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Ramsey Number of Wheels Versus Cycles and Trees

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Abstract. Let $G_1, G_2, \dots, G_t$ be arbitrary graphs. The Ramsey number $R(G_1, G_2, \dots, G_t)$ is the smallest positive integer n such that if the edges of the complete graph K_n are partitioned into t disjoint color classes giving t graphs $H_1, H_2, \dots, H_t$, then at least one H_i has a subgraph isomorphic to G_i. In this paper, we provide the exact value of the $R(T_n, W_m)$ for odd m, $n \ge m-1$, where T_n is either a caterpillar, a tree with diameter at most four, or a tree with a vertex adjacent to at least $\lceil \frac{n}{2} \rceil - 2$ leaves, and W_n is the wheel on $n+1$ vertices. Also, we determine $R(C_n, W_m)$ for even integers n and m, $n \ge m + 500$, which improves a result of Shi and confirms a conjecture of Surahmat et al. In addition, the multicolor Ramsey number of trees versus an odd wheel is discussed in this paper.

1 Introduction

In this paper, we are only concerned with undirected simple finite graphs, and we follow [1] for terminology and notations not defined here. For a graph G, we denote its vertex set, edge set, minimum degree, and chromatic number by $V(G)$, $E(G)$, $\delta(G)$, and $\chi(G)$, respectively. If $v \in V(G)$, we use $\deg_G(v)$ (or simply $\deg(v)$) and $N(v)$ to denote the degree and neighbors of v in G, respectively. The complement graph of a graph G is denoted by $\overline{G}$, and as usual, a complete graph, cycle, path, and a star on n vertices are denoted by K_n, C_n, P_n, and $K_{1,n-1}$, respectively. We also use T_n to denote an arbitrary tree on n vertices. The *wheel* W_m is the graph on $m+1$ vertices obtained from the cycle C_m by adding one vertex x, called the *hub* of the wheel, and making x adjacent to all vertices of C_m, called the *rim* of the wheel. The wheel W_m is called even (odd) if m is even (odd).

For given graphs $G_1, G_2, \dots, G_t$, the *Ramsey number* $R(G_1, G_2, \dots, G_t)$ is the smallest integer n such that if the edges of a complete graph K_n are partitioned into t disjoint color classes giving t graphs $H_1, H_2, \dots, H_t$, then at least one H_i has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's classical result [13]. Since the 1970's, Ramsey theory has grown into one of the most active areas of research within combinatorics, overlapping variously with graph theory, number theory, geometry and logic. For a summary, we refer the reader to the regularly updated survey by Radziszowski [12].

In this paper, we study the Ramsey numbers of odd wheels versus trees and also the Ramsey number of even wheels versus even cycles. Recently, the Ramsey numbers

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of wheels versus trees and cycles have been investigated by several authors. It was shown [2] that $R(T_n, W_5) = 3n - 2$ for $n \geq 4$ and any tree T_n that is not an star. In [5] it was proved that $R(P_n, W_m) = 3n - 2$ for m odd and $n \geq m - 1 \geq 2$ and for the Ramsey number of a star versus a wheel, Chen et al. proved that $R(K_{1,n-1}, W_m) = 3n - 2$ for m odd and $n \geq m - 1 \geq 2$. Furthermore, Baskoro et al. [2] posed the following conjecture.

Conjecture 1.1 *If m is odd and $n \geq m - 1 \geq 6$, then $R(T_n, W_m) = 3n - 2$.*

Also Surahmat et al. [15–17] showed that $R(C_n, W_4) = 2n - 1$ and $R(C_n, W_5) = 3n - 2$ for $n \geq 5$, and in general, $R(C_n, W_m) = 2n - 1$ for even m and $n \geq \frac{5m}{2} - 1$. In view of these results, Surahmat et al. [16, 17] posed the following conjecture.

Conjecture 1.2 *If m is even and $n \geq m \geq 5$, then $R(C_n, W_m) = 2n - 1$.*

In [14], the author improved the result of Surahmat et al. [16, 17] by reducing the lower bound of n from $\frac{5m}{2} - 1$ to $\frac{3m}{2} + 1$, *i.e.*, it is proved that $R(C_n, W_m) = 2n - 1$ for m even and $n \geq \frac{3m}{2} + 1$.

