DOUBLE MS_n-ALGEBRAS AND DOUBLE $K_{n,m}$-ALGEBRAS

by M. SEQUEIRA†

(Received 21 November, 1991)

0. Abstract. The variety $\mathbf{O}_2$ of all algebras $(L; \wedge, \vee, f, g, 0, 1)$ of type $(2,2,1,1,0,0)$ such that $(L; \wedge, \vee, f, 0, 1)$ and $(L; \wedge, \vee, g, 0, 1)$ are Ockham algebras is introduced, and, for n, $m \in \mathbb{N}$, its subvarieties $\mathbf{DMS}_n$, of double MS_n-algebras, and $\mathbf{DK}_{n,m}$, of double $K_{n,m}$-algebras, are considered. It is shown that $\mathbf{DK}_{n,m}$ has equationally definable principal congruences: a description of principal congruences on double $K_{n,m}$-algebras is given and simplified for double MS_n-algebras. A topological duality for O_2-algebras is developed and used to determine the subdirectly irreducible algebras in $\mathbf{DK}_{n,m}$ and in $\mathbf{DMS}_n$. Finally, MS_n-algebras which are reduct of a (unique) double MS_n-algebra are characterized.

1. Preliminaries. Algebras $(L; \wedge, \vee, f, 0, 1)$ of type $(2,2,1,0,0)$ such that $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and f is a dual endomorphism of $(L; \wedge, \vee, 0, 1)$ are called distributive *Ockham algebras* and form a variety. In [**1**], for $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, the subvariety of Ockham algebras characterized by the equation $f^{2n+m}(x) = f^m(x)$ is denoted by $\mathbf{K}_{n,m}$. Notice that $\mathbf{K}_{n,m} \subseteq \mathbf{K}_{n',m'}$ if and only if $n \mid n'$ and $m \leq m'$, [**11**].

A topological duality for Ockham algebras based on Priestley's duality for bounded distributive lattices was established in [**13**]. The duality was used to describe the subdirectly irreducible algebras and several subvarieties including $\mathbf{K}_{n,m}$ (denoted $\mathcal{P}_{2n+m,m}$ in [**13**]). In particular, each $\mathbf{K}_{n,m}$ is generated by a single algebra, $\mathcal{L}_{2n+m,m}$, which is subdirectly irreducible.

The variety **MS** of *MS-algebras*, [**4**], is the subvariety of Ockham algebras characterized by $x \leq f^2(x)$. For $n \in \mathbb{N}$, we denote by $\mathbf{MS}_n$ the variety of Ockham algebras satisfying $x \leq f^{2n}(x)$, [**12**], (these varieties appeared in [**11**] denoted by $\mathbf{K}^{\leq}_{n,0}$). Obviously, $\mathbf{MS}_1 = \mathbf{MS}$. We have $\mathbf{K}_{n,0} \subset \mathbf{MS}_n \subset \mathbf{K}_{n,1}$; besides, $\mathbf{MS}_n \subseteq \mathbf{MS}_{n'}$ if and only if $n \mid n'$, [**11**]. If $(L; \wedge, \vee, f, 0, 1)$ is an MS_n-algebra, f^{2n} is both an endomorphism and a closure operator on $(L; \wedge, \vee, 0, 1)$.

The notion of double MS-algebra, introduced by T. Blyth and J. Varlet in [**5**], was inspired by the properties of double Stone algebras. A *double MS-algebra* $(L; \wedge, \vee, f, g, 0, 1)$ is an algebra of type $(2,2,1,1,0,0)$ such that $(L; \wedge, \vee, f, 0, 1)$ and $(L; \wedge, \vee, g, 0, 1)$ are Ockham algebras and f, g satisfy $x \leq f^2(x)$, $g^2(x) \leq x$, $gf(x) = f^2(x)$, $fg(x) = g^2(x)$, $\forall x \in L$. **DMS** denotes the variety of double MS-algebras. Each algebra $(L; \wedge, \vee, f, g, 0, 1) \in \mathbf{DMS}$ is associated with an MS-algebra and a dual MS-algebra; gf and fg are, respectively, a closure and a dual closure on $(L; \wedge, \vee, 0, 1)$.

2. The variety $\mathbf{O}_2$ and the subvarieties $\mathbf{DK}_{n,m}$ and $\mathbf{DMS}_n$. We shall consider algebras of type $(2,2,1,1,0,0)$ which are associated with Ockham algebras.

Definition. [**12**] An *O_2-algebra* is an algebra $\mathcal{L} = (L; \wedge, \vee, f, g, 0, 1)$ of type $(2,2,1,1,0,0)$ such that $(L; \wedge, \vee, f, 0, 1)$ and $(L; \wedge, \vee, g, 0, 1)$ are Ockham algebras.

† Work done within the activities of Centro de Álgebra da Universidade de Lisboa, I.N.I.C.

The class of all O_2-algebras is a variety and we denote it by $\mathbf{O}_2$. For brevity, we write $\mathscr{L}=(L,f,g)\in\mathbf{O}_2$, (L,f), (L,g) for Ockham algebras and L for the underlying bounded distributive lattice.

For each $n\in\mathbb{N}$, we introduce a subvariety of $\mathbf{O}_2$ which is related to $\mathbf{MS}_n$ in the same way that double MS-algebras are related to MS-algebras. The fact that, for $(L,f)\in\mathbf{MS}_n$, the mapping f^{2n} is a closure on L leads to the following definition.

Definition [12]. A *double MS_n-algebra* is an algebra $\mathscr{L}=(L,f,g)\in\mathbf{O}_2$ such that

$$fg=g^{2n}\leq \mathrm{id}\leq f^{2n}=gf.$$

The variety of double MS_n-algebras is denoted by $\mathbf{DMS}_n$. Now $\mathbf{DMS}_1=\mathbf{DMS}$, and it is easy to check that $\mathbf{DMS}_n\subseteq\mathbf{DMS}_{n'}$ if $n\mid n'$. Notice that, if $(L,f)\in\mathbf{K}_{n,0}$ and $g=f^{2n-1}$, we obtain $(L,f,g)\in\mathbf{DMS}_n$. Hence, if $\mathbf{DMS}_n\subseteq\mathbf{DMS}_{n'}$, extending $\mathscr{L}_{2n,0}$ to a double MS_n-algebra, we conclude that $\mathbf{K}_{n,0}\ (=V(\{\mathscr{L}_{2n,0}\}))\subset\mathbf{MS}_{n'}\subset\mathbf{K}_{n',1}$ and $n\mid n'$.

Let $n,m\in\mathbb{N}$. For each $(L,f)\in\mathbf{K}_{n,m}$, we have $f^{2n+k}=f^k$, $\forall k\in\mathbb{N}$, $k\geq m$. If $2n\geq m$, the map $g=f^{2n-1}$ is a dual endomorphism of L satisfying $g^{2n}=f^{2n}$ and $g^{2n+m}=g^m$; hence $gf=f^{2n}$ and $fg=g^{2n}$. In general, if z is the smallest integer such that $2zn\geq m$, i.e. $z=\lceil m/2n\rceil$ ($\lceil x\rceil$ stands for the smallest integer greater than or equal to x), the dual endomorphism $g=f^{2zn-1}$ of L satisfies $g^{2n+m}=g^m$ and $g^{2zn}=f^{2zn}$; therefore $gf=f^{2zn}$ and $fg=g^{2zn}$.

Definition. Let $n,m\in\mathbb{N}$ and $z=\lceil m/2n\rceil$. We denote by $\mathbf{DK}_{n,m}$ the class of all algebras $\mathscr{L}=(L,f,g)\in\mathbf{O}_2$ such that

$$f^{2n+m}=f^m,\quad g^{2n+m}=g^m,\quad gf=f^{2zn},\quad fg=g^{2zn}.$$

If $\mathscr{L}\in\mathbf{DK}_{n,m}$, we say that $\mathscr{L}$ is a *double $K_{n,m}$-algebra*. For $m=1$, we get the double $K_{n,1}$-algebras introduced in [12]. Clearly, $\mathbf{DMS}_n\subset\mathbf{DK}_{n,1}$. The varieties $\mathbf{DK}_{n,m}$, n, $m\in\mathbb{N}$, are related in the following way.

Proposition 1. *Let n, n', m, $m'\in\mathbb{N}$.*
(i) *If $n\mid n'$, then $\mathbf{DK}_{n,m}\subseteq\mathbf{DK}_{n',m}$.*
(ii) *If $m\leq m'$, then $\mathbf{DK}_{n,m}\subseteq\mathbf{DK}_{n,m'}$.*
(iii) *$\mathbf{DK}_{n,m}\subseteq\mathbf{DK}_{n',m'}$ if and only if $n\mid n'$ and $m\leq m'$.*

Proof. Recall that $\mathbf{K}_{n,m}\subseteq\mathbf{K}_{n',m'}$ if and only if $n\mid n'$ and $m\leq m'$.

