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Some Functional Inequalities on Polynomial Volume Growth Lie Groups

Diego Chamorro

Abstract. In this article we study some Sobolev-type inequalities on polynomial volume growth Lie groups. We show in particular that improved Sobolev inequalities can be extended to this general framework without the use of the Littlewood–Paley decomposition.

1 Introduction

Classical Sobolev inequalities provide us with a family of *a priori* estimates of the following form:

$$\|f\|_{L^q} \leq C\|\nabla f\|_{L^p}, \quad \text{where } q = np/(n-p). \tag{1.1}$$

Initially stated over $\mathbb{R}^n$, they were successively generalized to other settings, such as manifolds and Lie groups; see, for example, [1, 15] for such generalizations.

Since the work of P. Gérard, Y. Meyer, and F. Oru [9], we know that it is possible to improve the classical Sobolev inequalities (1.1) by introducing a well-suited Besov space, and it is worth knowing if these improved inequalities can be generalized to some special Lie groups. For example, in the case of the Heisenberg group, which is the simplest example of a stratified Lie group, this was done by H. Bahouri, P. Gérard and C-J Xu [2], following essentially the same ideas of the original paper by P. Gérard, Y. Meyer, and F. Oru. For general stratified Lie groups, the task was achieved in [3] using some different techniques. In this special setting we obtained a family of Sobolev-type inequalities. Namely, for $G = (\mathbb{R}^n, \cdot, \delta)$ a stratified Lie group, and for f, a function such that $f \in \dot{W}^{s,p}(G)$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(G)$, we have

$$\|f\|_{\dot{W}^{s,q}} \leq C\|f\|^{\theta}_{\dot{W}^{s_1,p}}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{1.2}$$

where the parameters p, q, and the indices θ, β, s, and s_1 are related in a specific manner. See Section 5 below for the definitions of these functional spaces.

This type of Lie group is a generalization of $\mathbb{R}^n$ via modifying dilations; in this setting, the mathematical objects we are dealing with are constructed in such a way as to respect the homogeneity induced by these dilations. Therefore, many properties

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of these objects (operators, functional spaces) are very similar to the Euclidean case. See [7, 14] and the references therein for more details.

If we want to study inequalities of type (1.1) and (1.2) in a more abstract framework, it is possible to consider Lie groups without a dilation structure, and in this case we have several possibilities: a first example is given by *nilpotent Lie groups*, which are a generalization of stratified Lie groups (recall that every stratified Lie group is nilpotent) but these groups are not necessarily endowed with a dilation structure; see more details in [6]. A second example is given by *polynomial volume growth Lie groups*, where we have useful polynomial estimates for the Haar measure of a ball. Some other examples can be considered, such as *exponential growth Lie groups*; see [15] for definitions and some related results for the last case.

Classical Sobolev inequalities have been extensively studied in the three previous frameworks, and a detailed account can be found in [15].

In this article we will focus on *polynomial volume growth Lie groups*, and our main purpose is to study improved Sobolev inequalities of type (1.2) in this very particular setting. Our results are as follows.

Theorem 1.1 *Let G be a polynomial volume growth Lie group.*

- *Strong inequalities $p > 1$: If $f \in \dot{W}^{s_1,p}(G)$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(G)$, then we have*

$$\|f\|_{\dot{W}^{s,q}} \leq C\|f\|^{\theta}_{\dot{W}^{s_1,p}}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{1.3}$$

  where $1 < p < q < +\infty$, $\theta = p/q$, $s = \theta s_1 - (1-\theta)\beta$, and $-\beta < s < s_1$.
- *Strong inequalities $p = 1$: If $\nabla f \in L^1(G)$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(G)$, then we have*

$$\|f\|_{L^q} \leq C\|\nabla f\|^{\theta}_{L^1}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{1.4}$$

  where $1 < q < +\infty$, $\theta = 1/q$ and $\beta = \theta/(1-\theta)$.
- *Weak inequalities $p = 1$: If $\nabla f \in L^1(G)$ and $f \in \dot{B}^{-\beta,\infty}_{\infty}(G)$, then we have*

$$\|f\|_{\dot{W}^{s,q}_{\infty}} \leq C\|\nabla f\|^{\theta}_{L^1}\|f\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{1.5}$$

  where $1 < q < +\infty$, $0 < s < 1/q < 1$, $\theta = 1/q$ and $\beta = \frac{1-sq}{q-1}$.

For a precise definition of these functional spaces, refer to Section 5.

Let us make some remarks concerning the techniques used in the proof of these inequalities. For the proof of (1.3), we will not make use of the Littlewood–Paley decomposition as was done in [2, 9]. This estimate will be obtained in a more straightforward way applying the sub-Laplacian's fractional powers properties together with some properties of the Besov spaces.

Concerning strong inequalities (1.4) and weak inequalities (1.5), the proof follows a very different path, and we will see that these two estimates rely on the modified Poincaré pseudo-inequality stated in Theorem 6.1. Observe in particular that it is the use of this special inequality that suggested to us the definition of the weak Sobolev spaces $\dot{W}^{s,q}_{\infty}$ in the estimates (1.5).

To finish the preliminary remarks, let us stress here that the role of the polynomial growth geometry will be clearly identified in the estimates available for the heat kernel H_t associated with the sub-Laplacian $\mathcal{J}$ and in the operator's properties built from the spectral decomposition of the sub-Laplacian $\mathcal{J}$. These estimates and properties will be decisive for the proof of Theorem 1.1.

The plan of the article is the following: Section 2 is devoted to a short introduction of polynomial volume growth Lie groups. Section 3 gives some important estimates for the Heat kernel. Section 4 presents some results concerning spectral theory. Section 5 gives the precise definition of functional spaces involved in the inequalities above, and Section 6 presents the proof of Theorem 1.1.

2 Polynomial Volume Growth Lie Groups

Let G be a connected unimodular Lie group endowed with its Haar measure dx. Denote by $\mathfrak{g}$ the Lie algebra of G and consider a family $\mathbf{X} = \{X_1, \dots, X_k\}$ of left-invariant vector fields on G satisfying the Hörmander condition, which means that the Lie algebra generated by the X_j for $1 \le j \le k$ is $\mathfrak{g}$.