The aim of this paper is to improve the result of Shi [14] for the Ramsey numbers of even wheels versus even cycles by reducing the lower bound of n from $\frac{3m}{2}+1$ to $m+500$, which confirms Conjecture 1.2 for even wheel W_m and even cycle C_n, $n \geq m + 500$. In addition, we provide the exact value of the $R(T_n, W_m)$ for m odd and $n \geq m - 1$, where either T_n has diameter at most four or has a vertex with at least $\lceil \frac{n}{2} \rceil - 2$ leaf neighbors. In Section 3, we will consider the multicolored Ramsey number of trees versus an odd wheel and determine $R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, W_m)$ for odd m and $m \leq \Sigma_{i=1}^{t}(n_i - 1) + 2$.

2 Main Results

We begin with some notation and definitions. A graph G is called *H-free* if it does not contains H as a subgraph. The notation $ex(p, H)$ is defined as the maximum number of edges in a H-free graph on p vertices. The exact value of the $ex(p, C_n)$ is known in some cases. The following theorem can be found in the [1, appendix IV].

Theorem 2.1 ([1]) *If n and p are positive integers such that $n \leq \frac{1}{2}(p + 3)$, then $ex(p, C_n) = \lfloor \frac{p^2}{4} \rfloor$.*

In 1959, Erdős and Gallai [8] proved that every graph G on p vertices and at least $\frac{(n-2)}{2}p + 1$ edges contains a path of order n, *i.e.*, $ex(p, P_n) \leq \frac{(n-2)}{2}p$. Motivated by this result, Erdős and Sós conjectured that if G is a graph on p vertices and more than $\frac{(n-2)}{2}p$ edges, then G contains every tree T on n vertices. In other words, $ex(p, T_n) \leq \frac{(n-2)}{2}p$. Various specific cases of this conjecture have already been proved. Let $\mathcal{F}$ denote the set of all trees satisfying this conjecture. It is proved in [7, 10, 11] that trees with diameter at most four, caterpillars and trees containing a vertex with at least $\lceil \frac{n}{2} \rceil - 2$ leaf neighbors are all in $\mathcal{F}$. Now, we begin with the following theorem. Before that, we note that any graph G with $\delta(G) \geq n - 1$ contains every tree T_n as a subgraph [1].

Theorem 2.2 *If $n \geq m-1$, m odd and $T_n \in \mathcal{F}$, then $R(T_n, W_m) = 3n-2$.*

Proof To see that $R(T_n, W_m) \geq 3n-2$, let $G = 3K_{n-1}$. Clearly G contains no copy of T_n, and $\overline{G}$ contains no copy of W_m, since $\chi(W_m) = 4$ and $\chi(\overline{G}) = 3$.

To see the reverse inequality, we first prove that $R(T_n, C_m) \leq 2n-1$. For this purpose, assume that $H = K_{2n-1}$ is edge-colored red and blue and H^r and H^b are the subgraphs of G induced by the red and blue edges, respectively. We prove that $T_n \subseteq H^r$ or $C_m \subseteq H^b$. Since $T_n \in \mathcal{F}$, thus we may assume that

$$|E(H^r)| \leq \frac{(n-2)}{2}(2n-1),$$

otherwise $T_n \subseteq H^r$. Also, by Theorem 2.1, we may assume that $|E(H^b)| \leq \frac{(2n-1)^2}{4}$. One can easily check that

$$|E(H^r)| + |E(H^b)| < |E(H)| = (2n-1)(n-1),$$

which means that $R(T_n, C_m) \leq 2n-1$.

To prove $R(T_n, W_m) \leq 3n-2$, let $G = K_{3n-2}$ be edge-colored red and blue and let G^r and G^b be subgraphs of G induced by the edges of colors red and blue, respectively. We claim that $T_n \subseteq G^r$ or $W_m \subseteq G^b$. If there exists a vertex $v \in V(G)$ such that $\deg_{G^b}(v) \geq 2n-1$, then $G[N(v)]$ contains a red copy of T_n or a blue copy of C_m, since $R(T_n, C_m) \leq 2n-1$. This yields a red copy of T_n or a blue copy of W_m with hub v in G. Thus, we may assume that $\deg_{G^b}(v) < 2n-1$, for each vertex $v \in V(G)$. This means that $\deg_{G^r} \geq n-1$, and hence we obtain that G^r contains a copy of T_n. This observation completes the proof. ∎

The following corollary follows from Theorem 2.2, which gives some classes of trees satisfying Conjecture 1.1.