(i) Let $n'=nk$, $z=\lceil m/2n\rceil$, $z'=\lceil m/2n'\rceil$ and $\mathscr{L}=(L,f,g)\in\mathbf{DK}_{n,m}$. Then $kz'\geq z$ and, for $\rho\in\{f,g\}$, we have $\rho^{2z'n'}=\rho^{2(kz'-z)n+2zn}=\rho^{2zn}$. Hence $\mathscr{L}\in\mathbf{DK}_{n',m}$.

(ii) If $m\leq m'$ and $\mathscr{L}=(L,f,g)\in\mathbf{DK}_{n,m}$, we have $z=\lceil m/2n\rceil\leq z'=\lceil m'/2n\rceil$ and, for $\rho\in\{f,g\}$, $\rho^{2z'n}=\rho^{2(z'-z)n+2zn}=\rho^{2zn}$. Hence $\mathscr{L}\in\mathbf{DK}_{n,m'}$.

(iii) If $n\mid n'$ and $m\leq m'$, then $\mathbf{DK}_{n,m}\subseteq\mathbf{DK}_{n',m'}$ by (i) and (ii). Conversely, if $\mathbf{DK}_{n,m}\subseteq\mathbf{DK}_{n',m'}$, it suffices to extend the algebra $\mathscr{L}_{2n+m,m}$ (which generates $\mathbf{K}_{n,m}$) to a double $K_{n,m}$-algebra to conclude that $\mathbf{K}_{n,m}\subseteq\mathbf{K}_{n',m'}$, hence $n\mid n'$ and $m\leq m'$.

The process that motivates the definition of $\mathbf{DK}_{n,m}$ is not, in general, the only one that allows us to obtain a double $K_{n,m}$-algebra from a given algebra in $\mathbf{K}_{n,m}$. For instance, the Stone algebra $\mathscr{S}=(S,f)$, where S is the chain $0<a<1$ and f is defined by $f(0)=1$, $f(a)=f(1)=0$, yields two algebras in $\mathbf{DK}_{1,1}$: letting $g_1(0)=g_1(a)=1$, $g_1(1)=0$, we get $(S,f,g_1)\in\mathbf{DMS}_1$; taking $g_2=f$, we get $(S,f,g_2)\in\mathbf{DK}_{1,1}\backslash\mathbf{DMS}_1$.

Let $\mathscr{L}=(L,f,g)\in \mathbf{DK}_{n,m}$. Then $f^{2n+k}=f^k$, $g^{2n+k}=g^k$, $\forall k\geq m$. Denote by $r(t)$ the remainder of the integer t on division by $2n$. For $1\leq i,j\leq 2n+m-1$, let $z_{i,j}=m+r(j-i-m)$ (then $m\leq z_{i,j}\leq 2n+m-1$).

Proposition 2. *Let n, $m\in\mathbb{N}$, $z=\lceil m/2n\rceil$ and $\mathscr{L}=(L,f,g)\in\mathbf{DK}_{n,m}$. Then*
(i) $g^if^i=f^{2zn}$, $f^ig^i=g^{2zn}$, $1\leq i\leq 2n+m-1$.
(ii) $g^if^j=f^{z_{i,j}}$, $f^jg^i=g^{z_{j,i}}$, $1\leq i,j\leq 2n+m-1$.
(iii) $\operatorname{Im} f^m=\operatorname{Im} g^m$.

Proof. (i) Use induction on i and the fact that $2zn\geq m$.
(ii) For $1\leq i,j\leq 2n+m-1$, we have $|i-j|\leq 2n+m-2\leq 2n+2zn-2<4zn$. We consider three cases:
(a) $i=j$. Now, $z_{i,j}=2zn$ and $g^if^j=f^{2zn}$, by (i).
(b) $i<j$. We have $g^if^j=g^if^if^{j-i}=f^{2zn}f^{j-i}=f^{2zn+j-i}=f^{2zn+m+(j-i-m)}=f^{z_{i,j}}$.
(c) $i>j$. Now, $g^if^j=g^{i-j}g^jf^j=g^{i-j}f^{2zn}$. If $i-j<2zn$, we get $g^if^j=f^{z_{i,j}}$ by (b). If $i-j=2zn$, we have $g^if^j=g^{2zn}f^{2zn}=f^{2zn}=f^{z_{i,j}}$. If $i-j>2zn$, we get $g^if^j=g^{i-j-2zn}g^{2zn}f^{2zn}=g^{i-j-2zn}f^{2zn}=f^{z_{i,j}}$ using (b), since $i-j-2zn<2zn$.
(iii) Just notice that $f^m=g^mf^{m+r(m)}$ and $g^m=f^mg^{m+r(m)}$.

Corollary 3 [**12**, Lemma 5.3]. *Let $n\in\mathbb{N}$ and $\mathscr{L}=(L,f,g)\in\mathbf{DMS}_n$. Then*
(i) $g^if^i=f^{2n}$, $f^ig^i=g^{2n}$, $1\leq i\leq 2n$;
(ii) $g^if^j=f^{r(j-i)}$, $f^jg^i=g^{r(i-j)}$, $1\leq i,j\leq 2n$, $i\neq j$;
(iii) $\operatorname{Im} f=\operatorname{Im} g$;
(iv) $f^{2k+1}(x)\leq g^{2n-2k-1}(x)$, $g^{2n-2k}(x)\leq f^{2k}(x)$, $\forall x\in L$, $0\leq k\leq n-1$.

Proof. Since $\mathbf{DMS}_n\subset\mathbf{DK}_{n,1}$, (i), (ii) and (iii) follow from Proposition 2.
(iv) We have $g^{2n}(x)\leq x$, $\forall x\in L$. For $0\leq k\leq n-1$, using (ii) and the fact that f^{2k+1} is a dual endomorphism of L, we get $f^{2k+1}(x)\leq f^{2k+1}g^{2n}(x)=g^{2n-2k-1}(x)$. Again by (ii) and as f^{2k} is an endomorphism of L, we have $f^{2k}(x)\geq f^{2k}g^{2n}(x)=g^{2n-2k}(x)$.

3. Principal congruences. For $\mathscr{L}=(L,f,g)\in\mathbf{O}_2$, we denote by $\operatorname{Con}(\mathscr{L})$ the congruence lattice of $\mathscr{L}$ and by $\operatorname{Con}_D(\mathscr{L})$ the congruence lattice of the D_{01}-lattice L. For a, $b\in L$, $\theta(a,b)$, resp. $\theta_D(a,b)$, is the smallest element of $\operatorname{Con}(\mathscr{L})$, resp. $\operatorname{Con}_D(\mathscr{L})$, collapsing a and b. It suffices to consider $\theta(a,b)$ for $a<b$, since, if $\theta\in\operatorname{Con}(\mathscr{L})$ and x, $y\in L$, we have $(x,y)\in\theta$ if and only if $(x\wedge y,x\vee y)\in\theta$.

It is easy to see that, for $\mathscr{L}=(L,f,g)\in\mathbf{DK}_{n,m}$ and a, $b\in L$, $a<b$, the principal congruence $\theta(a,b)$ is given by

$$\theta(a,b)=\theta_D(a,b)\vee\bigvee_{i=1}^{2n+m-1}\theta_D(f^i(a),f^i(b))\vee\bigvee_{j=1}^{2n+m-1}\theta_D(g^j(a),g^j(b)).$$

Now, by [**2**, Th. 1.3], we conclude that $\mathbf{DK}_{n,m}$ has equationally definable principal congruences and, hence, satisfies the congruence extension property, [**8**, Corollary 2].

The description of a principal congruence as a join of congruences of a distributive lattice and [**9**, Lemma 2] allow us to conclude that each principal congruence in a double $K_{n,m}$-algebra can be defined by $2^{4n+2m-1}$ equations.

https://doi.org/10.1017/S0017089500009745 Published online by Cambridge University Press

For n, $m \in \mathbb{N}$, define

$$T_{n,m} = \left\{0, 1, 2, \ldots, n + \left\lfloor \frac{m-1}{2} \right\rfloor\right\}, \qquad T'_{n,m} = \left\{0, 1, 2, \ldots, n + \left\lfloor \frac{m-2}{2} \right\rfloor\right\},$$

$$T''_{n,m} = T_{n,m} \setminus \{0\}$$

($\lfloor x \rfloor$ stands for the greatest integer less than or equal to x).

