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REAL RANK OF C^*-ALGEBRAS ASSOCIATED WITH GRAPHS

JA A JEONG

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Abstract

For a locally finite directed graph E, it is known that the graph C^*-algebra $C^*(E)$ has real rank zero if and only if the graph E satisfies the loop condition (K). In this paper we extend this to an arbitrary directed graph case using the desingularization of a graph due to Drinen and Tomforde.

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1. Introduction

The Cuntz-Krieger algebra $\mathcal{O}_A$ associated with a finite $\{0, 1\}$-matrix A has been introduced in [5], and it is proved that $\mathcal{K} \otimes \mathcal{O}_A$ is an invariant of the isomorphism type of an irreducible topological Markov chain together with the automorphisms $(\mathrm{id} \otimes \lambda_z)_{z\in\mathbb{T}}$, where $(\lambda_z)_{z\in\mathbb{T}}$ is the gauge action on $\mathcal{O}_A$. If each entry of an $n \times n$ matrix A is one then the algebra is the Cuntz algebra $\mathcal{O}_n$ $(n > 2)$, and it is well known that for each $n > 2$, $\mathcal{O}_n$ has real rank zero. But for Cuntz-Krieger algebras it is not hard to find a matrix A for which the algebra $\mathcal{O}_A$ has nonzero real rank. Nevertheless one may still expect that many of the Cuntz-Krieger algebras have real rank zero since they already contain enough projections and partial isometries as generators. The Cuntz-Krieger algebra $\mathcal{O}_A$ is now well understood as a graph C^*-algebra $C^*(E)$ where the matrix A is the vertex matrix of a finite directed graph E, or if B is the edge matrix of E then $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_B$. The graph C^*-algebra $C^*(E)$ is generated by a family of partial isometries and projections satisfying the relations determined by the graph E and thus it would be useful if one can find a necessary and sufficient condition for $RR(\mathcal{O}_A) = 0$ (or $RR(C^*(E)) = 0$) in terms of the matrix A (or the graph E) itself.

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Recently, for graph C^*-algebras their ideal structures, simplicity criteria, and their K-theory have been studied by many authors (see [1, 2, 6, 8–11, 13] among others), and we know from [9] and [8] that if E is a locally finite directed graph then its graph C^*-algebra $C^*(E)$ has real rank zero exactly when the graph E satisfies loop condition (K), which implies that for a Cuntz-Krieger algebra $\mathcal{O}_B$, where B is the edge matrix of a finite graph E, $RR(\mathcal{O}_B) = 0$ if and only if the matrix B satisfies condition (II) considered in [4]. The purpose of this paper is to generalize this result to an arbitrary graph E (see Theorem 3.5). To prove the theorem we need to understand the ideal structure of graph C^*-algebras but the description of the ideal structure of a directed graph which is not row-finite is quite complicated [1] while it seems natural and convenient to work with row-finite graphs. Since the property of having real rank zero is preserved under strong Morita equivalence (equivalently, under stable isomorphism for separable C^*-algebras) it suffices to prove our theorem only for row-finite graphs with no sinks by virtue of the result of Drinen and Tomforde ([6, Theorem 2.11]): For any directed graph E, the graph C^*-algebra $C^*(E)$ is a full corner of $C^*(F)$, where F, the *desingularization* of E, is a row-finite graph with no sinks.

In view of the fact that the C^*-algebras $C^*(E)$ associated with arbitrary graphs are much harder to understand than those associated with locally finite ones it would be useful to present the proof of Theorem 3.5 here applying several recently known generalized facts for arbitrary graphs (or row-finite graphs) to handle the general case even though the main idea of the proof is basically same as in [8, 9].