Corollary 2.3 *If m is odd and $n \geq m-1$, then $R(T_n, W_m) = 3n-2$, where T_n is either a caterpillar, a star, a tree with diameter at most four, or a tree with a vertex adjacent to t leaves where $t \geq \lceil \frac{n}{2} \rceil - 2$.*

For a graph G, the *circumference* of G, denoted by $c(G)$, is the length of its longest cycle, and the *girth* of G, denoted by $g(G)$, is the length of its shortest cycle. A graph on n vertices is *Hamiltonian* if the circumference of G is n. A graph is called *weakly pancyclic* if it contains cycles of every length between the girth and the circumference. A graph is *pancyclic* if it is weakly pancyclic with girth 3 and circumference n. A graph G of order n is called *panconnected* if every pair of vertices in G is joined by a path of length k for all $1 < k < n$.

In the rest of this section, we prove that $R(C_n, W_m) = 2n-1$, for even integers n, m with $n \geq m+500$. The following results will be used in the proof.

Theorem 2.4 (Brandt et al. [3]) *Let G be a 2-connected non-bipartite graph of order n with minimum degree $\delta(G) \geq \frac{n}{4} + 250$. Then G is weakly pancyclic unless G has odd girth 7, in which case it has cycles of every length from 4 up to its circumference except the pentagon.*

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Theorem 2.5 (Dirac [6]) *Let G be a 2-connected graph of order $n \geq 3$ with $\delta(G) = \delta$. Then $c(G) \geq \min\{2\delta, n\}$.*

Theorem 2.6 (Faudree and Schelp [9]) $R(C_n, C_m) = n + \frac{m}{2} - 1$, *$n$ and m even, and $n \geq m \geq 6$.*

Theorem 2.7 (Faudree and Schelp [9]) *If G is a graph of order n with $\delta(G) \geq n/2+1$, then G is panconnected*

Now, we are ready to determine $R(C_n, W_m)$ when m and n are even integers with $n \geq m + 500$.

Theorem 2.8 *If m and n are even and $n \geq m + 500$, then $R(C_n, W_m) = 2n - 1$.*

Proof To see that $R(C_n, W_m) \geq 2n - 1$, let $G = 2K_{n-1}$. Clearly, G contains no copy of C_n, and $\overline{G}$ contains no copy of W_m, since $\chi(W_m) = 3$ and $\chi(\overline{G}) = 2$. To see the reverse inequality, assume that $G = K_{2n-1}$ is 2-edge colored red and blue and G^r and G^b are subgraphs of G induced by the red and blue edges, respectively. We prove that $C_n \subseteq G^r$ or $W_m \subseteq G^b$. Note that if G^r is bipartite, then one partite set has at least $n \geq m+1$ vertices which implies that $W_m \subseteq G^b$. Thus, we may assume that G^r is a non-bipartite graph. Also if there exists a vertex $v \in V(G)$ such that $\deg_{G^b}(v) \geq n + \frac{m}{2} - 1$, then $G[N(v)]$ contains a red copy of C_n or a blue copy of C_m, since by Theorem 2.6, $R(C_n, C_m) = n + \frac{m}{2} - 1$. This yields a red copy of C_n or a blue copy of W_m with hub v in G.

Therefore, we may assume that $\deg_{G^b}(v) < n + \frac{m}{2} - 1$ for each vertex $v \in V(G)$. This implies that $\delta(G^r) \geq n - \frac{m}{2}$. If G^r is 2-connected, then by Theorem 2.5, $c(G^r) \geq 2n - m \geq n$. Since $\delta(G^r) \geq n - \frac{m}{2} \geq (2n-1)/4 + 250$, by Theorem 2.4 we have $C_n \subseteq G^r$. Thus, we may assume that G^r is not 2-connected. Note that if B is a block of G^r with at least three vertices, then $|V(B)| \geq \frac{m}{2} + 500$, since $\delta(G^r) \geq n - \frac{m}{2}$ and $n \geq m + 500$. Now, we consider the following cases.

