107.24 How to cut cubes into dodecahedra and icosahedra

Introduction

There are only five regular polyhedra; tetrahedron, cube, octahedron, dodecahedron and icosahedron. Johannes Kepler (1571 – 1630) was interested in these five regular polyhedra together with a rhombic dodecahedron, and explained them in detail in *Epitome of Copernican Astronomy*, published in approximately 1620. (See pages 863 – 868 of [1]. See page 154, [2] for a proof of the existence of only five regular polyhedra. Alsina and Nelsen use the same Kepler's explanations for these polyhedra on page 68 of [3].) In *Epitome*, Kepler constructs two types of dodecahedra; a rhombic dodecahedron (Figure 4b) and a pentagonal regular dodecahedron (see Figures 2b and 4a below) by attaching a roof over each face of the cubes. These descriptions are nice, but the actual construction of a regular dodecahedron by this method is not obvious since there are infinitely many non-regular pentagonal dodecahedra. An icosahedron is described as "dual" to a dodecahedron in *Epitome*.

Notation 1.1: Let Σ be the $2 \times 2 \times 2$ cube bounded by the six planes $x = \pm 1, y = \pm 1$ and $z = \pm 1$.

It is not difficult to make paper regular polyhedral models. But describing how to cut the cube Σ by planes to make them is not easy. A regular tetrahedron can be inscribed in Σ by joining vertices (1,1,1), (−1, −1,1), (1, −1, −1) and (−1,1, −1). And a regular octahedron can also be inscribed in the cube Σ by joining six vertices $\left(1, \frac{1}{2}, 1\right)$, $\left(\frac{1}{2}, 1, -1\right)$, $\left(-1, -\frac{1}{2}, -1\right)$, $\left(-\frac{1}{2}, -1, 1\right)$, $\left(-1, 1, \frac{1}{2}\right)$ and $\left(1, -1, -\frac{1}{2}\right)$. So, a regular tetrahedron and octahedron can be obtained by cutting Σ by planes. This leads us to the following two questions:

Question 1: Is there a nice way to cut the cube Σ by twelve planes to obtain a regular dodecahedron? Similarly, is there a nice way to cut the cube Σ by twenty planes to obtain a regular icosahedron?

We will answer *yes* to these questions in Theorems 1 and 2.

This Note may appear to be totally unrelated to the golden ratio $\varphi = \frac{1}{2}(1 + \sqrt{5})$. However, as you will see, it is rather surprising that the golden ratio appears frequently in this paper.

Notation 1.2: Let $\varphi = \frac{1}{2}(1 + \sqrt{5})$, the golden ratio, and $\bar{\varphi} = \frac{1}{2}(1 - \sqrt{5})$. Both are the roots of the polynomial equation $x^2 - x - 1 = 0$, $\varphi \approx 1.618$ and $\bar{\varphi} \approx -0.618$.

Notation 1.3: The resulting regular dodecahedron by cutting the cube Σ by twelve planes is denoted by $K(\varphi)$ (see Theorem 1). The resulting regular icosahedron by cutting the cube Σ by twenty planes is denoted by $\hat{K}(\varphi^2)$ (see Theorem 2).

Optimising polyhedra inside of a cube has a history. Interested readers may wish to web-search Prince Rupert's problem. Also, see Sections 4.2 and 6.9 in [3]. The largest regular tetrahedron and octahedron contained in the

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cube Σ are the ones described above. We think $\hat{K}(\varphi^2)$ is the largest regular icosahedron inscribed in the cube Σ. So, the following question seems natural.

Question 2: Can $K(\varphi)$ and the rhombic dodecahedron $K(1)$ (see Notation 2.1) be contained in smaller cubes than Σ?

We will answer *yes* to this question in Theorems 3 and 4.

Dodecahedron

The idea to answer Question 1 came from the intersection of three square-cylinders $|y| + |z| = 1$, $|z| + |x| = 1$ and $|x| + |y| = 1$. The intersection is a dodecahedron with 12 identical rhombic faces as in Figure 4b.

Notation 2.1: We call the intersection of three square-cylinders $|y| + |z| = 1$, $|z| + |x| = 1$ and $|x| + |y| = 1$ the *rhombic dodecahedron*, and denote it by $K(1)$ which is the same rhombic dodecahedron described in *Epitome* and this explains how to cut Σ with twelve planes to make $K(1)$.