2. Directed graphs and their C^*-algebras

A *directed graph* $E = (E^0, E^1, r, s)$ consists of the set of countable vertices E^0, the set of countable edges E^1, and range, source maps $r, s : E^1 \to E^0$. A vertex v is called a *sink* if it emits no edges, $|s^{-1}(v)| = 0$. Following terminology in [6], we call a vertex v an *infinite-emitter* if it emits infinitely many edges, and *singular* if it is either a sink or an infinite-emitter. A graph E with no infinite-emitters is said to be *row-finite*, and if in addition $r^{-1}(v)$ is finite for each v then we call E *locally finite*. If $e_1, \dots, e_n$ $(n \geq 2)$ are edges with $r(e_i) = s(e_{i+1})$, $1 \leq i \leq n-1$, one can form a (finite) path $\alpha = (e_1, \dots, e_n)$ of *length* $|\alpha| = n$, and extend the maps r, s by $r(\alpha) = r(e_n)$, $s(\alpha) = s(e_1)$. We denote the set of all finite paths by E^* and infinite paths by E^∞. Note that vertices are regarded as finite paths of length zero. A *loop* at a vertex v is a finite path α with $|\alpha| > 0$ such that $s(\alpha) = r(\alpha) = v$. A graph E is said to satisfy *condition* (L) if every loop in E has an exit, and *condition* (K) if for each vertex v on a loop there exist at least two distinct loops based at v. Note that condition (K) is stronger than (L).

For a directed graph E, a *Cuntz-Krieger* E-family consists of partial isometries

$\{S_e \mid e \in E^1\}$ and mutually orthogonal projections $\{P_v \mid v \in E^0\}$ satisfying the relations:

$$S_e^* S_e = P_{r(e)}, \quad S_e S_e^* \leq P_{s(e)}, \quad \text{and}$$
$$P_v = \sum_{s(e)=v} S_e S_e^* \quad \text{if } v \text{ is not a singular vertex.}$$

For a row-finite directed graph E, the existence of a universal C^*-algebra generated by a Cuntz-Krieger E-family $\{s_e, p_v \mid e \in E^1, v \in E^0\}$ is proved in [10, Theorem 1.2]; there is a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s_e, p_v \mid e \in E^1, v \in E^0\}$ of nonzero elements such that for every Cuntz-Krieger E-family $\{S_e, P_v \mid e \in E^1, v \in E^0\}$ in a C^*-algebra A, there is a $*$-homomorphism $\pi : C^*(E) \to A$ such that $\pi(s_e) = S_e, \pi(p_v) = P_v$ for all $e \in E^1$, $v \in E^0$. For arbitrary directed graphs E and their associated universal C^*-algebras $C^*(E)$ (see [1, 7]).

Let $\{s_e, p_v \mid e \in E^1, v \in E^0\}$ be a Cuntz-Krieger E-family generating the C^*-algebra $C^*(E)$. Then for each $z \in \mathbb{T}$ we have another Cuntz-Krieger E-family $\{zs_e, p_v \mid e \in E^1, v \in E^0\}$ in $C^*(E)$, and by the universal property of $C^*(E)$ there exists an isomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$. In fact, $\gamma : z \mapsto \gamma_z \in \mathrm{Aut}(C^*(E))$ is a strongly continuous action of $\mathbb{T}$ on $C^*(E)$ and it is called the *gauge action*. It is known (see [1, 2]) that for a C^*-algebra $C^*(S_e, P_v)$ generated by a Cuntz-Krieger E-family of nonzero elements, the existence of the gauge action on $C^*(S_e, P_v)$ implies that $C^*(S_e, P_v) \cong C^*(E)$. Also the ideal structure of $C^*(E)$ is analyzed in [1, 2], which is essential to prove our theorem.

3. Real rank of graph C^*-algebras

Recall that a C^*-algebra A is said to have *real rank zero*, $RR(A) = 0$, if the set of all invertible self-adjoint elements in $\tilde{A}$ is dense in the set of all self adjoint elements of $\tilde{A}$, where $\tilde{A}$ is the smallest unitization of A. It then turns out [3] that $RR(A) = 0$ if and only if every nonzero hereditary C^*-subalgebra of A contains an approximate identity of projections, hence every hereditary C^*-subalgebra of A with $RR(A) = 0$ always has real rank zero.