Case 1. G^r contains exactly two blocks. Let B_1, B_2 be blocks of G^r with $B_1 \cap B_2 = \{x\}$. Since $\delta(G^r) \geq n - \frac{m}{2}$, we conclude that each of B_1, B_2 contains at least three vertices. Let each B_i, $i = 1, 2$, have b_i vertices and without lose of generality, let $b_1 > b_2$. Since G contains $2n - 1$ vertices, thus $b_1 \geq n$. If $\delta(B_1 \setminus \{x\}) \geq (b_1 - 1)/2 + 1$, then by Theorem 2.7, $B_1 \setminus \{x\}$ is panconnected, and so there exists a path of length $n - 2$, say P, between any two vertices $u, w \in N(x) \cap B_1$. Clearly, $xuPw$ form a copy of C_n in G^r, *i.e.,* $C_n \subseteq B_1 \subseteq G^r$. Thus, we may assume that $B_1 \setminus \{x\}$ contains a vertex v such that $\deg_{B_1 \setminus \{x\}}(v) \leq (b_1 - 1)/2$. Let X be the set of non-neighbors of v in $B_1 \setminus \{x\}$. Clearly, any $m/2$ vertices of $B_2 \setminus \{x\}$ and any $m/2$ vertices of X together with v form a blue copy of W_m with hub vertex v. This means that $W_m \subseteq G^b$.

Case 2. G^r contains at least three blocks. If G^r contains at least three blocks B_1, B_2, B_3 with $|V(B_i)| \geq 3$, $i = 1, 2, 3$, then any $m/2$ vertices of B_1 and B_2 and a vertex $v \in B_3$ form a blue copy of W_m with the hub vertex v. Thus, we have a copy of W_m in G^b unless $G^r = B_1 \cup B_2 \cup B_3$, where B_1 and B_3 are blocks with $|V(B_1)| \geq |V(B_2)| \geq 3$ and B_3 is an edge $e = xy$ joining B_1 and B_3. Let $x \in B_1 \cap B_3$ and each B_i, $i = 1, 2$, has b_i vertices and

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$b_1 > b_2$. Thus, $b_1 \geq n$, since G contains $2n-1$ vertices. If $\delta(B_1 \setminus \{x\}) \geq (b_1-1)/2+1$, then by Theorem 2.7, $B_1 \setminus \{x\}$ is panconnected, and so there exists a path of length $n-2$, say P, between any two vertices $u, w \in N(x) \cap B_1$. Clearly, $xuPw$ is a copy of C_n in G^r, *i.e.*, $C_n \subseteq B_1 \subseteq G^r$. Thus, we may assume that $B_1 \setminus \{x\}$ contains a vertex v such that $\deg_{B_1 \setminus \{x\}}(v) \leq (b_1-1)/2$. Let X be the set of non-neighbors of v in $B_1 \setminus \{x\}$. Clearly, any $m/2$ vertices of $B_2 \setminus \{x\}$ and any $m/2$ vertices of X together with v form a blue copy of W_m with hub vertex v. This means that $W_m \subseteq G^b$, which completes that proof. ∎

3 Multicolored Version

In this section, we will consider the multicolor Ramsey number of trees versus an odd wheel, and we determine $R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, W_m)$ where m is odd and $m \leq \Sigma_{i=1}^{t}(n_i-1)+2$. The exact value of the multicolor Ramsey number for stars is calculated in [4], which is given in the following theorem.

Theorem 3.1 ([4]) *If $R = R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t})$ and $\Sigma = \Sigma_{i=1}^{t}(n_i-1)$, then*

(i) $R = \Sigma + 2$ *if either Σ is odd or Σ is even and all n_i's are odd;*

(ii) $R = \Sigma + 1$ *if Σ is even and at least one n_i is even.*

Hereafter, for given positive integers $n_1, n_2, \ldots, n_t$ we use Σ to denote $\Sigma_{i=1}^{t}(n_i-1)$. In the following theorem, we determine $R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, C_m)$ for odd m, $m \leq \Sigma + 2$.

Theorem 3.2 *Let m be odd, $\Sigma \geq m-2$ and $r = R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t})$. Then*

$$R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, C_m) = 2r - 1.$$

Proof By the definition, there exists a t-edge coloring, say c, of K_{r-1} such that the i-th color class, $1 \leq i \leq t$, contains no copy of K_{1,n_i}. Let A and B be two disjoint copies of K_{r-1} whose edges are colored by t colors $\alpha_1, \alpha_2, \ldots, \alpha_t$ according to c. Now, color the remaining edges of $2K_{r-1}$ (edges between A and B) by an additional color α_{t+1}. Clearly, the induced graph on edges with color α_{t+1} is a bipartite graph and so cannot contain C_m, because m is odd. This observation shows that $R \geq 2r-1$.