Notation 2.2: The intersection of three rhombic cylinders $|y| + \frac{1}{a}|z| = 1$, $|z| + \frac{1}{a}|x| = 1$ and $|x| + \frac{1}{a}|y| = 1$, $a > 1$, is a dodecahedron with pentagonal faces, and we denote it by $K(a)$.

Visualising $K(a)$ is rather difficult. Figure 2a is a sketch of the three rhombic cylinders $|y| + \frac{1}{a}|z| = 1$, $|z| + \frac{1}{a}|x| = 1$ and $|x| + \frac{1}{a}|y| = 1$. And Figure 2b is the sketch of $K(a)$.

Figure 2c is the part of $K(a)$ in the first octant. It is bounded by planes $x = 0$, $y = 0$, $z = 0$, $y + \frac{z}{a} = 1$, $z + \frac{x}{a} = 1$ and $x + \frac{y}{a} = 1$. The point $A = \left(\frac{a}{a+1}, \frac{a}{a+1}, \frac{a}{a+1}\right)$ is the intersection of the three planes $y + \frac{z}{a} = 1$, $z + \frac{x}{a} = 1$ and $x + \frac{y}{a} = 1$. The point $B = \left(0, \left(1 - \frac{1}{a}\right), 1\right)$ is the intersection of the three planes $x = 0$, $y + \frac{z}{a} = 1$ and $z + \frac{x}{a} = 1$. The points $C\left(1, 0, \left(1 - \frac{1}{a}\right)\right)$ and $D = \left(\left(1 - \frac{1}{a}\right), 1, 0\right)$ in Figure 2c can be obtained similarly.

The part of $K(a)$ in the second octant, drawn in Figure 2d, is bounded by planes $x = 0$, $y = 0$, $z = 0$, $y + \frac{z}{a} = 1$, $z - \frac{x}{a} = 1$ and $-x + \frac{y}{a} = 1$. Here, $A' = \left(-\frac{a}{a+1}, \frac{a}{a+1}, \frac{a}{a+1}\right)$, $C' = \left(-1, 0, \left(1 - \frac{1}{a}\right)\right)$ and $D' = \left(-\left(1 - \frac{1}{a}\right), 1, 0\right)$.

Two solids in Figures 2c and 2d are mirror images of each other. Hence, the solid bounded by the three rhombic cylinders is made up with four of the solids in Figure 2c, and four of the solids in Figure 2d. This shows that $K(a)$ is a pentagonal dodecahedron having identical pentagonal faces (not necessarily regular) in Figure 2b fitting tangentially inside the $2 \times 2 \times 2$ cubic box Σ. Eight of the 20 vertices of this dodecahedron are $\left(\pm\frac{a}{a+1}, \pm\frac{a}{a+1}, \pm\frac{a}{a+1}\right)$. The remaining twelve vertices are $\left(\pm\left(1 - \frac{1}{a}\right), \pm 1, 0\right)$, $\left(\pm 1, 0, \pm\left(1 - \frac{1}{a}\right)\right)$ and $\left(0, \pm\left(1 - \frac{1}{a}\right), \pm 1\right)$. This also shows that this pentagonal dodecahedron is constructed by building a roof over each of the six faces of the cube having vertices $\left(\pm\frac{a}{a+1}, \pm\frac{a}{a+1}, \pm\frac{a}{a+1}\right)$. Kepler's description (see page 866 of [1] or Section 4.3 of [3]) of a regular dodecahedron is misleading since it seems to suggest that only a regular dodecahedron $K(\varphi)$ and a rhombic dodecahedron $K(1)$ can be constructed by raising roofs over faces of a cube. However, $K(a)$ for any $a > 1$ can be constructed this way. Because of the symmetry, all twelve pentagonal faces of $K(a)$ are identical even though they are not regular pentagons in general.