In this setting we have at our disposal the Carnot–Carathéodory metric associated with $\mathbf{X}$ defined as follows: let $\ell\colon [0,1] \to G$ be an absolutely continuous path. We say that ℓ is admissible if there exist measurable functions $\gamma_1, \dots, \gamma_k\colon [0,1] \to \mathbb{C}$ such that, for almost every $t \in [0,1]$, we have $\ell'(t) = \sum_{j=1}^{k} \gamma_j(t) X_j(\ell(l))$. If ℓ is admissible, define the length of ℓ by $\|\ell\| = \int_0^1 (\sum_{j=1}^{k} |\gamma_j(t)|^2)^{1/2}\, dt$. Then, for all $x, y \in G$, the distance between x and y is the infimum of the lengths of all admissible curves joining x to y. We will denote $\|x\|$ the distance between the origin e and x and $\|y^{-1} \cdot x\|$ the distance between x and y.

For $r > 0$ and $x \in G$, denote by $B(x,r)$ the open ball with respect to the Carnot–Carathéodory metric centered in x and of radius r, and by $V(r)$ the Haar measure of any ball of radius r. When $0 < r < 1$, there exist $d \in \mathbb{N}^*$, c_l and $C_l > 0$ such that for all $0 < r < 1$ we have

$$c_l r^d \le V(r) \le C_l r^d.$$

The integer d is the *local* dimension of $(G, \mathbf{X})$. When $r \ge 1$, either of the following two situations may occur independently of the choice of the family $\mathbf{X}$.

(i) G has polynomial volume growth, and there exist $D \in \mathbb{N}^*$, c_∞ and $C_\infty > 0$ such that for all $r \ge 1$ we have

$$c_\infty r^D \le V(r) \le C_\infty r^D. \tag{2.1}$$

(ii) G has exponential volume growth, which means that there exist $c_e, C_e, \alpha, \beta > 0$ such that for all $r \ge 1$ we have

$$c_e e^{\alpha r} \le V(r) \le C_e e^{\beta r}.$$

When G has polynomial volume growth, the integer D in (2.1) is called the dimension at infinity of G. Recall that nilpotent groups have polynomial volume growth and that a strict subclass of the nilpotent groups consists of stratified Lie groups. See [15] for more details.

We will assume from now on that G is a connected unimodular polynomial volume Lie group with local dimension d and dimension D at infinity.

3 Sub-Laplacian and Heat Kernel

Once we have fixed the family $\mathbf{X}$, we define the gradient on G by $\nabla = (X_1, \dots, X_k)$ and we consider a sub-Laplacian $\mathcal{J}$ on G defined by $\mathcal{J} = -\sum_{j=1}^{k} X_j^2$, which is a positive self-adjoint, hypo-elliptic operator since $\mathbf{X}$ satisfies the Hörmander's condition. Its associated heat operator on $G \times]0, +\infty[$ is given by $\partial_t + \mathcal{J}$. We recall in the next theorem some well-known properties of the semi-group H_t obtained from the sub-Laplacian $\mathcal{J}$. See[15] and the references therein for a proof.

Theorem 3.1 *There exists a unique family of continuous linear operators $(H_t)_{t>0}$ defined on $L^1 + L^\infty(G)$ with the semi-group property $H_{t+s} = H_t H_s$ for all $t, s > 0$ and* $H_0 = \mathrm{Id}$, *such that*

(i) *the sub-Laplacian $\mathcal{J}$ is the infinitesimal generator of the semi-group $H_t = e^{-t\mathcal{J}}$;*
(ii) *H_t is a contraction operator on $L^p(G)$ for $1 \le p \le +\infty$ and for $t > 0$;*
(iii) *the semi-group H_t admits a convolution kernel $H_t f = f * h_t$, where h_t is the heat kernel;*
(iv) *$\|H_t f - f\|_{L^p} \to 0$ if $t \to 0$ for $f \in L^p(G)$ and $1 \le p < +\infty$;*
(v) *if $f \in L^p(G)$, $1 \le p \le +\infty$, then the function $u(x, t) = H_t f(x)$ is a solution of the heat equation.*

We obtain, in particular, that H_t is a symmetric diffusion semi-group as considered by Stein [13] with infinitesimal generator $\mathcal{J}$.

We need to fix some terminology. Note that associated with the family $\mathbf{X}$ we also have a family of right-invariant vector fields $\{Y_1, \dots, Y_k\}$ with similar properties. Let $I = (j_1, \dots, j_\beta) \in \{1, \dots, k\}^\beta$ ($\beta \in \mathbb{N}$) be a multi-index. We set $|I| = \beta$ and define X^I and Y^I by the formula $X^I = X_{j_1} \cdots X_{j_\beta}$ ($Y^I = Y_{j_1} \cdots Y_{j_\beta}$, resp.) with the convention $X^I = \mathrm{Id}$ if $\beta = 0$. The interaction of operators X^I and Y^I with convolutions is clarified by the following identities:

$$X^I(f * g) = f * (X^I g), \quad Y^I(f * g) = (Y^I f) * g, \quad (X^I f) * g = f * (Y^I g).$$

In particular, we have $(\nabla f) * g = f * (\tilde{\nabla} g)$, where $\tilde{\nabla} = (Y_1, \dots, Y_k)$.

We will say now that $\varphi \in \mathcal{C}^\infty(G)$ belongs to the Schwartz class $\mathcal{S}(G)$ if

$$N_{\alpha,I}(\varphi) = \sup_{x \in G} (1 + \|x\|)^\alpha |X^I \varphi(x)| < +\infty. \quad (\alpha \in \mathbb{N}, I \in \bigcup_{\beta \in \mathbb{N}} \{1, \dots, k\}^\beta).$$

Remark 1 To characterize the Schwartz class $\mathcal{S}(G)$ we can replace vector fields X^I in the semi-norms $N_{k,I}$ above by right-invariant vector fields Y^I.