Let $\mathscr{L} = (L, f, g) \in \mathbf{DK}_{n,m}$, $a, b \in L$, $a < b$. Then $(x, y) \in \theta(a, b)$ if and only if

$$(x \wedge d_{F,G,H,J}(a,b)) \vee e_{F,G,H,J}(a,b) = (y \wedge d_{F,G,H,J}(a,b)) \vee e_{F,G,H,J}(a,b) \quad (\dagger)$$

for each F, G, H, J such that $F \subseteq T_{n,m}$; $G, J \subseteq T'_{n,m}$; $H \subseteq T''_{n,m}$ and where

$$d_{F,G,H,J}(a,b) = \bigwedge_{i \in F} f^{2i}(a) \wedge \bigwedge_{j \in G} f^{2j+1}(b) \wedge \bigwedge_{k \in H} g^{2k}(a) \wedge \bigwedge_{l \in J} g^{2l+1}(b),$$

$$e_{F,G,H,J}(a,b) = \bigvee_{q \in T_{n,m} \setminus F} f^{2q}(b) \vee \bigvee_{r \in T'_{n,m} \setminus G} f^{2r+1}(a) \vee \bigvee_{s \in T''_{n,m} \setminus X} g^{2s}(b) \vee \bigvee_{t \in T'_{n,m} \setminus J} g^{2t+1}(a).$$

(The process used for obtaining these equations is described in [**12**, Theorem 6.4].)

For double MS_n-algebras, this description can be simplified since some of the 2^{4n+1} equations ($\dagger$) obtained for algebras in $\mathbf{DK}_{n,1}$ hold trivially for algebras in $\mathbf{DMS}_n$. Let $\mathscr{L} = (L, f, g) \in \mathbf{DMS}_n$ and $x \in L$. Then $x \leq f^{2n}(x)$; and, for each $i \in T''_{n,1}$ and each $j \in T'_{n,1}$, we have $g^{2i}(x) \leq f^{2n-2i}(x)$ and $f^{2j+1}(x) \leq g^{2n-2j-1}(x)$ (Corollary 3(iv)).

For F, G, H, J such that $F \subseteq T_{n,1}$; $G, J \subseteq T'_{n,1}$; $H \subseteq T''_{n,1}$, define

$$T''_{F,H} = \{s \in T''_{n,1} \mid s \in H, n - s \notin F\}, \qquad T'_{G,J} = \{t \in T'_{n,1} \mid t \in G, n - 1 - t \notin J\}.$$

We say that

the pair (F, H) satisfies the condition $(0'')$ if $T''_{F,H} = \varnothing$, $n \notin F$ and $0 \in F$:
the pair (F, H) satisfies the condition (i''), for $i \in T''_{n,1}$, if $T''_{F,H} \neq \varnothing$ and $i = \min T''_{F,H}$;
the pair (G, J) satisfies the condition (j'), for $j \in T'_{n,1}$, if $T'_{G,J} \neq \varnothing$ and $j = \min T'_{G,J}$.

THEOREM 4 [**12**, Theorem 6.5]. *Let $\mathscr{L} = (L, f, g) \in \mathbf{DMS}_n$ and $a, b \in L$, $a < b$. Then the principal congruence $\theta(a, b)$ is defined by the equations ($\dagger$) in which (F, H) does not satisfy (i''), $i \in T_{n,1}$, and (G, J) does not satisfy (j'), $j \in T'_{n,1}$.*

Proof. Since $\mathbf{DMS}_n \subset \mathbf{DK}_{n,1}$, the congruence $\theta(a, b)$ is defined by the 2^{4n+1} equations ($\dagger$) above. Consider the following cases.

(a) (F, H) satisfies $(0'')$. Then $0 \in F$, $n \notin F$, hence

$$d_{F,G,H,J}(a,b) \leq a \leq f^{2n}(a) \leq f^{2n}(b) \leq e_{F,G,H,J}(a,b).$$

(b) $\exists i \in T''_{n,1} : (F, H)$ satisfies (i''). Since $i \in H$ and $n - i \notin F$, we get

$$d_{F,G,H,J}(a,b) \leq g^{2i}(a) \leq g^{2i}(b) \leq f^{2n-2i}(b) \leq e_{F,G,H,J}(a,b).$$

(c) $\exists j \in T'_{n,1} : (G, J)$ satisfies (j'). Now, $j \in G$ and $n - 1 - j \notin J$, hence

$$d_{F,G,H,J}(a,b) \leq f^{2j+1}(b) \leq f^{2j+1}(a) \leq g^{2n-2j-1}(a) \leq e_{F,G,H,J}(a,b).$$

In each case, we have $(z \wedge d_{F,G,H,J}(a,b)) \vee e_{F,G,H,J}(a,b) = e_{F,G,H,J}(a,b)$, $\forall z \in L$, therefore the corresponding equation ($\dagger$) holds trivially in L.

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Observe that, if $F \subseteq T_{n,1}$; $H \subseteq T''_{n,1}$ and (F, H) satisfies $(0'')$, then each k such that $1 \le k \le n-1$ satisfies exactly one of the following: $k \in H$, $n-k \in F$; $k \notin H$, $n-k \in F$; $k \notin H$, $n-k \notin F$; moreover, we have either $n \in H$, $0 \in F$ or $n \notin H$, $0 \in F$, and, besides, $n \notin F$. Therefore the number of pairs (F, H) that satisfy $(0'')$ is $\alpha_{n,0} = 3^{n-1} \,.\, 2$.

Also, if (F, H) satisfies (s''), for a given $s \in T''_{n,1}$, we have $s \in H$, $n-s \notin F$ and, for each $k \in T''_{n,1}$ with $k < s$, exactly one of the above cases holds. Therefore there exist $\alpha_{n,s} = 3^{s-1} \,.\, 2^{n+1-s} \,.\, 2^{n-s} = 3^{s-1} \,.\, 2^{2n+1-2s}$ pairs (F, H) satisfying (s'').

Similarly we conclude that the number of pairs (G, J), with $G, J \subseteq T'_{n,1}$, that satisfy (t'), for a given $t \in T'_{n,1}$, is $\beta_{n,t} = 3^t \,.\, 2^{n-1-t} \,.\, 2^{n-1-t} = 3^t \,.\, 2^{2n-2-2t}$.

Corollary 5. [**12**, Corollary 6.6] *Let* $\mathscr{L} = (L, f, g) \in \mathbf{DMS}_n$ and $a, b \in L$. *Then* $\theta(a, b)$ *can be described by* $2^2 \,.\, 3^{2n-1}$ *equations.*

Proof. Since $\theta(a, b) = \theta(a \wedge b, a \vee b)$, we simply consider the case $a < b$. Then $\theta(a, b)$ is defined by the equations (†) in the conditions of Theorem 4.

There are $\alpha_n = 2^{2n+1} - \sum_{s=0}^{n} \alpha_{n,s} = 2^2 \,.\, 3^{n-1}$ pairs (F, H) that do not satisfy (i''), $i \in T_{n,1}$; and there exist $\beta_n = 2^{2n} - \sum_{t=0}^{n-1} \beta_{n,t} = 3^n$ pairs (G, J) that do not satisfy (j'), $j \in T'_{n,1}$. Therefore $\theta(a, G)$ is defined by $\alpha_n \beta_n = 2^2 \,.\, 3^{2n-1}$.

A description of principal congruences in double MS-algebras by means of 12 equations is given in [**7**, Theorem 1].

4. A duality for O_2-algebras.

We develop a topological duality for O_2-algebras which is similar to the duality for Ockham algebras obtained in [**13**].

Definition [**12**]. $X = (X, \mathscr{T}, \le, \varepsilon, \gamma)$ is an O_2-*space* if $(X, \mathscr{T}, \le)$ is a Priestley space (i.e., a compact totally ordered disconnected space) and $\varepsilon, \gamma : X \to X$ are continuous antitone mappings.

Definition. The *dual space* of the algebra $\mathscr{L} = (L, f, g) \in \mathbf{O}_2$ is $Pr_2(\mathscr{L}) = (X_L, \mathscr{T}, \le, \varepsilon_f, \varepsilon_g)$ where

(i) X_L is the set of D_{01}-homomorphisms from L into the two-element chain $\{0, 1\}$;

(ii) $\mathscr{T}$ is the topology induced in X_L by the product topology of $\{0, 1\}^L$;

(iii) $\le$ is the order in X_L given by $h_1 \le h_2$ if and only if $h_1(a) \le h_2(a)$, $\forall a \in L$;

(iv) $\varepsilon_f(h) = chf$ and $\varepsilon_g(h) = chg$, $\forall h \in X_L$ (c denotes complementation in $\{0, 1\}$).