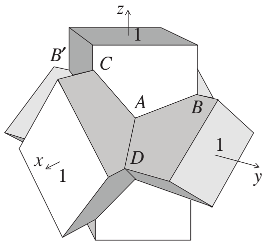

FIGURE 2a: Three intersecting rhombic cylinders

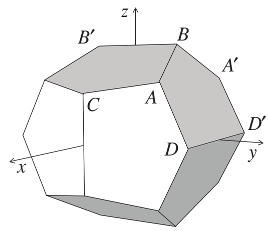

FIGURE 2b: A dodecahedron from Figure 2a

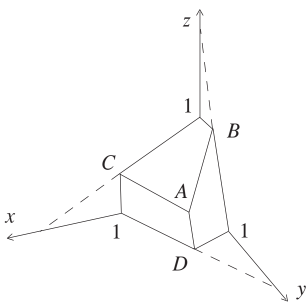

FIGURE 2c: One eighth of Figure 2b dodecahedron

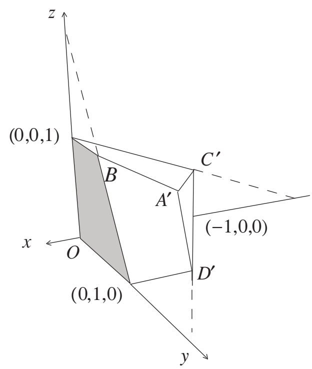

FIGURE 2d: Another one eighth of Figure 2b

Before we state the next theorem, it is useful to know some properties of the golden ratio φ.

Lemma 1:

(a) $\varphi^2 = \frac{1}{2}(3 + \sqrt{5})$,

(b) $\varphi^4 + \varphi^2 + 1 = 4\varphi^2$,

(c) $-\varphi^3 + \varphi^2 - \varphi = -2\varphi$.

Proofs are elementary, and we leave them to the readers.

Theorem 1: The dodecahedron $K(\varphi)$ is a regular dodecahedron with twenty vertices $(\pm\bar{\varphi}, \pm\bar{\varphi}, \pm\bar{\varphi})$, $(\pm\bar{\varphi}^2, \pm 1, 0)$, $(\pm 1, 0, \pm\bar{\varphi}^2)$ and $(0, \pm\bar{\varphi}^2, \pm 1)$. The edge-length of $K(\varphi)$ is $2\bar{\varphi}^2 = 3 - \sqrt{5}$.

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Proof: Let $B' = \left(0, -\left(1 - \frac{1}{a}\right), 1\right)$. Note that $\left\|\overrightarrow{BB'}\right\| = \left\|\overrightarrow{DD'}\right\|$ and $\left\|\overrightarrow{AB}\right\| = \left\|\overrightarrow{AC}\right\| = \left\|\overrightarrow{AD}\right\|$, etc. So, in order for $K(a)$ to be a regular dodecahedron, we must have $\left\|\overrightarrow{AB}\right\| = \left\|\overrightarrow{BB'}\right\|$. But

$$\overrightarrow{AB} = \left\langle -\frac{a}{a+1}, -\frac{1}{a(a+1)}, \frac{1}{a+1} \right\rangle = \frac{1}{a(a+1)} \left\langle -a^2, -1, a \right\rangle$$

so that $\left\|\overrightarrow{AB}\right\| = \frac{1}{a(a+1)}\sqrt{a^4 + a^2 + 1}$ and $\overrightarrow{BB'} = -\frac{2}{a}\langle 0, a - 1, 0 \rangle$ so that $\left\|\overrightarrow{BB'}\right\| = \frac{2}{a}(a - 1)$. Hence, $\left\|\overrightarrow{AB}\right\| = \left\|\overrightarrow{BB'}\right\|$ gives us the equation

$$\frac{1}{a(a+1)}\sqrt{a^4 + a^2 + 1} = \frac{2}{a}(a - 1).$$

This simplifies to $a^4 - 3a^2 + 1 = 0$. By noting that $a > 0$ and by the quadratic formula, the solution to this polynomial is $a = \sqrt{\dfrac{3 + \sqrt{5}}{2}} = \dfrac{1 + \sqrt{5}}{2} = \varphi$, by Lemma 1.

By assuming $a = \varphi$, we will show that each face is a regular pentagon. Let $\alpha = \angle BAC$ and $\beta = \angle ABB'$. Then the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel to $\langle -\varphi^2, -1, \varphi \rangle$ and $\langle \varphi, -\varphi^2, -1 \rangle$, respectively. Hence,

$$\cos\alpha = \frac{-\varphi^3 + \varphi^2 - \varphi}{\varphi^4 + \varphi^2 + 1} = \frac{-2\varphi}{4\varphi^2} = \frac{-1}{2\varphi} = \frac{1}{2}\bar{\varphi}$$

by the dot product and Lemma 1. The vectors $\overrightarrow{BB'}$ and $\overrightarrow{BA}$ are parallel to $\langle 0, -1, 0 \rangle$ and $\langle \varphi^2, 1, -\varphi \rangle$, respectively. Hence,

$$\cos\beta = \frac{-1}{\sqrt{\varphi^4 + \varphi^2 + 1}} = \frac{-1}{\sqrt{4\varphi^2}} = \frac{-1}{2\varphi} = \frac{1}{2}\bar{\varphi}.$$

This shows that $\alpha = \beta$. By the symmetry, we know that the face containing vertices A, B and C is a pentagon with equal edge-lengths and equal angles so that it is a regular pentagon. Since each face is congruent, this shows that the solid is a regular dodecahedron. Noting that $\dfrac{\varphi}{\varphi + 1} = \dfrac{1}{\varphi} = -\bar{\varphi}$ and $1 - \dfrac{1}{\varphi} = \bar{\varphi}^2$, we have proved the theorem.

