Theorem 5 is necessary. Another open question is whether some slightly stronger condition of the same type suffices for asymptotic stability.

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Postscript. The following example, supplied by Professor A. Strauss, shows that the condition (15) does not imply asymptotic stability of the solution $x = 0$ of (1). We choose a sequence $\{t_n\} \to \infty$, and corresponding to this sequence we define the functions g and h as follows. For each integer k, we define $g(t_{2k-2}) = 2$, make g decreasing on $[t_{2k-2}, t_{2k-1} - 1]$ with $g'(t) \geqslant -1/k$ on this interval, define $g(t) = 1/k$ on $[t_{2k-1} - 1, t_{2k-1} + 1]$, make g increasing on $[t_{2k-1} + 1, t_{2k}]$ with $g'(t) \leqslant 1/k$ on this interval, and define $g(t_{2k}) = 2$. We can make g continuously differentiable and $|g'(t)/g(t)| \leqslant 1$ for $0 \leqslant t < \infty$. For each integer k we define $h(t) = 1$ on $[t_{2k-1} - \frac{1}{2}, t_{2k-1} + \frac{1}{2}]$,

$$h(t) = 2^{-k}/(t_{2k+1} - t_{2k-1}) \quad \text{on} \quad [t_{2k-1} + 1, t_{2k+1} - 1],$$

and elsewhere we define h to be linear, so that h is continuous on $0 \leqslant t < \infty$. We can choose the sequence $\{t_n\}$ so that h is not integrable but g^2h is integrable on $0 \leqslant t < \infty$.

Now consider the first-order linear equation

$$x' = \frac{g'(t)}{g(t)}\, x, \tag{16}$$

whose solutions are given by $x(t) = x(0)g(t)$. Since g does not tend to zero, it is clear that the solution $x = 0$ of (16) is not asymptotically stable. The Lyapunov function

$$V(t, x) = \frac{x^2}{g^2(t)} \left[\int_t^\infty g^2(s)h(s)ds + \int_0^\infty g^2(s)h(s)ds \right]$$

is positive definite since

$$V(t, x) \geqslant \frac{x^2}{4} \int_0^\infty g^2(s)h(s)ds,$$

and its total derivative with respect to (16) is $V'(t, x) = -h(t)x^2$. Thus the hypotheses of Theorem 5 are satisfied, but the solution $x = 0$ of (16) is not asymptotically stable.