$Pr_2(\mathscr{L})$ is an O_2-space. For $\rho \in \{f, g\}$, $j \in \mathbb{N}$ and $h \in X_L$, we have $\varepsilon^j_\rho(h) = ch\rho^j$ if j is odd and $\varepsilon^j_\rho(h) = h\rho^j$ if j is even. If $\mathscr{L}$ is finite, then $\mathscr{T}$ is the discrete topology in X_L.

Definition. The *dual algebra* of the O_2-space $X = (X, \mathscr{T}, \le, \varepsilon, \gamma)$ is $\mathcal{O}_2(X) = (O(X), f_\varepsilon, f_\gamma)$ where $O(X)$ is the bounded distributive lattice of the clopen order filters of $(X, \mathscr{T}, \le)$, and f_β, $\beta \in \{\varepsilon, \gamma\}$, is the unary operation defined by $f_\beta(Y) = X \setminus \beta^{-1}(Y)$, $\forall Y \in O(X)$.

$\mathcal{O}_2(X)$ is an O_2-algebra. Given $\beta \in \{\varepsilon, \gamma\}$, $j \in \mathbb{N}$, $Y \in O(X)$, we have $f^j_\beta(Y) = X \setminus (\beta^j)^{-1}(Y)$ if j is odd and $f^j_\beta(Y) = (\beta^j)^{-1}(Y)$ if j is even.

For $\mathscr{L} \in \mathbf{O}_2$, the mapping $\Phi: L \to O(X_L)$, defined by $\Phi(a) = \{h \in X_L \mid h(a) = 1\}$,

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$\forall a \in L$, is an isomorphism of O_2-algebras from $\mathscr{L}$ into $\mathcal{O}_2(Pr_2(\mathscr{L}))$. If X is an O_2-space, the mapping $\Psi: X \to X_{O(X)}$, defined by

$$(\Psi(x))(Y) = \begin{cases} 1 & \text{if } x \in Y; \\ 0 & \text{if } x \notin Y. \end{cases}, \quad \forall x \in X, \quad \forall Y \in O(X),$$

is an O_2-homeomorphism between X and $Pr_2(\mathcal{O}_2(X))$.

Given $\mathscr{L}_1$, $\mathscr{L}_2 \in \mathbf{O}_2$, there exists a bijection between the set of homomorphisms from $\mathscr{L}_1$ into $\mathscr{L}_2$ and the set of O_2-continuous mappings from $Pr_2(\mathscr{L}_2)$ into $Pr_2(\mathscr{L}_1)$: just associate to each homomorphism $\phi: L_1 \to L_2$ the mapping $\sigma_\phi: X_{L_2} \to X_{L_1}$ defined by $\sigma_\phi(h) = h\phi$, $\forall h \in X_{L_2}$.

Therefore we have a duality between O_2-algebras (with O_2-homomorphisms) and O_2-spaces (with O_2-continuous mappings), (see [**13**, Theorems 1, 3, 4]).

Let X be a set, $Y \subseteq X$ and $\varepsilon, \gamma: X \to X$ mappings. We denote by $Y_{\varepsilon,\gamma}$ the smallest subset Z of X such that $Y \subseteq Z$, $\varepsilon(Z) \subseteq Z$ and $\gamma(Z) \subseteq Z$, and say that Y is *invariant under* ε *and* γ if $Y_{\varepsilon,\gamma} = Y$.

THEOREM 6. [**12**, Theorem 7.5] *The congruence lattice of an algebra* $\mathscr{L} = (L, f, g) \in \mathbf{O}_2$ *is dually isomorphic to the lattice of all closed subsets of* $Pr_2(\mathscr{L}) = (X_L, \mathscr{T}, \leq, \varepsilon_f, \varepsilon_g)$ *which are invariant under* ε_f *and* ε_g.

Proof. The proof is analogous to that of [**13**, Theorem 5]. Identify $\mathscr{L}$ and $\mathcal{O}_2(Pr_2(\mathscr{L}))$; for each closed invariant subset Y of $Pr_2(\mathscr{L})$, define the relation θ_Y on L by $(a, b) \in \theta_Y$ if and only if $Y \subseteq (a \cap b) \cup ((X_L \setminus a) \cap (X_L \setminus b))$. Then the correspondence associating Y to θ_Y is a dual isomorphism from the lattice of all closed invariant subsets of $Pr_2(\mathscr{L})$ into $\mathrm{Con}(\mathscr{L})$.

Note that, if $\mathscr{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $x, y \in L$ satisfy $\rho^i(x) = \rho^i(y)$, for some $\rho \in \{f, g\}$ and $m + 1 \leq i \leq 2n + m - 1$, then $\rho^m(x) = \rho^m(y)$. Now, for $1 \leq i, j \leq m$, we define the relation $\ker(f^i, g^j)$ on L by $(x, y) \in \ker(f^i, g^j)$ if and only if $f^i(x) = f^i(y)$, $g^j(x) = g^j(y)$. Using Proposition 2 we may easily prove that $\ker(f^i, g^j) \in \mathrm{Con}(\mathscr{L})$.

The results concerning subdirectly irreducible algebras in $\mathbf{O}_2$ are similar to those in [**13**, **2**].

LEMMA 7. [**12**, Lemma 7.6] *Let* $X = (X, \mathscr{T}, \leq, \varepsilon, \gamma)$ *be an* O_2*-space and* Y *a subset of* X.

(i) *If* Y *is invariant under* ε *and* γ, *then so is* $\bar{Y}$.

(ii) $\overline{Y_{\varepsilon,\gamma}}$ *is the smallest closed subset of* X *that contains* Y *and is invariant under* ε *and* γ.

THEOREM 8. [**12**, Theorem 7.7] *Let* $\mathscr{L} = (L, f, g) \in \mathbf{O}_2$ *and* $Pr_2(\mathscr{L}) = (X_L, \mathscr{T}, \leq \varepsilon_f, \varepsilon_g)$. *Then* $\mathscr{L}$ *is subdirectly irreducible if and only if* $\{x \in X_L \mid \overline{Ax}\}_{\varepsilon_f, \varepsilon_g} \neq X_L\}$ *is not dense in* $(X_L, \mathscr{T})$. *In particular, if* L *is finite*, $\mathscr{L}$ *is subdirectly irreducible if and only if there exists* $x \in X_L$ *such that* $\{x\}_{\varepsilon_f, \varepsilon_g} = X_L$.

5. Subdirectly irreducible algebras in $\mathbf{DK}_{n,m}$ and in $\mathbf{DMS}_n$. In order to apply the above duality to determine the subdirectly irreducible algebras in $\mathbf{DK}_{n,m}$ and in $\mathbf{DMS}_n$, we begin by characterizing the dual space of a double $K_{n,m}$-algebra and of a double MS_n-algebra.

THEOREM 9. *Let* $\mathcal{L}=(L,f,g)\in\mathbf{O}_2$ *and* $Pr_2(\mathcal{L})=(X_L,\mathcal{T},\le,\varepsilon_f,\varepsilon_g)$. *Let* $n,m\in\mathbb{N}$ *and* $z=\lceil m/2n\rceil$. *Then*

(i) $\mathcal{L}\in\mathbf{DK}_{n,m}$ *if and only if*

$$\begin{cases}\varepsilon_f^{2n+m}(h)=\varepsilon_f^m(h), & \varepsilon_g^{2n+m}(h)=\varepsilon_g^m(h)\\ \varepsilon_f\varepsilon_g(h)=\varepsilon_f^{2zn}(h), & \varepsilon_g\varepsilon_f(h)=\varepsilon_g^{2zn}(h)\end{cases},\ \forall h\in X_L.$$

(ii) $\mathcal{L}\in\mathbf{DMS}_n$ *if and only if* $\begin{cases}h\le\varepsilon_f^{2n}(h), & \varepsilon_g^{2n}(h)\le h\\ \varepsilon_f\varepsilon_g(h)=\varepsilon_f^{2n}(h), & \varepsilon_g\varepsilon_f(h)=\varepsilon_g^{2n}(h)\end{cases},\ \forall h\in X_L.$

Proof. Observe that $gf=f^{2zn}$ in L if and only if $\varepsilon_f\varepsilon_g=\varepsilon_f^{2zn}$ in X_L (similarly, $fg=g^{2zn}$ if and only if $\varepsilon_g\varepsilon_f=\varepsilon_g^{2zn}$). In fact, if $gf=f^{2zn}$ and $h\in X_L$, we get

$$\varepsilon_f\varepsilon_g(h)=\varepsilon_f(chg)=c(chg)f=h(gf)=hf^{2zn}=\varepsilon_f^{2zn}(h).$$

Conversely, suppose that $\varepsilon_f\varepsilon_g=\varepsilon_f^{2zn}$ and recall that $\mathcal{L}$ and $\mathcal{O}_2(Pr_2(\mathcal{L}))$ are isomorphic algebras. Let $Y\in O(X_L)$. Then

$$gf(Y)=g(X_L\setminus\varepsilon_f^{-1}(Y))=X_L\setminus\varepsilon_g^{-1}(X_L\setminus\varepsilon_f^{-1}(Y))=(\varepsilon_f\varepsilon_g)^{-1}(Y)=(\varepsilon_f^{2zn})^{-1}(Y)=f^{2zn}(Y).$$

Also, for $\rho\in\{f,g\}$ and $p,q\in\mathbb{N}_0$ such that $p\ne q$ and $|p-q|$ is even, $\rho^p(x)\le\rho^q(x)$ holds in L if and only if $\varepsilon_\rho^p(h)\le\varepsilon_\rho^q(h)$, $h\in X_L$, (see [**13**, Theorem 9]).