Remark 2.1: The rhombic dodecahedron $K(1)$ can be thought of as the limiting case of $K(a)$ as $a \to 1$. And $K(a)$ approaches Σ as $a \to \infty$. Note that, for any $a \geqslant 1$, $K(a)$ is contained in the $2 \times 2 \times 2$ cube Σ, and can be thought of as a dodecahedron obtained by raising a roof over each face of the cube having vertices $\left(\pm\frac{a}{a+1}, \pm\frac{a}{a+1}, \pm\frac{a}{a+1}\right)$.

Remark 2.2: (A construction of $K(a)$, $a \geqslant 1$, from a cube.) On the faces of a cube, draw three lines as in Figure 2e. Draw three additional lines on the opposite faces so the lines are parallel to the opposite ones. Draw slanted lines at the angle $\tan^{-1}(a)$ from the floor to the mid-line. (If the angle is $\tan^{-1}(\varphi)$, we obtain a *regular dodecahedron*. If the angle is $\tan^{-1}(1) = \frac{\pi}{4}$, we obtain a *rhombic dodecahedron*.) Repeat this on each face and connect

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the bottom of the slanted lines as in Figure 2f. Then cut along the angled lines in Figure 2f. As an intermediary step, you may get a solid as in Figure 2g. When you finish cutting all with *twelve* planes, you will obtain a dodecahedron as in Figure 2b.

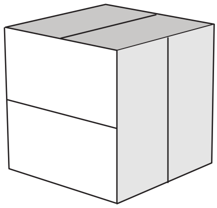

FIGURE 2e: A box with mid-line

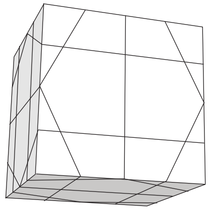

FIGURE 2f: Lines with the angle $\tan^{-1} a$

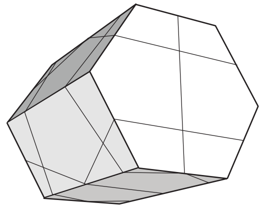

FIGURE 2g: After four cuts were made

Icosahedron

We will construct an icosahedron from $K(a)$. By the second Section, the vertices of $K(a)$ are $\left(\pm\frac{a}{a+1}, \pm\frac{a}{a+1}, \pm\frac{a}{a+1}\right)$, $\left(\pm\left(1-\frac{1}{a}\right), \pm1, 0\right)$, $\left(\pm1, 0, \pm\left(1-\frac{1}{a}\right)\right)$ and $\left(0, \pm\left(1-\frac{1}{a}\right), \pm1\right)$. Let A, B, B', C, C' and D, D' be all the same as in the second Section. Notice that the vertices B, C, D form an equilateral triangle BCD. So, we remove the tetrahedron $ABCD$ from $K(a)$ by cutting along the triangle BCD (see Figure 3b). By repeating this eight times in each octant, we remove eight tetrahedra from $K(a)$, each having one of

$$\left(\pm\frac{a}{a+1}, \pm\frac{a}{a+1}, \pm\frac{a}{a+1}\right)$$

as a vertex. The figure in Figure 3a is the dodecahedron $K(a)$ with lines of cuts drawn on its faces. The result is an icosahedron, denoted by $\hat{K}(a)$, in Figure 3c. It has twelve *isosceles* triangular faces inherited from the twelve faces of $K(a)$ congruent to the triangle BCB', and eight *equilateral* triangular faces congruent to the triangle BCD. Hence, $\hat{K}(a)$ is an icosahedron. Also, note that $\hat{K}(a)$ has twelve vertices since eight vertices $\left(\pm\frac{a}{a+1}, \pm\frac{a}{a+1}, \pm\frac{a}{a+1}\right)$ were eliminated from twenty vertices of $K(a)$. Figure 3c shows the icosahedron $\hat{K}(a)$ with twelve vertices $\left(\pm\left(1-\frac{1}{a}\right), \pm1, 0\right)$, $\left(\pm1, 0, \pm\left(1-\frac{1}{a}\right)\right)$ and $\left(0, \pm\left(1-\frac{1}{a}\right), \pm1\right)$.