Let E be a directed graph. Then for two vertices v, w we simply write $w \leq v$ if there is a path $\alpha \in E^*$ from v to w. A subset H of E^0 is said to be *hereditary* if $w \leq v \in H$ implies $w \in H$, and a subset H of E^0 is *saturated* if every vertex v such that $0 < |s^{-1}(v)| < \infty$ and $r(e) \in H$ for each $e \in s^{-1}(v)$ belongs to H. The *saturation* of a hereditary set H is the smallest saturated subset of E^0 containing H. Let H be a saturated hereditary subset of E^0. Then

$$I_H = \overline{\mathrm{span}}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H\}$$

is an ideal generated by the projections $\{p_v \mid v \in H\}$.

THEOREM 3.1 ([2, Theorem 4.1, 4.4]). *Let E be a row-finite directed graph.*

(a) *Let H be a saturated hereditary subset of E^0 and let F be the quotient graph $F = (F^0 := E^0 \setminus H,\ F^1 := \{e \in E^1 \mid r(e) \notin H\})$, then $C^*(E)/I_H \cong C^*(F)$.*

(b) *Let H be a hereditary subset of E^0, and let*

$$G = \big(G^0 := H,\ G^1 := \{e \in E^1 \mid s(e) \in H\}, r, s\big),$$

then $C^*(G)$ is isomorphic to the subalgebra $C^*\{s_e, p_v \mid e \in G^1, v \in H\}$ of $C^*(E)$, and this subalgebra is a full corner of the ideal I_H.

The above theorem for a locally finite graph was first proved in [11], and is generalized to arbitrary directed graphs in [1].

Let us recall [6] the definition of the desingularization F of a graph E: Let v_0 be a singular vertex of E. If v_0 is a sink then we add an (infinite) tail T to v_0,

$$T := \big(T^0 = \{v_i \mid i = 0, 1, \dots\},\ T^1 = \{e_i \mid s(e_i) = v_{i-1}, r(e_i) = v_i, i = 1, 2, \dots\}\big).$$

If v_0 is an infinite emitter, first list the edges $g_1, g_2, g_3, \dots$ that v_0 emits, then add a tail T to v_0, remove the edges $\{g_i\}$, and for each g_i draw an edge f_i from v_{i-1} to $r(g_i)$. This procedure is referred to as *adding a tail* to v_0. Then the *desingularization* F of a graph E is the graph obtained by adding a tail to each singular vertex of E. Then it is clear that the desingularization F is a row-finite graph with no sinks, and E and F share the same loop condition, that is, E satisfies condition (L) (respectively, (K)) if and only if F satisfies condition (L) (respectively, (K)). Moreover it is proved [6, Theorem 2.11] that the graph C^*-algebra $C^*(E)$ is isomorphic to a full corner $pC^*(F)p$ of $C^*(F)$, where $p := \sum_{v \in E^0} q_v$ is the projection in the multiplier algebra of $C^*(F)$ and $\{t_e, q_v \mid e \in F^1, v \in F^0\}$ is a Cuntz-Krieger F-family generating $C^*(F)$. Hence $RR(C^*(E)) = 0$ if and only if $RR(C^*(F)) = 0$.

Recall [3] that if I is an ideal of a C^*-algebra A then $RR(A) = 0$ if and only if $RR(I) = RR(A/I) = 0$ and every projection in A/I lifts to a projection in A. Then we have the following proposition.

PROPOSITION 3.2 (see [9, Theorem 4.3]). *If $RR(C^*(E)) = 0$ then E satisfies condition* (K).