For a proof of these facts and for further details, see [7, 13, 15] and the references therein.

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Theorem 3.2 *Let G be a polynomial volume growth Lie group. Then for every $j \in \{1, \dots, k\}$ there exists $C > 0$ such that*

$$|X_j h_t(x)| \le C t^{-1/2} V(\sqrt{t})^{-1} e^{-\frac{\|x\|^2}{ct}} \quad \text{for all } x \in G,\ t > 0.$$

This theorem implies the following proposition.

Proposition 3.3 *For every $j \in \{1, \dots, k\}$ and for all $p \in [1, +\infty]$ there exists a constant $C > 0$ such that*

$$\|X_j h_t(\cdot)\|_{L^p} \le C t^{-1/2} V(\sqrt{t})^{-1/p'}, \quad t > 0. \tag{3.1}$$

For a proof of Theorem 3.2 and Proposition 3.3, see [15, Chapter VIII].

4 Spectral Decomposition for the Sub-Laplacian

The use here of spectral resolution for the sub-Laplacian consists roughly in expressing this operator by the formula $\mathcal{J} = \int_0^{+\infty} \lambda \, dE_\lambda$ and, by means of this characterization, building a family of new operators $m(\mathcal{J})$ associated with a Borel function m. This kind of operator has some nice properties, as shown in the following propositions.

Proposition 4.1 *If G is a polynomial growth Lie group and if m is a bounded Borel function on $]0, +\infty[$, then the operator $m(\mathcal{J})$ defined by*

$$m(\mathcal{J}) = \int_0^{+\infty} m(\lambda) \, dE_\lambda \tag{4.1}$$

is bounded on $L^2(G)$ and admits a convolution kernel M, i.e., *$m(\mathcal{J})(f) = f * M$ for all $f \in L^2(G)$.*

Following [8, 10] we can improve the conclusion of the above proposition. Let $k \in \mathbb{N}$ and m be a function of class $\mathcal{C}^k(\mathbb{R}^+)$. We write

$$\|m\|_{(k)} = \sup_{\substack{\lambda > 0 \\ 1 \le r \le k}} (1 + \lambda)^k |m^{(r)}(\lambda)|.$$

This formula gives us a necessary condition to obtain some properties of the operators defined by (4.1).

Proposition 4.2 *Let G be polynomial volume growth Lie group with local dimension d. Let $j \in \{1, \dots, k\}$ and $p \in [1, +\infty]$. There is a constant $C > 0$ and an integer k such that, for any function $m \in \mathcal{C}^k(\mathbb{R}^+)$ with $\|m\|_{(k)} < +\infty$, the kernel M_t associated with the operator $m(t\mathcal{J})$ for $t > 0$ satisfies*

$$\|X_j M_t(\cdot)\|_{L^p} \le C t^{-(\frac{d}{2p'} + \frac{1}{2})} \|m\|_{(k)}. \tag{4.2}$$

where $1/p + 1/p' = 1$.

Proof Follow the steps of the proof of [8, Proposition 3.2] and use inequality (3.1). ■

Remark 2 Notice that when $0 < t \le 1$, in (4.2), we can replace X_j by X^I for some multi-index I.

5 Functional Spaces

In this section we give the precise definition of the functional spaces involved in Theorem 1.1. In a general way, given a norm $\|\cdot\|_E$, we will define the corresponding functional space $E(G)$ by $\{f \in \mathcal{S}'(G) : \|f\|_E < +\infty\}$. For the Lebesgue spaces $L^p(G)$ with $1 \leq p + \infty$, we will use the characterization

$$\|f\|_{L^p}^p = \int_0^{+\infty} p\sigma^{p-1}|\{x \in G : |f(x)| > \sigma\}|\, d\sigma,$$

and for the Lorentz spaces $L^{p,\infty}(G)$ we set

$$\|f\|_{L^{p,\infty}} = \sup_{\sigma>0}\{\sigma|\{x \in G : |f(x)| > \sigma\}|^{1/p}\}.$$

In order to define Sobolev spaces, we need to introduce the fractional powers $\mathcal{J}^s$ and $\mathcal{J}^{-s}$ with $s > 0$:

$$\mathcal{J}^s f(x) = \lim_{\varepsilon\to 0} \frac{1}{\Gamma(k-s)} \int_\varepsilon^{+\infty} t^{k-s-1}\mathcal{J}^k H_t f(x)\, dt, \tag{5.1}$$

$$\mathcal{J}^{-s} f(x) = \lim_{\eta\to +\infty} \frac{1}{\Gamma(s)} \int_0^{\eta} t^{s-1} H_t f(x)\, dt, \tag{5.2}$$

for all $f \in \mathcal{C}^\infty(G)$ with k the smallest integer greater than s. We consider then the Sobolev spaces with the norms

$$\|f\|_{\dot{W}^{s,p}} = \|\mathcal{J}^{s/2} f\|_{L^p}, \tag{5.3}$$

for $1 < p < +\infty$. When $p = s = 1$, we write

$$\|f\|_{\dot{W}^{1,1}} = \|\nabla f\|_{L^1}. \tag{5.4}$$

We will also need to define the weak Sobolev spaces $\dot{W}^{s,p}_\infty(G)$ used in (1.5) and we write here

$$\|f\|_{\dot{W}^{s,p}_\infty} = \|\mathcal{J}^{s/2} f\|_{L^{p,\infty}} \quad (1 < p < +\infty). \tag{5.5}$$

Finally, for Besov spaces of indices $(-\beta, \infty, \infty)$, which appear in all the inequalities (1.3)–(1.5), we have

$$\|f\|_{\dot{B}^{-\beta,\infty}_\infty} = \sup_{t>0} t^{\beta/2}\|H_t f\|_{L^\infty}. \tag{5.6}$$

The choice of this *thermic* definition for Besov spaces will be clarified in the next section. Observe that other equivalent characterizations do exist in the framework of polynomial volume growth Lie groups (see for example [8, 12]), but they are not as useful in our computations as the thermic one.

