SOME FINITENESS CONDITIONS FOR AUTOMORPHISM GROUPS

by SILVANA FRANCIOSI and FRANCESCO DE GIOVANNI

(Received 21 May, 1986)

1. Introduction. Many authors have investigated the behaviour of the elements of finite order of a group G when finiteness conditions are imposed on the automorphism group Aut G of G. The first result was obtained in 1955 by Baer [**1**], who proved that a torsion group with finitely many automorphisms is finite. This theorem was generalized by Nagrebeckiĭ in [**6**], where he proved that if the automorphism group Aut G is finite then the set of elements of finite order of G is a finite subgroup.

More generally Robinson and the authors [**4**] have recently shown that finiteness conditions on the ranks of the automorphism group Aut G give strong restrictions on the elements of finite order of G.

Other finiteness conditions on automorphism groups have been considered, for instance, in [**10**], [**14**] and [**3**].

The object of this paper is to study how the imposition of the maximal or minimal conditions on normal or subnormal subgroups of the automorphism group forces the set of elements of finite order of the group to be small.

Our main results are as follows.

THEOREM A. *Let G be a soluble-by-finite group.*

(i) *If* Aut G *satisfies the maximal condition on subnormal abelian subgroups with defect at most* 2, *then the Sylow subgroups of G have finite exponents.*

(ii) *If* Aut G *satisfies the minimal condition on subnormal subgroups with defect at most* 2, *then the elements of finite order of G form a subgroup of finite exponent.*

THEOREM B. *Let G be a torsion group.*

(i) *If G is soluble-by-finite and* Aut G *satisfies the maximal condition on normal subgroups, then G has finite exponent.*

(ii) *If G is nilpotent-by-finite and* Aut G *satisfies the minimal condition on normal subgroups, then G has finite exponent.*

In the situation of Theorem A(i) infinitely many Sylow subgroups can occur in the group G. Moreover in Theorem A and Theorem B the Sylow subgroups of G can be infinite. Finally an example will be given of a group G such that Aut G satisfies the minimal condition on subnormal abelian subgroups and $Z(G)$ is a p^∞-group.

We also obtain some information on torsion groups whose automorphism groups are metanilpotent and satisfy the minimal condition on normal subgroups (see Theorem C in §3).

For homological results we refer to [**5**] and [**13**].

Glasgow Math. J. **29** (1987) 259–265.

NOTATION

G_{ab}: the derived factor group G/G'.
$M(G)$: the Schur multiplicator of G.
$\pi(G)$: the set of prime divisors of the orders of elements of G.
$\operatorname{Inn} G$: the group of all inner automorphisms of G.
A_p: the p-component of the abelian group A.

2. The maximal and minimal conditions on subnormal subgroups.

LEMMA 1. *Let G be a group such that $G/Z(G)$ is a Černikov group and $G/Z_2(G)$ is finite. Then $G/Z(G)$ is finite.*

Proof. The finite residual $R/Z(G)$ of $G/Z(G)$ is a central subgroup of $G/Z(G)$ and so $[R, G]$ is a radicable subgroup of $G' \cap Z(G)$. Since $G/Z(G)$ is a Černikov group, $M(G/Z(G))$ is finite [9], and hence also its homomorphic image $G' \cap Z(G)$ is finite. Therefore $[R, G] = 1$ and $G/Z(G)$ is finite.

LEMMA 2. *Let G be a group satisfying the maximal condition on subnormal abelian subgroups with defect at most 2. Then every soluble normal subgroup of G is polycyclic.*

Proof. Let H be a soluble normal subgroup of G. Then by induction we may assume that H' is polycyclic. If $C = C_H(H')$, the group H/C is polycyclic (see [7, Part 1, Theorem 3.27]), and $[C', C] \leq [H', C] = 1$, so that C is nilpotent. Let A be a maximal normal abelian subgroup of C; then $A = C_C(A)$ and A is finitely generated since C is a nilpotent group satisfying the maximal condition on normal abelian subgroups. Therefore C/A is finitely generated and hence H is polycyclic.