Hence the double $K_{n,m}$-algebras (resp. double MS_n-algebras) are exactly the algebras $\mathcal{L}\in\mathbf{O}_2$ for which the conditions in (i) (resp. (ii)) hold in $Pr_2(\mathcal{L})$.

PROPOSITION 10. *Let* $\mathcal{L}=(L,f,g)\in\mathbf{DK}_{n,m}$ *and* $Pr_2(\mathcal{L})=(X_L,\mathcal{T},\le,\varepsilon_f,\varepsilon_g)$. *Then*

(i) $\varepsilon_f^j\varepsilon_g^i=\varepsilon_f^{z_{i,j}}$, $\varepsilon_g^i\varepsilon_f^j=\varepsilon_g^{z_{j,i}}$, $1\le i,\ j\le 2n+m-1$ (*in particular*, $\varepsilon_f^j\varepsilon_g^i=\varepsilon_f^{2zn}$, $\varepsilon_g^i\varepsilon_f^j=\varepsilon_g^{2zn}$).

(ii) $\{x\}_{\varepsilon_f,\varepsilon_g}=\{x,\varepsilon_f^i(x),\varepsilon_g^i(x)\mid 1\le i\le 2n+m-1\}$, $\forall x\in X_L$.

Proof. (i) Just translate the properties in Proposition 2 to the dual space of $\mathcal{L}$.

(ii) Apply (i) to check that $Y=\{x,\varepsilon_f^i(x),\ \varepsilon_g^i(x)\mid 1\le i\le 2n+m-1\}$ is invariant under ε and γ. Now it is clear that $\{x\}_{\varepsilon_f,\varepsilon_g}=Y$.

THEOREM 11. *Every subdirectly irreducible algebra in* $\mathbf{DK}_{n,m}$ *is finite. Up to isomorphism, there is only a finite number of subdirectly irreducible algebras in* $\mathbf{DK}_{n,m}$.

Proof. Let $\mathcal{L}=(L,f,g)\in\mathbf{DK}_{n,m}$ be subdirectly irreducible and $Pr_2(\mathcal{L})=(X_L,\mathcal{T},\le,\varepsilon_f,\varepsilon_g)$. By Theorem 8, we have $\overline{\{x\}_{\varepsilon_f,\varepsilon_g}}=X_L$, for some $x\in X_L$. By Proposition 10(ii), $\{x\}_{\varepsilon_f,\varepsilon_g}$ is finite. Hence $\mathcal{L}$ is finite. Since the cardinality of the dual space of a subdirectly irreducible algebra is not greater than $4n+2m-1$, the number of non-isomorphic subdirectly irreducible algebras in $\mathbf{DK}_{n,m}$ is finite.

PROPOSITION 12. *Every subalgebra of a subdirectly irreducible algebra* $\mathcal{L}\in\mathbf{DK}_{n,m}$ *is subdirectly irreducible.*

Proof. Let $\mathcal{L}=(L,f,g)\in\mathbf{DK}_{n,m}$ be subdirectly irreducible, $Pr_2(\mathcal{L})=(X_L,\mathcal{T},\le,\varepsilon_f,\varepsilon_g)$ and $\mathcal{L}_1$ a subalgebra of $\mathcal{L}$. Then $\mathcal{L}$ is finite and there exists $x_0\in X_L$ such that $X_L=\{x_0\}_{\varepsilon_f,\varepsilon_g}$. The inclusion $\mathrm{inc}:L_1\to L$ is an embedding, hence the corresponding O_2-continuous mapping $\sigma_{\mathrm{inc}}:X_L\to X_{L_1}$ is onto. It is easy to check that $X_{L_1}=\{\sigma_{\mathrm{inc}}(x_0)\}_{\varepsilon_f,\varepsilon_g}$. Therefore $\mathcal{L}_1$ is subdirectly irreducible.

https://doi.org/10.1017/S0017089500009745 Published online by Cambridge University Press

We are going to introduce an algebra whose role is particularly important in $\mathbf{DK}_{n,m}$. For each integer t, denote by $r(t)$ the remainder of t on division by $2n$ and let $s(t) = 4n + 2m - 2 - r(2m - 2 - t)$. Consider $X_{n,m} = \{0, 1, 2, \ldots, 4n + 2m - 2\}$ and define the mappings $\varepsilon, \gamma : X_{n,m} \to X_{n,m}$ by

$$\varepsilon(k) = \begin{cases} k - 1 & \text{if } 2n + 1 \leq k \leq 2n + m - 1; \\ r(k-1) & \text{otherwise.} \end{cases}$$

$$\gamma(k) = \begin{cases} k + 1 & \text{if } 2n + m - 1 \leq k \leq 2n + 2m - 3; \\ s(k+1) & \text{otherwise.} \end{cases}$$

Then $\varepsilon(X_{n,m}) = \{0, 1, 2, \ldots, 2n + m - 2\}$ and $\gamma(X_{n,m}) = \{2n + m, 2n + m + 1, \ldots, 4n + 2m - 2\}$.

Lemma 13. *For $j \in \mathbb{N}$ and $k \in X_{n,m}$,*

$$\varepsilon^j(k) = \begin{cases} k - j & \text{if } 2n + j \leq k \leq 2n + m - 1; \\ r(k-j) & \text{otherwise.} \end{cases}$$

$$\gamma^j(k) = \begin{cases} k + j & \text{if } 2n + m - 1 \leq k \leq 2n + 2m - 2 - j; \\ s(k+j) & \text{otherwise.} \end{cases}$$

If $j \geq m$, then $\varepsilon^j(k) = r(k - j)$, $\gamma^j(k) = s(k + j)$, $\forall k \in X_{n,m}$ (in particular, $\varepsilon^{2zn}(k) = r(k)$, $\gamma^{2zn}(k) = s(k)$, $\forall k \in X_{n,m}$).

Proof. By induction on j.

Note that, for $1 \leq j \leq m$,

$$\varepsilon^j(X_{n,m}) = \{k \in X_{n,m} \mid 0 \leq k \leq 2n + m - 1 - j\}$$

and

$$\gamma^j(X_{n,m}) = \{k \in X_{n,m} \mid 2n + m - 1 + j \leq k \leq 4n + m - 2\};$$

for $j \geq m$, $\varepsilon^j(X_{n,m}) = \varepsilon^m(X_{n,m})$ and $\gamma^j(X_{n,m}) = \gamma^m(X_{n,m})$.

Let $\mathbf{X}_{n,m} = (X_{n,m}, \mathcal{T}_d, \leq_T, \varepsilon, \gamma)$ where $X_{n,m} = \{0, 1, 2, \ldots, 4n + 2m - 2\}$, $\mathcal{T}_d$ is the discrete topology, $\leq_T$ is the trivial order and $\varepsilon, \gamma : X_{n,m} \to X_{n,m}$ are the mappings defined above. It is obvious that $\mathbf{X}_{n,m}$ is an O_2-space. Denote by $\mathcal{D}_{n,m}$ the dual algebra of $\mathbf{X}_{n,m}$: the D_{01}-reduct of $\mathcal{D}_{n,m}$ is the lattice $\mathcal{P}(X_{n,m})$ of all subsets of $X_{n,m}$, and the unary operations f_β, $\beta \in \{\varepsilon, \gamma\}$, are defined by $f_\beta(Y) = X_{n,m} \setminus \beta^{-1}(Y)$, $\forall Y \subseteq X_{n,m}$.