6 Improved Sobolev Inequalities on Stratified Groups: The Proofs

We will divide the proof of the Theorem 1.1 into two steps following the values of the parameter p used in the Sobolev spaces that appear on the right-hand side of inequalities (1.3)–(1.5). This separation of the proof in the cases when $p > 1$ and when $p = 1$ is due to the definition of Sobolev spaces given by the formulas (5.3) and (5.4) and is independent from the underlying geometry. Thus, we first study the inequality (1.3), and then we prove the strong inequality (1.4) and the weak inequality (1.5).

6.1 The General Improved Sobolev Inequalities ($p > 1$)

We start the proof by observing that the operator $\mathcal{J}^{s/2}$ carries out an isomorphism between the spaces $\dot{B}^{-\beta,\infty}_{\infty}(G)$ and $\dot{B}^{-\beta-s,\infty}_{\infty}(G)$. This fact follows from the thermic definition of Besov spaces (see [12] for a proof and more details). We can rewrite the inequality (1.3) as

$$\|\mathcal{J}^{\frac{s-s_1}{2}} f\|_{L^q} \leq C\|f\|_{L^p}^{\theta}\|f\|^{1-\theta}_{\dot{B}^{-\beta-s_1,\infty}_{\infty}},$$

where $1 < p < q < +\infty$, $\theta = p/q$, $s = \theta s_1 - (1-\theta)\beta$, and $-\beta < s < s_1$. Using the sub-Laplacian fractional powers characterization (5.2), we have the identity

$$\begin{aligned}(6.1)\qquad \mathcal{J}^{\frac{-\alpha}{2}} f(x) &= \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^{+\infty} t^{\frac{\alpha}{2}-1}H_t f(x)\,dt\\ &= \frac{1}{\Gamma(\frac{\alpha}{2})}\Big(\int_0^{T} t^{\frac{\alpha}{2}-1}H_t f(x)dt + \int_T^{+\infty} t^{\frac{\alpha}{2}-1}H_t f(x)\,dt\Big),\end{aligned}$$

where $\alpha = s_1 - s > 0$ and where $T > 0$ is a parameter that will be fixed in the sequel.

We will use the following estimates to study each of these integrals.

$$|H_t f(x)| \leq |f(x)|,$$

$$|H_t f(x)| \leq C t^{\frac{-\beta-s_1}{2}}\|f\|_{\dot{B}^{-\beta-s_1,\infty}_{\infty}} \quad \text{(by the thermic definition of Besov spaces).}$$

Then applying these inequalities in (6.1), we obtain

$$|\mathcal{J}^{\frac{-\alpha}{2}} f(x)| \leq \frac{c_1}{\Gamma(\frac{\alpha}{2})}T^{\frac{\alpha}{2}}|f(x)| + \frac{c_2}{\Gamma(\frac{\alpha}{2})}T^{\frac{\alpha-\beta-s_1}{2}}\|f\|_{\dot{B}^{-\beta-s_1,\infty}_{\infty}}.$$

We fix now

$$T = \Big(\frac{\|f\|_{\dot{B}^{-\beta-s_1,\infty}_{\infty}}}{|f(x)|}\Big)^{2/(\beta+s_1)}$$

and we get

$$|\mathcal{J}^{\frac{-\alpha}{2}} f(x)| \leq \frac{c_1}{\Gamma(\frac{\alpha}{2})}|f(x)|^{1-\frac{\alpha}{\beta+s_1}}\|f\|^{\alpha/(\beta+s_1)}_{\dot{B}^{-\beta-s_1,\infty}_{\infty}} + \frac{c_2}{\Gamma(\frac{\alpha}{2})}|f(x)|^{1-\frac{\alpha}{\beta+s_1}}\|f\|^{\alpha/(\beta+s_1)}_{\dot{B}^{-\beta-s_1,\infty}_{\infty}}.$$

Since $\alpha/(\beta+s_1) = 1-\theta$ and $\theta = p/q$, we have

$$|\mathcal{J}^{-\alpha/2} f(x)| \leq \frac{c}{\Gamma(\frac{\alpha}{2})}|f(x)|^{\theta}\|f\|^{1-\theta}_{\dot{B}^{-\beta-s_1,\infty}_{\infty}}.$$

We finally obtain

$$\|\mathcal{J}^{\frac{-\alpha}{2}} f\|_{L^q} \leq c\|f\|_{L^p}^{\theta}\|f\|_{\dot{B}_{\infty}^{-\beta-s_1,\infty}}^{1-\theta}$$

and we are done. ∎

6.2 Strong and Weak Inequalities ($p = 1$)

We treat now the inequalities (1.4) and (1.5). For this we will need the following result.

Theorem 6.1 (Modified Poincaré pseudo-inequality) *Let f be a function such that $\nabla f \in L^1(G)$. We have the following estimate for $0 \leq s < 1$ and for $t > 0$:*

$$\|\mathcal{J}^{s/2} f - H_t \mathcal{J}^{s/2} f\|_{L^1} \leq C\, t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}. \tag{6.2}$$

Let us make some remarks. This theorem is crucial for proving strong and weak inequalities when $p = 1$, mainly because this estimate is especially well suited for matching with the Besov space thermic definition. Note also that when $s = 0$, we have some alternative proofs of (6.2) depending on the framework and its underlying geometry. See [11] for details. In the general case exposed in Theorem 6.1, the role of the geometry is given in the L^p-estimates available for the Heat kernel.