LEMMA 3 (see [4, Lemma 4]). *Let G be a group and let C be a central subgroup of G. Assume that G/C is soluble-by-finite with finite abelian section rank and C is an infinite p-group of finite exponent. Then G has an infinite direct factor contained in C.*

Proof of Theorem A. (i) Let $C = Z(G)$ and $Q = G/C$.

Firstly we will prove that the torsion subgroup T of C is reduced. Suppose by contradiction that the p-component T_p of T is not reduced for some prime p, and denote by A the divisible part of T_p. Then $C = A \times K$ for some subgroup K and so $H^2(Q, C) \simeq H^2(Q, A) \oplus H^2(Q, K)$. By the Universal Coefficients Theorem we have $H^2(Q, A) \simeq \operatorname{Hom}(M(Q), A)$, and since $Q \simeq \operatorname{Inn} G$ is polycyclic-by-finite by Lemma 2, the Schur multiplicator $M(Q)$ is finitely generated and $H^2(Q, A)$ is a torsion group. If Δ is the cohomology class of the central extension

$$C \rightarrowtail G \twoheadrightarrow Q$$

we can write $\Delta = \Delta_0 + \Delta_1$, where $\Delta_0 \in H^2(Q, A)$ and $\Delta_1 \in H^2(Q, K)$, so that $e\Delta_0 = 0$ for some $e > 0$. For each p-adic integer $\alpha \equiv 0 \pmod p$, the automorphism τ of C defined by $a\tau = a^{1+e\alpha}$, $k\tau = k$ $(a \in A, k \in K)$ can be extended to an automorphism of G acting trivially on Q since $e\Delta_0 = 0$. Therefore the group Γ of all automorphisms of G acting trivially on G/C and C/A, and as p-adic integers on A, is an uncountable normal subgroup of Aut G. The derived subgroup Γ' of Γ stabilizes the series $G \geqslant C \geqslant A \geqslant 1$, so that Γ is soluble and therefore countable by Lemma 2, a contradiction. Therefore T is reduced.

Suppose that T_p is infinite for some prime p. Then T_p/T_p^p is infinite and so C/C^p is an infinite group. The group G' is polycyclic-by-finite (see [**7**, p. 115]), and so $G/G'C^p$ has an elementary abelian infinite p-quotient. Since the group $\mathrm{Hom}(G/G'C^p, C^p)$ is isomorphic with a normal abelian subgroup of Aut G, it is countable, and this forces $(C^p)_p = 1$. Therefore T_p has exponent p, if it is infinite. It follows that the Sylow subgroups of G have finite exponents.

(ii) Let $C = Z(G)$ and $Q = G/C$. Since the factors of the derived series of the soluble radical of Q satisfy the minimal condition, then Q is a Černikov group, and hence the Schur multiplicator $M(Q)$ is finite [**9**]. Then we can prove as in (i) that the Sylow subgroups of C have finite exponents.

Firstly we will prove that Q is finite. Suppose that this is false. Then by Lemma 1 the group $Q/Z(Q)$ is infinite. If R denotes the finite residual of Q, there exists a prime p such that the p-component S of $[R, Q]$ is non-trivial. As in the proofs of Lemma 1 and Lemma 2 of [**10**] it can be proved that there exists an automorphism of infinite order α of G acting trivially on C and Q/S, and as the multiplication by a fixed integer on S. The group Γ of all automorphisms of G acting trivially on C and Q/S, and as multiplications by fixed integers on S, is a normal subgroup of Aut G, and Γ' is nilpotent since it acts trivially on S. It follows that Γ is a Černikov group, a contradiction since $\alpha \in \Gamma$. Therefore Q is finite.

By contradiction assume that the torsion subgroup T of C has infinite exponent; then the set $\pi_0 = \pi(C)\backslash\pi(Q)$ is infinite. If $p_1 < p_2 < \ldots < p_n < \ldots$ is the ordered sequence of primes in π_0, we can write $G = (C_{p_1} \times \ldots \times C_{p_n}) \times K_n$ for all n; therefore the automorphism λ_n of T which acts as the inversion on $C_{p_1} \times \ldots \times C_{p_n}$ and as the identity on $\mathrm{Dr}_{h>n}\, C_{p_h}$ can be extended to an automorphism of G, and so $\langle \lambda_n \mid n \in \mathbb{N} \rangle$ is an infinite elementary abelian group contained in the centre of the group Λ of all automorphisms induced by Aut G on T. This is impossible since Λ satisfies the minimal condition on normal subgroups. Therefore T has finite exponent and the result follows.