Theorem 14. *For $n, m \in \mathbb{N}$, $\mathcal{D}_{n,m}$ is a subdirectly irreducible double $K_{n,m}$-algebra.*

Proof. Let $n, m \in \mathbb{N}$ and $\mathbf{X}_{n,m} = (X_{n,m}, \mathcal{T}_d, \leq_T, \varepsilon, \gamma)$. By Lemma 13 we have $\varepsilon^{2n+m} = \varepsilon^m$, $\gamma^{2n+m} = \gamma^m$, $\varepsilon\gamma = \varepsilon^{2zn}$, $\gamma\varepsilon = \gamma^{2zn}$. Therefore $\mathcal{D}_{n,m} \in \mathbf{DK}_{n,m}$ by Theorem 9(i). Now, $\{2n + m - 1\}_{\varepsilon,\gamma} = X_{n,m}$, hence $\mathcal{D}_{n,m}$ is subdirectly irreducible (Proposition 10(ii), Theorem 8).

We can, in fact, describe $\operatorname{Con}(\mathcal{D}_{n,m})$.

Theorem 15. *Let $n, m \in \mathbb{N}$.*

(i) *Besides $\emptyset$ and $X_{n,m}$, the (closed) subsets of $X_{n,m}$ which are invariant under ε and γ are exactly the sets $Y_{i,j} = \varepsilon^i(X_{n,m}) \cup \gamma^j(X_{n,m})$, $1 \leq i, j \leq m$.*

(ii) $\operatorname{Con}(\mathcal{D}_{n,m}) \cong 1 \oplus (\underline{m} \times \underline{m}) \oplus 1$.

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Proof. (i) Let $1 \le i, j \le m$ and $Y_{i,j} = \varepsilon^i(X_{n,m}) \cup \gamma^j(X_{n,m})$. Then

$$\varepsilon(Y_{i,j}) = \varepsilon^{i+1}(X_{n,m}) \cup \varepsilon^{z_{j,1}}(X_{n,m}) = \varepsilon^{i+1}(X_{n,m}) \cup \varepsilon^m(X_{n,m}) \subseteq \varepsilon^i(X_{n,m}) \subseteq Y_{i,j}.$$

Similarly, $\gamma(Y_{i,j}) \subseteq Y_{i,j}$. Hence $Y_{i,j}$ is invariant under ε and γ. Let $Y \subseteq X_{n,m}$ be a nonempty set invariant under ε and γ. If $2n+m-1 \in Y$, then $X_{n,m} = \{2n+m-1\}_{\varepsilon,\gamma} \subseteq Y$, i.e. $Y = X_{n,m}$. If $2n+m-1 \notin Y$, let $Y_1 = \{k \in Y \mid k \le 2n+m-2\}$ and $Y_2 = \{k \in Y \mid k \ge 2n+m\}$. Then $Y_1 \neq \varnothing$, $Y_2 \neq \varnothing$ and $Y = Y_1 \cup Y_2$. Notice that $\max Y_1 \ge 2n-1$ and $\min Y_2 \le 2n+2m-1$. Let $i = 2n+m-1-\max Y_1$, $j = \min Y_2 - (2n+m-1)$. Now we have $1 \le i, j \le m$, $Y_1 = \varepsilon^i(X_{n,m})$, $Y_2 = \gamma^j(X_{n,m})$; hence $Y = Y_{i,j}$.

(ii) By Theorem 6, $\mathrm{Con}(\mathcal{D}_{n,m})$ is dually isomorphic to the lattice of all (closed) subsets of $X_{n,m}$ which are invariant under ε and γ. The set $\{Y_{i,j} \mid 1 \le i, j \le m\}$, partially ordered by inclusion, is lattice-isomorphic to $\underline{m} \times \underline{m}$ and its non-trivial $\vee$-irreducibles are $Y_{i,m}$, $Y_{m,j}$, $1 \le i, j \le m-1$. Therefore, both $\mathrm{Con}(\mathcal{D}_{n,m})$ and the lattice of invariant subsets of $X_{n,m}$ are isomorphic to the self-dual lattice $1 \oplus (\underline{m} \times \underline{m}) \oplus 1$. (Note that, for $1 \le i, j \le m$, the congruence $\theta_{Y_{i,j}}$ associated with $Y_{i,j}$ in Theorem 6 is just $\ker(f^i_\varepsilon, f^j_\gamma)$).

The importance of $\mathcal{D}_{n,m}$ in $\mathbf{DK}_{n,m}$ is evident in the following result.

THEOREM 16. *Up to isomorphism, each double $K_{n,m}$-algebra is a subalgebra of a direct product of copies of $\mathcal{D}_{n,m}$, i.e., $\mathbf{DK}_{n,m} = SP(\{\mathcal{D}_{n,m}\})$.*

Proof. Let $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $Pr_2(\mathcal{L}) = (X_L, \mathcal{T}, \le \varepsilon_f, \varepsilon_g)$. Identifying $\mathcal{L}$ and $\mathcal{O}_2(Pr_2(\mathcal{L}))$, we shall define an embedding of $\mathcal{L}$ into a direct product of copies of $\mathcal{D}_{n,m}$. For each $x \in X_L$ and $Y \in L$, consider

$$Y^x_{\varepsilon_f} = \{2n+m-1-k \mid 1 \le k \le 2n+m-1,\ \varepsilon^k_f(x) \in Y\},$$
$$Y^x_{\varepsilon_g} = \{2n+m-1+l \mid 1 \le l \le 2n+m-1,\ \varepsilon^l_g(x) \in Y\};$$

and define $\varphi_x : L \to \mathcal{P}(X_{n,m})$ by

$$\varphi_x(Y) = \begin{cases} Y^x_{\varepsilon_f} \cup Y^x_{\varepsilon_g} & \text{if } x \notin Y; \\ Y^z_{\varepsilon_f} \cup Y^x_{\varepsilon_g} \cup \{2n+m-1\} & \text{if } x \in Y. \end{cases}$$

Given $x \in X_L$, it is easily seen that φ_x is a D_{01}-homomorphism and, using Lemma 13, we conclude that $\varphi_x(f(Y)) = f_\varepsilon(\varphi_x(Y))$ and $\varphi_x(g(Y)) = f_\gamma(\varphi_x(Y))$, $\forall Y \in L$. Hence the mapping $\varphi : L \to \prod_{x \in X_L} \mathcal{P}(X_{n,m})$, defined by $\varphi(Y) = (\varphi_x(Y))_{x \in X_L}$, $\forall Y \in L$, is an O_2-homomorphism. For $Y_0, Y_1 \in L$, $Y_0 \neq Y_1$, there exist $i \in \{0, 1\}$ and $x \in X_L$ such that $x \in Y_i$, $x \notin Y_{1-i}$. Then $2n+m-1 \in \varphi_x(Y_i)$ and $2n+m-1 \notin \varphi_x(Y_{1-i})$, i.e., $\varphi(Y_0) \neq \varphi(Y_1)$. Therefore φ is injective.

THEOREM 17. *Up to isomorphism, the subdirectly irreducible algebras in $\mathbf{DK}_{n,m}$ are exactly the subalgebras of $\mathcal{D}_{n,m}$.*

Proof. Since $\mathcal{D}_{n,m}$ is subdirectly irreducible, so are all its subalgebras (Proposition 12). It follows immediately from Theorem 16 that each subdirectly irreducible algebra in $\mathbf{DK}_{n,m}$ is isomorphic to a subalgebra of $\mathcal{D}_{n,m}$.

In order to obtain the subdirectly irreducible algebras in $\mathbf{DMS}_n$, observe that every algebra $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,1}$ has, at least, a subalgebra in $\mathbf{DMS}_n$; the universe of the greatest subalgebra of $\mathcal{L}$ in $\mathbf{DMS}_n$ is $\{x \in L \mid g^{2n}(x) \le x \le f^{2n}(x)\}$. Since $\mathbf{DK}_{n,1}$ is generated by a single subdirectly irreducible algebra, the same is true for $\mathbf{DMS}_n$. Denote by $\mathcal{D}'_n$ the greatest subalgebra of $\mathcal{D}_{n,1}$ that belongs to $\mathbf{DMS}_n$.

https://doi.org/10.1017/S0017089500009745 Published online by Cambridge University Press

COROLLARY 18. *The subdirectly irreducible algebras in* $\mathbf{DMS}_n$ *are, up to isomorphism, the subalgebras of* $\mathcal{D}'_n$. *Therefore* $\mathbf{DMS}_n$ *is generated by* $\mathcal{D}'_n$.

Proof. It follows immediately from Theorem 17: each subalgebra $\mathcal{L}$ of $\mathcal{D}'_n$ is a subalgebra of $\mathcal{D}_{n,1}$, hence $\mathcal{L}$ is subdirectly irreducible; on the other hand, every subdirectly irreducible algebra in $\mathbf{DMS}_n$ is a subalgebra of $\mathcal{D}_{n,1}$ and, hence, of $\mathcal{D}'_n$.