Proof To begin the proof, we observe that the following identity occurs:

$$(\mathcal{J}^{s/2} f - H_t \mathcal{J}^{s/2} f)(x) = \Big(\int_0^{+\infty} m(t\lambda) dE_\lambda\Big)\, t^{1-s/2} \mathcal{J} f(x),$$

where we noted that $m(\lambda) = \lambda^{s/2-1}(1 - e^{-\lambda})$ for $\lambda > 0$. Note that m is a bounded function which tends to 0 at infinity since $s/2 - 1 < 0$. We break up this function by writing

$$m(\lambda) = m_0(\lambda) + m_1(\lambda) = m(\lambda)\theta_0(\lambda) + m(\lambda)\theta_1(\lambda),$$

where we chose the auxiliary functions $\theta_0(\lambda), \theta_1(\lambda) \in \mathcal{C}^\infty(\mathbb{R}^+)$ defined by

$$\theta_0(\lambda) = 1 \text{ on }]0, 1/2[\quad \text{and} \quad 0 \text{ on }]1, +\infty[,$$

$$\theta_1(\lambda) = 0 \text{ on }]0, 1/2[\quad \text{and} \quad 1 \text{ on }]1, +\infty[,$$

so that $\theta_0(\lambda) + \theta_1(\lambda) \equiv 1$. Then we obtain the formula

$$\begin{aligned}(\mathcal{J}^{s/2} f &- H_t \mathcal{J}^{s/2} f)(x) \\ &= \Big(\int_0^{+\infty} m_0(t\lambda) dE_\lambda\Big)\, t^{1-s/2} \mathcal{J} f(x) + \Big(\int_0^{+\infty} m_1(t\lambda) dE_\lambda\Big)\, t^{1-s/2} \mathcal{J} f(x).\end{aligned}$$

If we denote by $M_t^{(i)}$ the kernel of the operator fixed by $\int_0^{+\infty} m_i(t\lambda) dE_\lambda$ for $i = 0, 1$, we have

$$(\mathcal{J}^{s/2} f - H_t \mathcal{J}^{s/2} f)(x) = t^{1-s/2} \mathcal{J} f * M_t^{(0)}(x) + t^{1-s/2} \mathcal{J} f * M_t^{(1)}(x).$$

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We obtain the inequality

$$(6.3)\qquad \int_G |\mathcal{J}^{s/2} f - H_t \mathcal{J}^{s/2}|\,dx \\ \leq \int_G |t^{1-s/2}\mathcal{J} f * M_t^{(0)}(x)|\,dx + \int_G |t^{1-s/2}\mathcal{J} f * M_t^{(1)}(x)|\,dx.$$

We will now estimate the right side of the above inequality by the following two propositions.

Proposition 6.2 *For the first integral in the right-hand side of* (6.3) *we have the inequality*

$$\int_G |t^{1-s/2}\mathcal{J} f * M_t^{(0)}(x)|\,dx \leq C t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}.$$

Proof The function m_0 is the restriction on $\mathbb{R}^+$ of a function belonging to the Schwartz class. This function satisfies the assumptions of Proposition 4.2, which we apply after having noticed the identity

$$I = \int_G |t^{1-s/2}\mathcal{J} f * M_t^{(0)}(x)|\,dx = \int_G |t^{1-s/2}\nabla f * \tilde{\nabla} M_t^{(0)}(x)|\,dx,$$

where we noted $\tilde{\nabla}$, the gradient formed by the vectors fields $(Y_j)_{1\leq j\leq k}$. We have then

$$I \leq \int_G\int_G t^{1-s/2} |\nabla f(y)|\,|\tilde{\nabla} M_t^{(0)}(y^{-1}\cdot x)|\,dxdy \leq t^{1-s/2}\|\nabla f\|_{L^1}\|\tilde{\nabla} M_t^{(0)}\|_{L^1}.$$

Using inequality (4.2), we obtain

$$\int_G |t^{1-s/2}\mathcal{J} f * M_t^{(0)}(x)|\,dx \leq C t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}. \qquad \blacksquare$$

Proposition 6.3 *For the last integral of* (6.3) *we have the inequality*

$$\int_G |t^{1-s/2}\mathcal{J} f * M_t^{(1)}(x)|\,dx \leq C t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}.$$

Proof Here it is necessary to make an additional step. We cut out the function m_1 in the following way:

$$m_1(\lambda) = \Big(\frac{1-e^{-\lambda}}{\lambda}\Big)\theta_1(\lambda) = m_a(\lambda) - m_b(\lambda),$$

where $m_a(\lambda) = \frac{1}{\lambda}\theta_1(\lambda)$ and $m_b(\lambda) = \frac{e^{-\lambda}}{\lambda}\theta_1(\lambda)$. We will denote by $M_t^{(a)}$ and $M_t^{(b)}$ the associated kernels of these two operators. We thus obtain the estimate

$$(6.4)\qquad \int_G |t^{1-s/2}\mathcal{J} f * M_t^{(1)}(x)|\,dx \\ \leq \int_G \Big|t^{1-s/2}\mathcal{J} f * M_t^{(a)}(x)\Big|\,dx + \int_G \Big|t^{1-s/2}\mathcal{J} f * M_t^{(b)}(x)\Big|\,dx. \qquad \blacksquare$$

We have the next lemma for the last integral in (6.4).

Lemma 6.4

$$\int_G |t^{1-s/2}\mathcal{J} f * M_t^{(b)}(x)|\, dx \leq Ct^{\frac{1-s}{2}} \|\nabla f\|_{L^1}.$$

Proof Observe that $m_b \in \mathcal{S}(\mathbb{R}^+)$. Then the proof is straightforward and follows the same steps as those of Proposition 6.2. ∎

We treat the other part of (6.4) with the following lemma.

Lemma 6.5

$$\int_G |t^{1-s/2}\mathcal{J} f * M_t^{(a)}(x)|\, dx \leq Ct^{\frac{1-s}{2}} \|\nabla f\|_{L^1} \tag{6.5}$$

Proof We consider the auxiliary function $\psi(\lambda) = \theta_0(\lambda/2) - \theta_0(\lambda) = \theta_1(\lambda) - \theta_1(\lambda/2)$ in order to obtain the identity

$$\sum_{j=0}^{+\infty} \psi(2^{-j}\lambda) = \theta_1(\lambda).$$

We then have

$$m_a(t\lambda) = \frac{1}{t\lambda} \sum_{j=0}^{+\infty} \psi(2^{-j}t\lambda) = \sum_{j=0}^{+\infty} 2^{-j}\phi(2^{-j}t\lambda),$$

where $\phi(\lambda) = \frac{\psi(\lambda)}{\lambda}$ is a function in $\mathcal{C}_0^\infty(\mathbb{R}^+)$. Then from the point of view of operators, one has

$$M_t^{(a)} = \sum_{j=0}^{+\infty} 2^{-j} K_{j,t}, \tag{6.6}$$

where $K_{j,t} = \phi(2^{-j}t\mathcal{J})$. With formula (6.6) we return to the left side of (6.5):

$$\int_G |t^{1-s/2}\mathcal{J} f * M_t^{(a)}(x), dx \leq \sum_{j=0}^{+\infty} 2^{-j} \int_G |t^{1-s/2}\mathcal{J} f * K_{j,t}(x)|\, dx. \tag{6.7}$$