We have a stronger result for purely-non-abelian groups (recall that a group is *purely-non-abelian* if it has no non-trivial abelian direct factors).

COROLLARY. *Let G be a soluble-by-finite group which is also purely-non-abelian.*

(i) *If* Aut G *satisfies the maximal condition on subnormal abelian subgroups with defect at most* 2, *then the Sylow subgroups of G are finite.*

(ii) *If* Aut G *satisfies the minimal condition on subnormal subgroups with defect at most* 2, *then the elements of finite order of G form a finite subgroup.*

Proof. (i) By Theorem A(i) the Sylow subgroups of G have finite exponents. By contradiction suppose that the p-component C_p of $C = Z(G)$ is infinite for some prime p. We have $C = C_p \times K$ for some subgroup K and by Lemma 3 there exists an infinite direct factor H/K of G/K contained in C/K. Then we have $G = HL$ with $H \cap L = K$, and

$H = K \times (C_p \cap H)$, so that $G = HL = (C_p \cap H)L = (C_p \cap H) \times L$. It follows that $C_p \cap H = 1$ and $H = K$, a contradiction.

(ii) By the proof of Theorem A(ii) it follows that $G/Z(G)$ is finite and the set of all elements of finite order of G is a subgroup of finite exponent. Then the result can be obtained as in (i).

REMARK 1. In the hypothesis of Theorem A(i) infinitely many Sylow subgroups can occur in the group G. In fact in Theorem 3(ii) of **[4]** there is constructed a soluble group G such that Aut G is polycyclic and $\pi(G)$ is infinite.

REMARK 2. In the hypothesis of Theorem A the Sylow subgroups of G can be infinite. In fact, let G be a countable elementary abelian p-group and put $\Gamma = \text{Aut}\, G$. If Γ^* is the group of all automorphisms γ of G such that $\ker(\gamma - 1)$ has finite codimension, then Γ/Γ^* is finite-by-simple (see **[11]**) and Γ^* is finite-by-simple-by-finite (see **[2]**). Therefore Aut G satisfies the maximal and minimal conditions on subnormal subgroups.

PROPOSITION. *There exists a metabelian group G such that* Aut G *satisfies the minimal condition on subnormal abelian subgroups, and the centre of G is a p^∞-group.*

Proof. Let p be a prime and $A = B \times C$, where $B = \langle b_n \mid n \in \mathbb{N}_0,\ b_{n+1}^p = b_n,\ b_0 = 1\rangle$ and $C = \langle c_n \mid n \in \mathbb{N}_0,\ c_{n+1}^p = c_n,\ c_0 = 1\rangle$ are p^∞-groups. If α is a p-adic integer of infinite multiplicative order, the automorphism x of A defined by $b_n^x = b_n^\alpha c_n$, $c_n^x = c_n (n \in \mathbb{N}_0)$ has infinite order. Let $G = \langle x \rangle \ltimes A$; then $Z(G) = C$ and $A = C[A, x]$.

Firstly we will prove that the Baer radical Γ of Aut G (i.e. the subgroup of Aut G generated by all subnormal abelian subgroups) stabilizes the series $G > A > 1$. If γ is any element of Γ, $\langle \gamma \rangle$ is subnormal in Aut G and so it acts trivially on the infinite cyclic group G/A. Moreover $[A/C, \gamma] \neq A/C$ and $[A/C, \gamma]$ is divisible; therefore γ acts trivially on A/C and, by a similar argument, γ acts trivially on C. If $a \in A$, then $[a, \gamma^{-1}, x] = 1$ and $[a, [x^{-1}, \gamma]] = 1$. It follows that $[x, a^{-1}, \gamma] = 1$. Thus γ acts trivially on $[A, x]$ and on C and hence on the whole of A.

Therefore the mapping $\gamma \mapsto [x, \gamma]$ is a monomorphism of Γ into A. It follows that Γ is a Černikov group and so Aut G satisfies the minimal condition on subnormal abelian subgroups.