PROOF. By considering the desingularization of E we may assume that E is a row-finite graph with no sinks. If E has a simple loop $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ with no exits, the vertex subset $H = \{s(\alpha_j) \mid j = 1, \dots, n\}$ is hereditary and by Theorem 3.1 (b) the C^*-algebra $C^*(G)$ is a full corner of the ideal I_H, where

$$G := \big(G^0 = H, G^1 = \{e \in E^1 \mid s(e) \in G^0\}\big).$$

But G consists of the single loop α and hence $C^*(G)$ is isomorphic to $C(\mathbb{T}) \otimes M_n$ and so $RR(C^*(G)) \neq 0$. Thus $RR(I_H) \neq 0$, which contradicts to the assumption that $RR(C^*(E)) = 0$. Therefore E must satisfy condition (L). Suppose E does not satisfy condition (K), and let v be a vertex lying only one loop α. Let H be the smallest saturated hereditary vertex subset containing the range vertices of the exits of α. Then clearly H is a nonempty proper subset of E^0. Then the quotient algebra $C^*(E)/I_H$ is isomorphic to the graph algebra $C^*(F)$ by Theorem 3.1 (a), where $F = (E^0 \setminus H, \{e \mid r(e) \notin H\})$. Since F has a loop α which has no exits in F, we see from the first argument that $RR(C^*(F)) \neq 0$. Therefore $RR(C^*(E)) \neq 0$. □

Let E be a subgraph of G. Then the subgraph E_e of G obtained by adding to E all the exits, that is, the edges $e \in G^1 \setminus E^1$ such that $s(e) = s(f)$ for some $f \in E^1$ and their range vertices $r(e)$ is called the *exit completion* of E.

PROPOSITION 3.3 (see [9, Theorem 4.6]). *Let E satisfy condition* (K). *If $C^*(E)$ has only finitely many ideals then $RR(C^*(E)) = 0$.*

PROOF. Let F be the desingularization of E. Then F also satisfies condition (K) and $C^*(F)$ contains the same number of ideals as $C^*(E)$. Since $RR(C^*(F)) = 0$ implies $RR(C^*(E)) = 0$ we may assume that the given graph E is row-finite and has no sinks.

We prove the assertion by induction on n, the number of nonzero ideals of $C^*(E)$. If $n = 1$, that is, $C^*(E)$ is simple then $RR(C^*(E)) = 0$ by [6, Remark 2.16]. Now let $n > 1$, and I_H be a maximal ideal of $C^*(E)$ corresponding to a saturated hereditary vertex subset H of E^0. Then $C^*(E)/I_H \cong C^*(F)$, where $F = (E^0 \setminus H, \{e \mid r(e) \notin H\})$. Let F_e be the exit completion of F in E. Then one can prove that $RR(C^*(F_e)) = 0$ by the same argument as in the proof of [9, Theorem 4.6]. Since F_e satisfies condition (K), the subalgebra B of $C^*(E)$ generated by $\{p_v, s_e \mid v \in (F_e)^0, e \in (F_e)^1\}$ is isomorphic to $C^*(F_e)$ by [2, Theorem 3.1]. Hence $RR(B) = 0$. But $C^*(E) = I_H + B$ and $RR(I_H) = 0$ by induction hypothesis. Therefore $C^*(E) = 0$ by [3, Proposition 3.18]. □

LEMMA 3.4 (see [8, Proposition 4.1]). *Let E be a graph and H be a saturated hereditary vertex subset of E^0. If $RR(I_H)=RR(C^*(E)/I_H)=0$, then $RR(C^*(E))=0$.*

PROOF. We show that $RR(C^*(F)) = 0$, where F is the desingularization of E. Since $C^*(E) \cong pC^*(F)p$ for a full projection p in the multiplier algebra of $C^*(F)$, there is an inclusion preserving bijection ρ between the sets of ideals such that for an ideal I in $C^*(E)$, $\rho(I)$ ($C^*(F)/\rho(I)$, respectively) is strong Morita equivalent to I ($C^*(E)/I$, respectively), in fact, the isomorphism ρ is given by $\rho(I) = pIp$. Also by