The subdirectly irreducible algebras in $\mathbf{DMS}_1 = \mathbf{DMS}$ were determined in [**6**, Theorem 2.7].

We describe the algebra $\mathcal{D}'_n$. Recall that $\mathcal{D}_{n,1}$ is the dual algebra of $X_{n,1} = (X_{n,1}, \mathcal{T}_d, \leq_T, \varepsilon, \gamma)$ where $X_{n,1} = \{0, 1, 2, \ldots, 4n\}$, and $\varepsilon, \gamma : X_{n,1} \to X_{n,1}$ are defined by $\varepsilon(k) = r(k-1)$ and $\gamma(k) = s(k+1)$, $\forall k \in X_{n,1}$. Then $\mathcal{D}_{n,1} = (\mathcal{P}(X_{n,1}), f_\varepsilon, f_\gamma)$ where f_ε and f_γ are the dual endomorphisms of $\mathcal{P}(X_{n,1})$ induced, respectively, by

$$f_\varepsilon(\{i\}) = \begin{cases} X_{n,1} \setminus \{i+1, i+1+2n\} & \text{if } 0 \leq i \leq 2n-2; \\ X_{n,1} \setminus \{0, 2n, 4n\} & \text{if } i = 2n-1; \\ X_{n,1} & \text{if } 2n \leq i \leq 4n. \end{cases}$$

$$f_\gamma(\{i\}) = \begin{cases} X_{n,1} & \text{if } 0 \leq i \leq 2n; \\ X_{n,1} \setminus \{0, 2n, 4n\} & \text{if } i = 2n+1; \\ X_{n,1} \setminus \{i-1, i-1-2n\} & \text{if } 2n+2 \leq i \leq 4n. \end{cases}$$

$\mathcal{D}'_n$ is the subalgebra of $\mathcal{D}_{n,1}$ whose universe is $D'_n = \{Y \in \mathcal{P}(X_{n,1}) \mid f_\gamma^{2n}(Y) \subseteq Y \subseteq f_\varepsilon^{2n}(Y)\}$. For $Y \in \mathcal{P}(X_{n,1})$, we have

(i) $Y \subseteq f_\varepsilon^{2n}(Y) \Leftrightarrow (\forall k \in X_{n,1}, k \in Y \Rightarrow \varepsilon^{2n}(k) \in Y)$
$\Leftrightarrow (\forall k \in X_{n,1}, k \in Y \Rightarrow r(k) \in Y)$.

(ii) $f_\gamma^{2n}(Y) \subseteq Y \Leftrightarrow (\forall k \in X_{n,1}, \gamma^{2n}(k) \in Y \Rightarrow k \in Y)$
$\Leftrightarrow (\forall k \in X_{n,1}, s(k) \in Y \Rightarrow k \in Y)$.

We say that $Z \subseteq X_{n,1}$ satisfies $(*)$ if $Z = Z' \cup Z'' \cup Z'''$ where

$$Z' \subseteq \{2n+1, 2n+2, \ldots, 4n-1\}, \qquad Z'' = \{r(k) \mid k \in Z'\},$$
$$Z''' \subseteq \{1, 2, \ldots, 2n-1\} \setminus Z''.$$

The elements of $\mathcal{P}(X_{n,1})$ in case (i) are

$$Z,\ Z \cup \{0\},\ Z \cup \{0, 2n\},\ Z \cup \{0, 4n\},\ Z \cup \{0, 2n, 4n\} \text{ where } Z \text{ satisfies } (*);$$

the subsets of $X_{n,1}$ in case (ii) are

$$Z,\ Z \cup \{0\},\ Z \cup \{2n\},\ Z \cup \{0, 2n\},\ Z \cup \{0, 2n, 4n\} \text{ where } Z \text{ satisfies } (*).$$

Hence $D'_n = \{Z, Z \cup \{0\}, Z \cup \{0, 2n\}, Z \cup \{0, 2n, 4n\} \mid Z \text{ satisfies } (*)\}$ and $\mathcal{D}'_n = (D'_n, f_\varepsilon, f_\gamma)$ where f_ε and f_γ are the dual endomorphisms of D'_n whose restriction to $J(D'_n)$ is, respectively,

$$f_\varepsilon(\{0\}) = f_\varepsilon(\{0, 2n\}) = f_\varepsilon(\{0, 2n, 4n\}) = X_{n,1} \setminus \{1, 2n+1\},$$

$$f_\varepsilon(\{i\}) = f_\varepsilon(\{i, i+2n\}) = \begin{cases} X_{n,1} \setminus \{i+1, i+2n+1\} & \text{if } 1 \leq i \leq 2n-2; \\ X_{n,1} \setminus \{0, 2n, 4n\} & \text{if } i = 2n-1. \end{cases}$$

$$f_\gamma(\{i, i+2n\}) = \begin{cases} X_{n,1} \setminus \{0, 2n, 4n\} & \text{if } i = 1; \\ X_{n,1} \setminus \{i-1, i+2n-1\} & \text{if } 2 \leq i \leq 2n-1. \end{cases}$$

$$f_\gamma(\{0, 2n, 4n\}) = X_{n,1} \setminus \{2n-1, 4n-1\},$$

$$f_\gamma(\{i\}) = f_\gamma(\{0, 2n\}) = X_{n,1} \quad 0 \leq i \leq 2n-1.$$

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6. MS_n-algebras which are reduct of double MS_n-algebras. We already observed that each algebra $(L,f) \in \mathbf{K}_{n,m}$ can be extended to, at least, one double $K_{n,m}$-algebra. On the contrary, not every MS_n-algebra can be extended to a double MS_n-algebra, but, whenever it is possible, the extension is unique.

The MS-algebras which are reduct of a double MS-algebra are characterized in [**5**, Theorem 2.2]. We obtain a similar result for MS_n-algebras, $n \in \mathbb{N}$, and the central point is the fact that, for $(L,f,g) \in \mathbf{DMS}_n$, the closure f^{2n} is residuated.

We recall a few notions from [**3**]. Let E, F be partially ordered sets. A mapping $\varphi: E \to F$ is said to be *residuated* if it is isotone and there exists a (unique) isotone mapping $\psi: F \to E$ such that $\psi\varphi \geq \mathrm{id}_E$ and $\varphi\psi \leq \mathrm{id}_F$. The mapping ψ is called the *residual* of φ and is given by $\psi(y) = \max\{x \in E \mid \varphi(x) \leq y\}$, $\forall y \in F$. Moreover, φ preserves suprema and ψ preserves infima. If $E = F$ and φ is a residuated closure, we have $\varphi(x) = \min([x) \cap \mathrm{Im}\,\varphi)$ and $\psi(x) = \max((x] \cap \mathrm{Im}\,\varphi)$, $\forall x \in E$; besides, ψ is a dual closure on E and $\mathrm{Im}\,\psi = \mathrm{Im}\,\varphi$.

A nonempty subset Z of E is said to be *bicomplete* if, for each $x \in E$, $[x) \cap Z$ has a smallest element and $(x] \cap Z$ has a greatest element. The bicomplete subsets of E are exactly the sets $\mathrm{Im}\,\varphi$, where φ is a residuated closure on E. Let Z be a bicomplete subset of E and $\nu; E \to E$ the mapping defined by $\nu(x) = \max((x] \cap Z)$, $\forall x \in E$; then we say that Z is *strong* if ν preserves suprema, [**5**]. Clearly, Z is a strong bicomplete subset if and only if $Z = \mathrm{Im}\,\varphi$ for a residuated closure φ whose residual ψ preserves suprema. Moreover, the following result holds.

LEMMA 19 [**12**, Lemma 5.4]. *Let E be a distributive lattice and φ be a closure on E. Then the following are equivalent*:

(i) $\mathrm{Im}\,\varphi$ *is a strong bicomplete subset of E.*

(ii) $\mathrm{Im}\,\varphi$ *is a bicomplete subset of E and, for every $x \in \mathrm{Im}\,\varphi$, if $x = y \vee z$, with y, $z \in E$, then $x = \psi(y) \vee \psi(z)$.*

Proof. See the proof of the equivalence of the statements (2) and (3) in [**5**, Theorem 2.2]: only properties of closure operators are used, not the particular closure involved.

Note that (i) $\Rightarrow$ (ii) holds in every partially ordered set E, but the converse is not true in general. Consider the lattice E whose Hasse diagram is

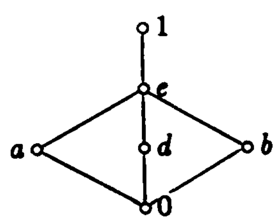

Figure 1.