Now, assume that K_{2r-1} is edge-colored by colors $\alpha_1, \alpha_2, \ldots, \alpha_{t+1}$ and let H_i, $1 \leq i \leq t+1$, denote the subgraph of K_{2r-1} induced by edges with color α_i. Using Theorem 2.1 and the fact that $K_{1,n_i} \in \mathcal{F}$, $1 \leq i \leq t$, we may assume that

$$|E(H_i)| \leq \frac{n_i-1}{2}(2r-1), \quad |E(H_{t+1})| \leq \Big\lfloor \frac{(2r-1)^2}{4} \Big\rfloor.$$

Using Theorem 3.1, one can easily check that $\sum_{i=1}^{t+1}|E(H_i)| < |E(K_{2r-1})|$, which means that $R \leq 2r-1$. This observation completes that proof. ∎

Finally, using Theorem 3.2, we have the following theorem, which determines the exact value of the $R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, W_m)$ for odd m, $m \leq \Sigma + 2$.

Theorem 3.3 *If m is odd, $\Sigma \geq m-2$ and $r = R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t})$, then*

$$R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, W_m) = 3r - 2.$$

Proof By the definition, there exists a t-edge coloring of K_{r-1}, say c, such that the i-th color class, $1 \leq i \leq t$, contains no copy of K_{1,n_i}. Let A, B, and C be three disjoint copies of K_{r-1} whose edges are colored by t colors $\alpha_1, \alpha_2, \ldots, \alpha_t$ according to c. Now, color the remaining edges of $3K_{r-1}$ (edges between A, B, and C) by an additional color α_{t+1}. Clearly, the induced graph on edges with color α_{t+1} is tripartite and so cannot contain W_m, because $\chi(W_m) = 4$. This observation shows that $R \geq 3r - 2$.

Now, consider an arbitrary $(t+1)$-edge coloring of $G = K_{3r-2}$ by colors $\alpha_1, \alpha_2, \ldots,$ α_{t+1} and let H_i, $1 \leq i \leq t+1$, be the subgraph of K_{3r-2} induced by the edges of color α_i. We assume that $K_{1,n_i} \nsubseteq H_i$, $1 \leq i \leq t$, and we prove that $W_m \subseteq H_{t+1}$. Let H be the subgraph of K_{3r-2} induced by the edges with colors $\alpha_1, \alpha_2, \ldots, \alpha_t$.

Claim. $\delta(H) \leq r - 2$.

On the contrary, let $\delta(H) \geq r-1$. If either Σ is odd or Σ is even and all n_i are odd, then by Theorem 3.1, $r = \Sigma + 2$ and so $\delta(H) \geq \Sigma + 1$, which means that $K_{1,n_i} \subseteq H_i$ for some i, $1 \leq i \leq t$, a contradiction. Thus, let Σ be even and at least one n_i, say n_t, be even. In this case, by Theorem 3.1, $r = \Sigma + 1$ and so each vertex of H must have degree precisely Σ and each color appears exactly $n_i - 1$ times in each vertex of H, because $K_{1,n_i} \nsubseteq H_i$, $1 \leq i \leq t$. Now, H_t is a $(n_t - 1)$-regular graph on $3r-2$ vertices. Since Σ and n_t are even, we are seeking a regular graph of odd order and degree, a contradiction. This contradiction shows that $\delta(H) \leq r - 2$.

Let v be a vertex in H with $\deg_H(v) \leq r-2$ and $G' = G - (N(v) \cup \{v\})$. Clearly, G' has at least $2r-1$ vertices, and so by Theorem 3.2, we have a copy of C_m in color α_{t+1} in G' and hence a copy of W_m in H_{t+1} with the hub v. This observation shows that $R \leq 3r - 2$, which completes that proof. ■

For odd m, one can easily check that if $r = R(T_{n_1}, T_{n_2}, \ldots, T_{n_k})$, then

$$R(T_{n_1}, T_{n_2}, \ldots, T_{n_k}, W_m) \geq 3r - 2.$$

It would be interesting to decide whether this natural lower bound is always the true value of this Ramsey number. We end this section by posing the following conjecture.

Conjecture 3.4 *Let $T_{n_1}, T_{n_2}, \ldots, T_{n_k}$ be trees and $r = R(T_{n_1}, T_{n_2}, \ldots, T_{n_k})$. If m is an odd integer and $m \leq r+1$, then $R(T_{n_1}, T_{n_2}, \ldots, T_{n_k}, W_m) = 3r - 2$.*

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