Using the sub-Laplacian definition and the vector fields properties, we have

$$\sum_{j=0}^{+\infty} 2^{-j} \int_G |t^{1-s/2}\mathcal{J} f * K_{j,t}(x)|\, dx \leq \sum_{j=0}^{+\infty} 2^{-j} t^{1-s/2} \|\nabla f\|_{L^1} \|\tilde{\nabla} K_{j,t}\|_{L^1}.$$

Now we apply Proposition 4.2 to obtain the estimate $\|\tilde{\nabla} K_{j,t}\|_{L^1} \leq C 2^{j/2} t^{-1/2}$. Then for (6.7) we have the following inequality:

$$\int_G |t^{1-s/2}\mathcal{J} f * M_t^{(a)}(x)|\, dx \leq C \sum_{j=0}^{+\infty} 2^{-j/2} t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}.$$

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Finally we get

$$\int_G \left| t^{1-s/2} \mathcal{J} f * M_t^{(a)}(x) \right| dx \leq C\, t^{\frac{1-s}{2}} \|\nabla f\|_{L^1},$$

which ends the proof of Lemma 6.5. ∎

With these last two lemmas we conclude the proof of Proposition 6.3. Now, getting back to the formula (6.3), with Propositions 6.2 and 6.3 we finally finish the proof of Theorem 6.1.

6.3 Weak Inequalities

To begin the proof, we again use the fact that operator $\mathcal{J}^{s/2}$ carries out an isomorphism between the spaces $\dot{B}_{\infty}^{-\beta,\infty}$ and $\dot{B}_{\infty}^{-\beta-s,\infty}$. Thus we rewrite inequality (1.5) as

$$\|\mathcal{J}^{s/2} f\|_{L^{q,\infty}} \leq C \|\nabla f\|_{L^1}^{\theta} \|\mathcal{J}^{s/2} f\|_{\dot{B}_{\infty}^{-\beta-s,\infty}}^{1-\theta}.$$

By homogeneity, we can suppose that the norm $\|\mathcal{J}^{s/2} f\|_{\dot{B}_{\infty}^{-\beta-s,\infty}}$ is bounded by 1; then we must show

$$\|\mathcal{J}^{s/2} f\|_{L^{q,\infty}} \leq C \|\nabla f\|_{L^1}^{\theta}.$$

We thus must evaluate the expression $|\{x \in G : |\mathcal{J}^{s/2} f(x)| > 2\alpha\}|$ for all $\alpha > 0$. If we use the thermic definition of the Besov space (5.6), we have

$$\|\mathcal{J}^{s/2} f\|_{\dot{B}_{\infty}^{-\beta-s,\infty}} \leq 1 \iff \sup_{t>0} \left\{ t^{(\beta+s)/2} \|H_t \mathcal{J}^{s/2} f\|_{L^\infty} \right\} \leq 1.$$

But if one fixes $t_\alpha = \alpha^{-(\frac{2}{\beta+s})}$, we obtain $\|H_{t_\alpha} \mathcal{J}^{s/2} f\|_{L^\infty} \leq \alpha$. Note also that with the definition of parameter β one has $t_\alpha = \alpha^{-\frac{2(q-1)}{(1-s)}}$. Therefore, since we have the following set inclusion

$$\{x \in G : |\mathcal{J}^{s/2} f(x)| > 2\alpha\} \subset \{x \in G : |\mathcal{J}^{s/2} f(x) - H_{t_\alpha} \mathcal{J}^{s/2} f(x)| > \alpha\},$$

the Chebyshev inequality implies

$$\alpha^q |\{x \in G : |\mathcal{J}^{s/2} f(x)| > 2\alpha\}| \leq \alpha^{q-1} \int_G |\mathcal{J}^{s/2} f(x) - H_{t_\alpha} \mathcal{J}^{s/2} f(x)| dx.$$

At this point, we use Theorem 6.1 to estimate the right side of the preceding inequality:

$$\alpha^q |\{x \in G : |\mathcal{J}^{s/2} f(x)| > 2\alpha\}|\ leq C \alpha^{q-1}\, t_\alpha^{\frac{1-s}{2}} \int_G |\nabla f(x)| dx. \tag{6.8}$$

But by the choice of t_α, one has $\alpha^{q-1} \alpha^{-\frac{2(q-1)}{(1-s)} \frac{(1-s)}{2}} = 1$. Then (6.8) implies the inequality

$$\alpha^q \left| \{x \in G : |\mathcal{J}^{s/2} f(x)| > 2\alpha\} \right| \leq C \|\nabla f\|_{L^1}$$

and, finally, using the definition (5.5) of weak Sobolev spaces, we obtain

$$\|\mathcal{J}^{s/2} f\|_{L^{q,\infty}}^q \leq C \|\nabla f\|_{L^1},$$

which is the desired result. ∎

6.4 Strong Inequalities

When $s = 0$ in the weak inequalities above, it is possible to obtain stronger estimates. To achieve this, we will need an intermediate step.

Proposition 6.6 *Let $1 < q < +\infty$, $\theta = \frac{1}{q}$ and $\beta = \theta/(1-\theta)$. Then we have*

$$\|f\|_{L^q} \leq C\|\nabla f\|_{L^1}^{\theta}\|f\|_{\dot{B}_\infty^{-\beta,\infty}}^{1-\theta},$$

when the three norms in this inequality are bounded.