The mapping φ defined by $\varphi(1) = \varphi(e) = \varphi(a) = \varphi(b) = 1$, $\varphi(d) = d$ and $\varphi(0) = 0$ is a residuated closure on E and its residual ψ is given by $\psi(1) = 1$, $\psi(e) = \psi(d) = d$ and $\psi(a) = \psi(b) = \psi(0) = 0$. Then $\mathrm{Im}\,\varphi = \{0, d, 1\}$ satisfies (ii) and does not satisfy (i) since $d = \psi(e) = \psi(a \vee b) > \psi(a) \vee \psi(b) = 0$.

Now, if $\mathscr{L} = (L,f,g) \in \mathbf{DMS}_n$, it follows from Corollary 3(i) that the closure operator f^{2n} is residuated, its residual being g^{2n}.

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For $(L,f) \in \mathbf{MS}n$, we have $\operatorname{Im} f^{2n} = \operatorname{Im} f$. We present a condition on $\operatorname{Im} f$ which is necessary and sufficient for (L,f) to be a reduct of a (unique) double MS_n-algebra.

THEOREM 20. [**12**, Theorem 5.6] *An algebra $(L,f) \in \mathbf{MS}_n$ can be extended to a double MS_n-algebra if and only if* $\operatorname{Im} f$ *is a strong bicomplete subset of L. In this case, we obtain $(L,f,g) \in \mathbf{DMS}_n$ where $g(x) = f^{2n-1}(\max((x] \cap \operatorname{Im} f))$, $\forall x \in L$.*

Proof. If (L,f) can be extended to a double MS_n-algebra (L,f,g), then f^{2n} is a residuated closure. Moreover, its residual, g^{2n}, is an endomorphism of L. Hence $\operatorname{Im} f$ is a strong bicomplete subset of L. For each $x \in L$, we have $\max((x] \cap \operatorname{Im} f) = g^{2n}(x)$ and, applying Corollary 3(ii), we have $f^{2n-1}(\max((x] \cap \operatorname{Im} f)) = g(x)$. Therefore (L,f) is the reduct of exactly one double MS_n-algebra.

Conversely, let (L,f) be an MS_n-algebra such that $\operatorname{Im} f$ is a strong bicomplete subset of L. Then the closure f^{2n} is residuated and its residual ψ is both an endomorphism and a dual closure on L. We have $\psi(x) = \max((x] \cap \operatorname{Im} f)$, hence $\psi(f(x)) = f(x)$, $\forall x \in L$. The mapping $g: L \to L$, defined by $g(x) = f^{2n-1}(\psi(x))$, $\forall x \in L$, is a dual endomorphism of L. Now $\psi(g(x)) = g(x)$, and $g^i(x) = f^{2n-i}(\psi(x))$, $1 \le i \le 2n$. Hence $g^{2n}(x) = \psi(x) \le x$, $gf(x) = f^{2n-1}(\psi(f(x))) = f^{2n}(x)$ and $fg(x) = f^{2n}(\psi(x)) = \psi(x) = g^{2n}(x)$ so that $(L,f,g) \in \mathbf{DMS}_n$.

From Theorem 20 and using Lemma 19 we now obtain the following corollary.

COROLLARY 21 [**12**, Corollary 5.7] *If $(L,f) \in \mathbf{MS}_n$ can be extended to a double MS_n-algebra, then every element of* $\operatorname{Im} f$ that is *$\vee$-reducible in L is also $\vee$-reducible in* $\operatorname{Im} f$.

Observe that the condition stated above is not sufficient for an MS_n-algebra to be a reduct of a double MS_n-algebra: if L is the chain $-\infty < \cdots < -2 < -1 < 0 < 1 < 2 < \cdots < z < +\infty$ and f is defined by $f(z) = -\infty$, $f(a) = -a$ if $a \ne z$, then $(L,f) \in \mathbf{MS}$ and $\operatorname{Im} f = L \setminus \{z\}$ is not bicomplete ($(z] \cap \operatorname{Im} f$ does not a have a greatest element).

EXAMPLES. (1) It was already pointed out that, if $(L,f) \in \mathbf{K}_{n,0}$, then $(L,f,f^{2n-1}) \in \mathbf{DMS}_n$.

(2) The (non-isomorphic) subdirectly irreducibles in $\mathbf{MS}_2 \setminus \mathbf{MS}$ are the algebras $\mathscr{A}$, $\mathscr{A}_i$, $1 \le i \le 5$, $\mathscr{C}$ and $\mathscr{C}_1$ depicted in [**11**, Theorem 1].

As $\mathscr{A}$, $\mathscr{C} \in \mathbf{K}_{2,0}$, $\mathscr{A}$, $\mathscr{C}$ are reducts of double MS_2-algebras; so are $\mathscr{A}_1$ and $\mathscr{A}_4$ (see [**12**, example 5.3]). The algebra $\mathscr{C}_1 = (C_1, f)$ that generates $\mathbf{MS}_2$ has the Hasse diagram shown in Figure 2 and can be extended to the double MS_2-algebra (C_1, f, g) where g is the dual endomorphism of C_1 induced by $g(a_i) = f^3(a_i)$, $0 \le i \le 3$, and $g(u) = f^3(\max((u] \cap \operatorname{Im} f)) = f^3(a_0 \vee a_2 \vee a_3) = a_0$.

The algebras $\mathscr{A}_2$, $\mathscr{A}_3$ and $\mathscr{A}_5$ are not extendable to double MS_2-algebras; just apply Corollary 21: the element $b = y \vee k$ is $\vee$-reducible in A_2, but is $\vee$-irreducible in $\operatorname{Im} f$; a similar statement holds for the element $d = s \vee k$ both in A_3 and in A_5.

(3) Given $n \in \mathbb{N}$, let L be a direct product of $2n$ finite non-trivial chains. Let a_i, $0 \le i \le 2n-1$, be the maximal elements in $J(L)$ (i.e., the atoms of $C(L)$, the center of L) and consider the dual endomorphism f of L induced by $f(x) = c(a_{r(i+1)})$, $x \in J(L)$, $x \le a_i$, $0 \le i \le 2n-1$ ($c(z)$ denotes the complement of z). Then $(L,f) \in \mathbf{MS}_n$ and $\operatorname{Im} f = C(L)$.

For each $y \in L$, let $w_y = \bigvee \{a_i \mid a_i \le y\}$. It is obvious that $w_y \in (y] \cap \operatorname{Im} f$; if $a \in (y] \cap \operatorname{Im} f$, then $a = w_a \le w_y$, hence $w_y = \max((y] \cap \operatorname{Im} f)$. Moreover, for y, $z \in L$ and

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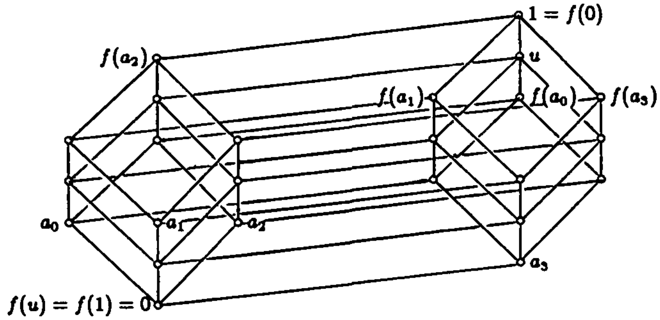

Figure 2.

since a_i, $0 \le i \le 2n-1$, is a $\vee$-irreducible element, we have $w_{y \vee z} = w_y \vee w_z$. Therefore $\operatorname{Im} f$ is a strong bicomplete subset of L. By Theorem 20, we obtain $(L, f, g) \in \mathbf{DMS}_n$ where $g(y) = f^{2n-1}(w_y)$, $\forall y \in L$. Since $w_{a_i} = a_i$, $0 \le i \le 2n-1$, and $w_x = 0$, $\forall x \in J(L) \setminus \{a_i \mid 0 \le i \le 2n-1\}$, we conclude that g is the dual endomorphism of L induced by $g(a_i) = c(a_{r(i-1)})$, $0 \le i \le 2n-1$, and $g(x) = 1$, $\forall x \in J(L) \setminus \{a_i \mid 0 \le i \le 2n-1\}$.

Note that, if $L = \mathbf{4} \times \mathbf{3}^{2n-1}$, the algebra (L, f, g) just described is isomorphic to $\mathcal{D}_n'$.

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Dep. Matemática-Fac. Ciências Lisboa
R. Ernesto de Vasconcelos, Bloco C1
1700 Lisboa
Portugal

Present address:
Dep. Matemática
Universidade do Minho
Largo do Pago
4719 Braga Codex
Portugal

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