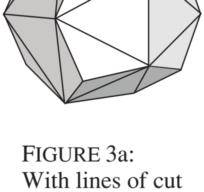

FIGURE 3a: With lines of cut

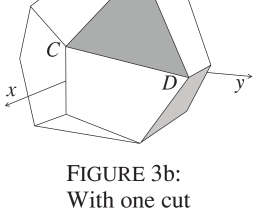

FIGURE 3b: With one cut

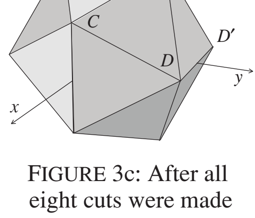

FIGURE 3c: After all eight cuts were made

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Remark 3.1: If we let $a \to 1$, then $\hat{K}(a)$ becomes the regular octahedron bounded by $|x| + |y| + |z| = 1$. Since $K(a) \to \Sigma$ as $a \to \infty$, $\hat{K}(a)$ becomes Σ as $a \to \infty$.

Theorem 2: The icosahedron $\hat{K}(\varphi^2)$ is regular. The twelve vertices of $\hat{K}(\varphi^2)$ are $(\pm\bar{\varphi}, \pm 1, 0)$, $(\pm 1, 0, \pm\bar{\varphi})$ and $(0, \pm\bar{\varphi}, \pm 1)$. The regular icosahedron $\hat{K}(\varphi^2)$ has edge-length $2|\bar{\varphi}| = \sqrt{5} - 1$.

Proof: In order for $\hat{K}(a)$ to be a regular icosahedron, all we have to do is to make all 12 isosceles triangular faces to be equilateral since two identical edges of an isosceles triangle are the edges of equilateral triangular faces. In particular, we want the isosceles triangle BCB' to be equilateral. Since

$$\left\|\overrightarrow{BC}\right\| = \left\|\left\langle 1, 0, 1 - \frac{1}{a}\right\rangle - \left\langle 0, 1 - \frac{1}{a}, 1\right\rangle\right\| = \left\|\left\langle 1, -1 + \frac{1}{a}, -\frac{1}{a}\right\rangle\right\|$$
$$= \tfrac{1}{a}\sqrt{2 - 2a + 2a^2}$$

and $\left\|\overrightarrow{BB'}\right\| = 2\left(1 - \frac{1}{a}\right)$ must be the same, we have $\frac{1}{a}\sqrt{2 - 2a + 2a^2} = 2\left(1 - \frac{1}{a}\right)$ or $a^2 - 3a + 1 = 0$. Solving it for $a > 1$, we have

$$a = \frac{3 + \sqrt{5}}{2} = \left(\frac{1 + \sqrt{5}}{2}\right)^2 = \varphi^2$$

by Lemma 1. Since $2 - \frac{1}{\varphi^2} = \varphi$ and $1 - \frac{1}{\varphi^2} = -\bar{\varphi}$, we have proved the theorem.

Remark 3.2: By joining each vertex to the origin, the regular icosahedron $\hat{K}(\varphi^2)$ can be thought of as a union of 20 identical tetrahedra. The distance from its vertex $(1, 0, -\bar{\phi})$ to the origin is $\sqrt{1 + \bar{\varphi}^2}$. The edge-length of $\hat{K}(\varphi^2)$ is $2|\bar{\varphi}|$. Since $\sqrt{1 + \bar{\varphi}^2} < 2|\bar{\varphi}|$, these 20 tetrahedra are not regular even though they are very close to being regular.

Remark 3.3: The description of $\hat{K}(a)$, $a > 1$, at the beginning of this section shows how to cut the dodecahedron $K(\varphi^2)$ by removing eight tetrahedra to obtain the regular icosahedron $\hat{K}(\varphi^2)$. The angle of cuts to obtain $K(\varphi^2)$ in Figure 2f is $\tan^{-1}(\varphi^2)$. It is interesting to note that the angles between two adjacent faces of a regular dodecahedron and icosahedron are given by $2\tan^{-1}(\varphi)$ and $2\tan^{-1}(\varphi^2)$, respectively.

Cubes smaller than Σ that contain $K(\varphi)$ and $K(1)$

We will show that the regular dodecahedron $K(1)$ and the rhombic dodecahedron can be contained in cubes slightly smaller than Σ. The proofs of Theorems 3 and 4 are rather technical.