3. The maximal and minimal conditions on normal subgroups.

Proof of Theorem B. (i) Let R be the maximum normal soluble subgroup of $I = \text{Inn}\, G$. If S is a factor of the derived series of R, the Sylow subgroups of S are Aut G-invariant, and so the set $\pi(S)$ is finite since Aut G satisfies the maximal condition on normal subgroups. By the same considerations it follows that the Sylow subgroups of S have finite exponents, so that I has finite exponent.

Write $C = Z(G)$ and $Q = G/C$. Then $Q \simeq I$ has finite exponent. Put $\pi_0 = \pi(C)\backslash\pi(Q)$; then the π_0-component H of C is a Hall subgroup of G and so $G = H \times K$ for some subgroup K. It follows that Aut G = Aut $H \times$ Aut K, and Aut $H \simeq \underset{p \in \pi_0}{\text{Cr}}$ Aut C_p is in Max-n, so that π_0 is finite and $\pi(G)$ is finite.

https://doi.org/10.1017/S0017089500006911 Published online by Cambridge University Press

Since Q is a soluble-by-finite group of finite exponent, an easy application of the Lyndon–Hochschild–Serre spectral sequence for homology shows that the Schur multiplicator $M(Q)$ of Q has finite exponent, and by the Universal Coefficients Theorem $H^2(Q, C) \simeq \mathrm{Ext}(Q_{\mathrm{ab}}, C) \oplus \mathrm{Hom}(M(Q), C)$ has finite exponent e.

By contradiction assume that the p-component C_p of C has infinite exponent for some prime p. If Δ denotes the cohomology class of the central extension $C \rightarrowtail G \twoheadrightarrow Q$, we have $e\Delta = 0$ and so, for any p-adic integer $\alpha \equiv 0 \pmod p$, the mapping $x \mapsto x^{1+e\alpha}$ is an automorphism γ_α of C_p which extends to a central automorphism of G acting trivially on $C_{p'}$.

Let Γ be the group of all automorphisms induced by Aut G on C_p; then Γ satisfies the maximal condition on normal subgroups and so $Z(\Gamma)$ is countable. This is a contradiction, since each γ_α belongs to $Z(\Gamma)$.

Therefore each primary component of C has finite exponent and so G has finite exponent.

(ii) Let F be the maximum normal nilpotent subgroup of G and put $A = Z(F)$. Then $Z(G) \leqslant A$ and for each positive integer i the group $Z_{i+1}(F)/Z_i(F)$ is reduced and satisfies the minimal condition on characteristic subgroups; therefore $Z_{i+1}(F)/Z_i(F)$ has finite exponent if $i \geqslant 1$, and so G/A has finite exponent. The group A is a trivial F/A-module, and so the group $H^2(F/A, A)$ has finite exponent by the Universal Coefficients Theorem. Since F has finite index in G, the Lyndon–Hochschild–Serre spectral sequence for cohomology shows that $H^2(G/A, A)$ has finite exponent e.

As in the proof of (i) we can now prove that each primary component of A has finite exponent. Moreover it is easily seen that the set $\pi(G/Z(G))$ is finite and so, as in the proof of (i), the set $\pi(G)$ is finite. The result follows.

Remark. In the hypothesis of Theorem B the Sylow subgroups of G can be infinite, as proved by a countably infinite elementary abelian p-group.

Finally we have the following result.

Theorem C. *Let G be a torsion group. If* Aut G *is metanilpotent and satisfies the minimal condition on normal subgroups, then the centre of G is reduced.*

Proof. Let $C = Z(G)$ and $Q = G/C$.

By contradiction assume that C contains a p^∞-subgroup P. If R/Q' is the p'-component of Q_{ab}, then Q/R is an abelian p-group satisfying the minimal condition on characteristic subgroups, and so it is the direct product of a group of finite exponent and a radicable group. The group $\mathrm{Hom}(Q/R, C)$ is isomorphic with a subgroup of Aut G, and hence it follows that Q/R is finite since Aut G is countable (see [**12**]).

Since Aut G is metanilpotent, there exists an integer $i \geqslant 1$ such that $A/C = \gamma_i(Q)$ is nilpotent. Then A is a characteristic nilpotent subgroup of G and $C \leqslant Z(A)$, so that P is contained in the p-component B of $Z(A)$.