Proof We will follow closely [11]. Just as in the preceding theorem, we will start by supposing that $\|f\|_{\dot{B}_\infty^{-\beta,\infty}} \leq 1$. Thus, we must show the estimate

$$\|f\|_{L^q} \leq C\|\nabla f\|_{L^1}^{\theta}.$$

Let us fix t in the following way: $t_\alpha = \alpha^{-2(q-1)/q}$, where $\alpha > 0$. Then by the thermic definition of Besov spaces, we have the estimate $\|H_t f\|_{L^\infty} \leq \alpha$. Now we use the characterization of Lebesgue space given by the distribution function

$$\frac{1}{5^q}\|f\|_{L^q}^q = \int_0^{+\infty} |\{x \in G : |f(x)| > 5\alpha\}|\, d(\alpha^q). \tag{6.9}$$

It now remains to estimate $|\{x \in G : |f(x)| > 5\alpha\}|$ and for this we introduce the following thresholding function:

$$\Theta_\alpha(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \alpha, \\ t-\alpha & \text{if } \alpha \leq t \leq M\alpha, \\ (M-1)\alpha & \text{if } t > M\alpha. \end{cases}$$

and $\Theta_\alpha(-t) = -\Theta_\alpha(t)$. Here M is a parameter depending on q and which we will suppose for the moment to be larger than 10.

This cut-off function enables us to define a new function posing $f_\alpha = \Theta_\alpha(f)$. In the next lemma we collect some significant properties of the function f_α.

Lemma 6.7 (i) *The set defined by $\{x \in G : |f(x)| > 5\alpha\}$ is included in the set $\{x \in G : |f_\alpha(x)| > 4\alpha\}$.*

(ii) *On the set $\{x \in G : |f(x)| \leq M\alpha\}$ one has the estimate $|f - f_\alpha| \leq \alpha$.*

(iii) *If $f \in \mathcal{C}^1(G)$, one has the equality $\nabla f_\alpha = (\nabla f)\mathbb{1}_{\{\alpha \leq |f| \leq M\alpha\}}$ almost everywhere.*

We leave the verification of this lemma to the reader.

Let us return now to (6.9). By the first point of Lemma 6.7 we have

$$\begin{aligned} &\int_0^{+\infty} |\{x \in G : |f(x)| > 5\alpha\}|\, d(\alpha^q) \\ &\qquad\qquad \leq \int_0^{+\infty} |\{x \in G : |f_\alpha(x)| > 4\alpha\}|\, d(\alpha^q) = I. \end{aligned} \tag{6.10}$$

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We denote

$$A_\alpha = \{x \in G : |f_\alpha(x)| > 4\alpha\},$$
$$B_\alpha = \{x \in G : |f_\alpha(x) - H_{t_\alpha}(f_\alpha)(x)| > \alpha\},$$
$$C_\alpha = \{x \in G : |H_{t_\alpha}(f_\alpha - f)(x)| > 2\alpha\}.$$

By linearity of H_t we can write $f_\alpha = f_\alpha - h_{t_\alpha}(f_\alpha) + h_{t_\alpha}(f_\alpha - f) + h_{t_\alpha}(f)$. Then, recalling the fact that $\|H_t f\|_{L^\infty} \le \alpha$, we obtain $A_\alpha \subset B_\alpha \cup C_\alpha$. Returning to (6.10), this set inclusion gives us the following inequality:

$$I \le \int_0^{+\infty} |B_\alpha|\, d(\alpha^q) + \int_0^{+\infty} |C_\alpha| d(\alpha^q). \tag{6.11}$$

We will study and estimate these two integrals, which we will call I_1 and I_2, respectively, by the two following lemmas.

Lemma 6.8 *For the first integral of* (6.11) *we have the estimate*

$$I_1 = \int_0^{+\infty} |B_\alpha|\, d(\alpha^q) \le C\, q \log(M) \|\nabla f\|_{L^1}.$$

Proof The Chebyshev inequality implies

$$|B_\alpha| \le \alpha^{-1} \int_G |f_\alpha(x) - H_{t_\alpha}(f_\alpha)(x)|\, dx.$$

Using Theorem 6.1 with $s = 0$ in the above integral, we obtain

$$|B_\alpha| \le C\alpha^{-1} t_\alpha^{1/2} \int_G |\nabla f_\alpha(x)|\, dx.$$

We remark that the choice of t_α fixed before gives $t_\alpha^{1/2} = \alpha^{1-q}$. Then we have

$$|B_\alpha| \le C\alpha^{-q} \int_{\{\alpha \le |f| \le M\alpha\}} |\nabla f(x)|\, dx.$$

Now we integrate the preceding expression with respect to $d(\alpha^q)$:

$$\begin{aligned} I_1 &\le C \int_0^{+\infty} \alpha^{-q} \Big(\int_{\{\alpha \le |f| \le M\alpha\}} |\nabla f(x)|\, dx \Big)\, d(\alpha^q) \\ &= C\, q \int_G |\nabla f(x)| \Big(\int_{\frac{|f|}{M}}^{|f|} \frac{d\alpha}{\alpha} \Big)\, dx. \end{aligned}$$

It follows then that $I_1 \le C\, q\, \log(M) \|\nabla f\|_{L^1}$, and one obtains the estimation needed for the first integral. ∎

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Lemma 6.9 *For the second integral of* (6.11) *one has the following result:*

$$I_2 = \int_0^{+\infty} |C_\alpha| \, d(\alpha^q) \le \frac{q}{q-1} \, \frac{1}{M^{q-1}} \|f\|_{L^q}^q.$$

Proof For the proof of this lemma, we write

$$|f - f_\alpha| = |f - f_\alpha| \mathbb{1}_{\{|f| \le M\alpha\}} + |f - f_\alpha| \mathbb{1}_{\{|f| > M\alpha\}}.$$