[6, Lemma 3.2] the ideal I_H of $C^*(E)$ is mapped to the ideal

$$J_{\tilde{H}} = \overline{\operatorname{span}}\{s_\alpha s_\beta^* \mid \alpha, \beta \in F^*, r(\alpha) = r(\beta) \in \tilde{H}\}$$

of $C^*(F)$, where $\tilde{H}$ is the saturated hereditary vertex subset of F^0 obtained from E^0 by adding the vertices on a tail added to each singular vertex in H. Now by Theorem 3.1 (a) the quotient algebra $C^*(F)/J_{\tilde{H}}$ is the graph algebra $C^*(G)$ for the quotient graph G. Since the K_0 group $K_0(C^*(F))$ is generated by the equivalence classes $\{[p_v] \mid v \in F^0\}$ subject to the relation $[p_v] = \sum_{s(e)=v}[p_{r(e)}]$ by [12, Theorem 3.1], it follows that the quotient map $C^*(F) \to C^*(F)/J_{\tilde{H}}$ induces the surjection from $K_0(C^*(F))$ onto $K_0(C^*(F)/J_{\tilde{H}})$. This implies that every projection in the quotient algebra $C^*(F)/J_{\tilde{H}}$ lifts to a projection in $C^*(F)$ [3, Proposition 3.15], and we conclude that $RR(C^*(F)) = 0$. □

Recall that for a subgraph F of E the *loop completion* $\ell_E(F)$ is the subgraph of E obtained by adding all the loops based at vertices of F^0 to the graph F, and the *loop contraction* $\ell c(F)$ of F is the graph obtained by shrinking each loop in F to a loop consisting of a single edge, [8, Definition 3.1]. Then F and $\ell c(F)$ have the same isomorphic lattice structure of hereditary subsets of vertices. Moreover [8, Lemma 3.1] holds for row-finite graphs.

THEOREM 3.5 (cf. [8, Theorem 4.1]). *Let E be a directed graph. Then the following are equivalent*:

(1) *$C^*(E)$ has real rank zero.*
(2) *E satisfies condition* (K).
(3) *$C^*(E)$ has no quotients containing a corner that is $*$-isomorphic to $M_n(C(\mathbb{T}))$.*

PROOF. (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) can be proved by the same arguments as in the proof of [8, Theorem 4.1].

(2) $\Rightarrow$ (1). By considering the desingularization we may assume that E is a row-finite graph with no sinks. Since the linear span of the elements of the form $s_\alpha s_\beta^*$ is dense in $C^*(E), \alpha, \beta \in E^*$, to prove $RR(C^*(E)) = 0$ it suffices to approximate a self-adjoint element $x = \sum_{\text{finite}} \lambda_{\alpha\beta} s_\alpha s_\beta^* + \lambda \cdot 1$ in $C^*(E)^\sim$ by invertible self-adjoint elements. Let F be the loop completion of the finite subgraph consisting of edges of $\alpha's$ and $\beta's$ in the expression of x and their source and range vertices, and let F_e be the exit completion of F. Then we only need to prove that $RR(C^*(F_e))(= RR(C^*(F_e)^\sim)) = 0$. But the same proof of [8, Theorem 4.1] applies to show the assertion. Then $C^*(F_e)^\sim$ can be identified with the unitization of the C^*-subalgebra of $C^*(E)^\sim$ generated by $\{s_e, p_v \mid v \in F_e^0, e \in F_e^1\}$ since the subgraph F_e satisfies condition (K) and hence the uniqueness theorem ([2, Theorem 3.1]) applies. We have shown that for each self-adjoint element $x \in C^*(E)^\sim$ there is a C^*-subalgebra $B(\cong C^*(F_e)^\sim)$ of $C^*(E)^\sim$ such that $x \in B$ and $RR(B) = 0$, which completes the proof. □

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BK21
Mathematical Sciences Division
Seoul National University
Seoul 151-742
Korea
e-mail: jajeong@math.snu.ac.kr