We will prove that the cohomology group $H^2(G/B, B)$ has finite exponent. If H/A is the p-complement of the nilpotent group G/A, then G/H is finite, and so the groups

$H^2(G/H, H^0(H/A, B))$ and $H^1(G/H, H^1(H/A, B))$ have finite exponents. Since $H^2(H/A, B) = 0$ (see [**8**, Corollary 4.2]), the Lyndon–Hochschild–Serre spectral sequence for cohomology shows that $H^2(G/A, B)$ has finite exponent. The reduced group $A/Z(A)$ satisfies the minimal condition on characteristic subgroups and hence it has finite exponent. It follows that $H^1(A/Z(A), B) \doteq \mathrm{Hom}(A/Z(A), B)$ has finite exponent and so $H^1(G/A, H^1(A/Z(A), B))$ has finite exponent. Moreover the group $H^2(A/Z(A), B) \simeq \mathrm{Ext}((A/Z(A))_{ab}, B) \oplus \mathrm{Hom}(M(A/Z(A)), B)$ has finite exponent. The Lyndon–Hochschild–Serre spectral sequence for cohomology associated with the extension

$$A/Z(A) \rightarrowtail G/Z(A) \twoheadrightarrow G/A$$

and the $G/Z(A)$-module B implies that $H^2(G/Z(A), B)$ has finite exponent. Finally $Z(A)/B$ is a p'-group, and hence we obtain easily that $H^2(G/B, B)$ has finite exponent e.

If Δ denotes the cohomology class of the extension

$$B \rightarrowtail G \twoheadrightarrow G/B$$

we have $e\Delta = 0$ and therefore, if α is a p-adic integer $\equiv 0 \pmod p$, the mapping $x \mapsto x^{1+e\alpha}$ is an automorphism of B which extends to an automorphism γ of G acting trivially on G/B. The automorphism γ has infinite order since B has infinite exponent, and this is a contradiction since Aut G is a torsion group (see [**7**, Part 1, Theorem 5.25]).

REFERENCES

1. R. Baer, Finite extensions of abelian groups with minimum condition, *Trans. Amer. Math. Soc.* **79** (1955), 521–540.

2. J. Dieudonné, Les determinants sur un corps non commutatif, *Bull. Soc. Math. France* **71** (1943), 27–45.

3. S. Franciosi and F. de Giovanni, A note on groups with countable automorphism groups, *Arch. Math.* (*Basel*), **47** (1986), 12–16.

4. S. Franciosi, F. de Giovanni and D. J. S. Robinson, On torsion in groups whose automorphism groups have finite rank, *Rocky Mountain J. Math.*, to appear.

5. S. MacLane, *Homology* (Springer, 1975).

6. V. T. Nagrebeckiĭ, On the periodic part of a group with a finite number of automorphisms, *Dokl. Akad. Nauk SSSR* **205** (1972), 519–521 = *Soviet Math. Dokl.* **13** (1972), 953–956.

7. D. J. S. Robinson, *Finiteness conditions and generalized soluble groups* (Springer, 1972).

8. D. J. S. Robinson, On the cohomology of soluble groups with finite rank, *J. Pure Appl. Algebra* **6** (1975), 155–164.

9. D. J. S. Robinson, Homology of group extensions with divisible abelian kernel, *J. Pure Appl. Algebra* **14** (1979), 145–165.

10. D. J. S. Robinson, Infinite torsion groups as automorphism groups, *Quart. J. Math. Oxford Ser.* (2) **30** (1979), 351–364.

11. A. Rosenberg, The structure of the infinite general linear group, *Ann. of Math.* **68** (1958), 278–294.

12. H. L. Silcock, Metanilpotent groups satisfying the minimal condition for normal subgroups, *Math. Z.* **135** (1974), 165–173.

13. U. Stammbach, *Homology in Group Theory* (Lecture Notes in Mathematics **359,** Springer, 1973).

14. J. Zimmerman, Countable torsion FC-groups as automorphism groups, *Arch. Math. (Basel)* **43** (1984), 108–116.

Dipartimento di Matematica "R. Caccioppoli"
Via Mezzocannone 8
80134 Napoli
Italy