As the distance between f and f_α is lower than α on the set $\{x \in G : |f(x)| \le M\alpha\}$, one has the inequality $|f - f_\alpha| \le \alpha + |f| \mathbb{1}_{\{|f|>M\alpha\}}$. By applying the heat semi-group to both sides of this inequality, we obtain $H_{t_\alpha}(|f - f_\alpha|) \le \alpha + H_{t_\alpha}(|f|\mathbb{1}_{\{|f|>M\alpha\}})$ and we then have the following set inclusion: $C_\alpha \subset \{x \in G : H_{t_\alpha}(|f|\mathbb{1}_{\{|f|>M\alpha\}}) > \alpha\}$. Thus, considering the measure of these sets and integrating with respect to $d(\alpha^q)$, we obtain

$$I_2 = \int_0^{+\infty} |C_\alpha| \, d(\alpha^q) \le \int_0^{+\infty} |\{H_{t_\alpha}(|f|\mathbb{1}_{\{|f|>M\alpha\}}) > \alpha\}| \, d(\alpha^q).$$

By applying the Chebyshev inequality, we now obtain the estimate

$$I_2 \le \int_0^{+\infty} \alpha^{-1} \Big(\int_G H_{t_\alpha}(|f|\mathbb{1}_{\{|f|>M\alpha\}}) \, dx \Big) \, d(\alpha^q).$$

Then by Fubini's theorem we have

$$\begin{aligned} I_2 &\le q \int_G |f(x)| \Big(\int_0^{+\infty} \mathbb{1}_{\{|f|>M\alpha\}} \alpha^{q-2} d\alpha \Big) \, dx \\ &= \frac{q}{q-1} \int_G |f(x)| \frac{|f(x)|^{q-1}}{M^{q-1}} \, dx = \frac{q}{q-1} \frac{1}{M^{q-1}} \|f\|_{L^q}^q. \end{aligned}$$

And this concludes the proof of this lemma. ■

We finish the proof of Proposition 6.6 by connecting together these two lemmas:

$$\frac{1}{5^q} \|f\|_{L^q}^q \le Cq \, \log(M) \|\nabla f\|_{L^1} + \frac{q}{q-1} \frac{1}{M^{q-1}} \|f\|_{L^q}^q.$$

Since we supposed all the norms bounded and $M \gg 1$, we finally have

$$\Big(\frac{1}{5^q} - \frac{q}{q-1} \frac{1}{M^{q-1}} \Big) \|f\|_{L^q}^q \le Cq \, \log(M) \|\nabla f\|_{L^1}. \qquad ■$$

The proof of Theorem 1.1 is not yet completely finished. The last step is provided by the following proposition.

Proposition 6.10 *In Proposition* 6.6 *it is possible to consider only the two assumptions* $\nabla f \in L^1(G)$ *and* $f \in \dot{B}_\infty^{-\beta,\infty}(G)$.

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Proof For the proof of this proposition we will build an approximation of f, writing:

$$f_j = \Big(\int_0^{+\infty} (\varphi(2^{-2j}\lambda) - \varphi(2^{2j}\lambda)) dE_\lambda \Big) (f),$$

where φ is a $\mathcal{C}^\infty(\mathbb{R}^+)$ function such that $\varphi = 1$ on $]0, 1/4[$ and $\varphi = 0$ on $[1, +\infty[$. ■

Lemma 6.11 *If $q > 1$, if $\nabla f \in L^1(G)$, and if $f \in \dot{B}_\infty^{-\beta,\infty}(G)$, then $\nabla f_j \in L^1(G)$, $f_j \in \dot{B}_\infty^{-\beta,\infty}(G)$, and $f_j \in L^q(G)$.*

Proof The fact that $\nabla f_j \in L^1(G)$ and $f_j \in \dot{B}_\infty^{-\beta,\infty}(G)$ is an easy consequence of the definition of f_j. For $f_j \in L^q(G)$ the starting point is given by the relation

$$f_j = \Big(\int_0^{+\infty} m(2^{-2j}\lambda)\, dE_\lambda \Big) 2^{-2j} \mathcal{J}(f),$$

where we noted

$$m(2^{-2j}\lambda) = \frac{\varphi(2^{-2j}\lambda) - \varphi(2^{2j}\lambda)}{2^{-2j}\lambda}.$$

Observe that the function m vanishes near the origin and satisfies the assumptions of Proposition 4.2. We obtain then the following identity where M_j is the kernel of the operator $m(2^{-2j}\mathcal{J})$:

$$f_j = 2^{-2j}\mathcal{J} f * M_j = 2^{-2j}\nabla f * \tilde{\nabla} M_j.$$

Using inequality (4.2), we estimate the norm $L^q(G)$ in the preceding identity:

$$\|f_j\|_{L^q} = \|2^{-2j}\nabla f * \tilde{\nabla} M_j\|_{L^q} \leq 2^{-2j} \|\nabla f\|_{L^1} \|\tilde{\nabla} M_j\|_{L^q}.$$

Finally, we obtain

$$\|f_j\|_{L^q} \leq C\, 2^{j(d(1-\frac{1}{q})-1)} \|\nabla f\|_{L^1} < +\infty.$$

■

Thanks to this estimate, we can apply Proposition 6.6 to f_j, whose $L^q(G)$ norm is bounded, and we obtain:

$$\|f_j\|_{L^q} \leq C \|\nabla f_j\|_{L^1}^{\theta} \|f_j\|_{\dot{B}_\infty^{-\beta,\infty}}^{1-\theta}.$$

Now, since $f \in \dot{B}_\infty^{-\beta,\infty}(G)$, we have $f_j \rightharpoonup f$ in the sense of distributions. It follows that

$$\|f\|_{L^q} \leq \liminf_{j\to+\infty} \|f_j\|_{L^q} \leq C \|\nabla f\|_{L^1}^{\theta} \|f\|_{\dot{B}_\infty^{-\beta,\infty}}^{1-\theta}.$$

We restricted ourselves to the two initial assumptions, namely $\nabla f \in L^1(G)$ and $f \in \dot{B}_\infty^{-\beta,\infty}(G)$. The strong inequalities (1.4) are now completely proved for stratified groups. ■

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Laboratoire d'Analyse et de Probabilités, Université d'Evry Val d'Essonne et ENSIIE, 91025 Evry Cedex
e-mail: diego.chamorro@m4x.org