Notation 4a: $\alpha = \sqrt{2} - 1$.

Theorem 3: The cubic box $\frac{2\varphi}{1 + \alpha\varphi} \times \frac{2\varphi}{1 + \alpha\varphi} \times \frac{2\varphi}{1 + \alpha\varphi}$ contains the regular dodecahedron $K(\varphi)$. Note that $\frac{2\varphi}{1 + \alpha\varphi} \approx 1.9375$.

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Proof: Let $A = (0, \bar{\varphi}^2, 1)$, $B = (0, -\bar{\varphi}^2, 1)$; and let the symmetric points with respect to the origin be $A' = (0, -\bar{\varphi}^2, -1)$, $B' = (0, \bar{\varphi}^2, -1)$.

Let $C = (-\bar{\varphi}, -\bar{\varphi}, -\bar{\varphi})$, $D = (\bar{\varphi}^2, 1, 0)$; $C = (\bar{\varphi}, \bar{\varphi}, \bar{\varphi})$, $D = (-\bar{\varphi}^2, -1, 0)$ and $E = (-\bar{\varphi}, \bar{\varphi}, -\bar{\varphi})$, $F = (\bar{\varphi}^2, -1, 0)$; $E' = (\bar{\varphi}, -\bar{\varphi}, \bar{\varphi})$, $F' = (-\bar{\varphi}^2, 1, 0)$. These are vertices of $K(\varphi)$ by Theorem 1. (See Figure 4a. This figure is the same as the one in Figure 2b. But vertices are labelled differently.)

The segments (AB and $A'B'$), (CD and $C'D'$) and (EF and $E'F'$) are pairwise parallel. Moreover, $\overrightarrow{AB} = -2\bar{\varphi}^2 \cdot \vec{u}$, $\overrightarrow{CD} = \bar{\varphi} \cdot \vec{v}$ and $\overrightarrow{EF} = \bar{\varphi} \cdot \vec{w}$, where $\vec{u} = \langle 0, 1, 0 \rangle$, $\vec{v} = \left\langle \bar{\varphi} + 1, \frac{1 + \bar{\varphi}}{\bar{\varphi}}, 1 \right\rangle = \langle \bar{\varphi}^2, \bar{\varphi}, 1 \rangle$ and $\vec{w} = \langle \bar{\varphi}^2, -\bar{\varphi}, 1 \rangle$. Let $\alpha = \sqrt{2} - 1$, $\vec{p} = \langle \alpha, 0, -\varphi \rangle$, $\vec{q} = \langle \varphi, 1 + \alpha\varphi, \alpha \rangle$ and $\vec{r} = \langle \varphi, -(1 + \alpha\varphi), \alpha \rangle$.

(*) Note that $\varphi\bar{\varphi} = -1$ and $\alpha^2 + \varphi^2 = \frac{1}{2}(9 + \sqrt{5} - 4\sqrt{2}) = (1 + \alpha\varphi)^2$.

Using (*), we can verify the following dot products:

$$\vec{u} \cdot \vec{p} = \vec{v} \cdot \vec{q} = \vec{w} \cdot \vec{r} = \vec{p} \cdot \vec{q} = \vec{q} \cdot \vec{r} = \vec{r} \cdot \vec{p} = 0.$$

The part $\vec{u} \cdot \vec{p} = \vec{v} \cdot \vec{q} = \vec{w} \cdot \vec{r} = 0$ shows that

(1) the segment AB is contained in the plane $\Gamma : \alpha x - \varphi(z - 1) = 0$ that passes through A with the normal vector $\vec{p}$,

(2) segment $A'B'$ is contained in the plane $\hat{\Gamma} : \alpha x - \varphi(z + 1) = 0$ that passes through A' with the normal vector $\vec{p}$,

(3) the segment CD is contained in the plane

$$\Lambda : \varphi(x + \bar{\varphi}) + (1 + \alpha\varphi)(y + \bar{\varphi}) + \alpha(z + \bar{\varphi}) = 0$$

through C with the normal vector $\vec{q}$,

(4) segment $C'D'$ is contained in the plane

$$\hat{\Lambda} : \varphi(x - \bar{\varphi}) + (1 + \alpha\varphi)(y - \bar{\varphi}) + \alpha(z - \bar{\varphi}) = 0$$

through C' with the normal vector $\vec{q}$,

(5) the segment EF is contained in the plane

$$\Omega : \varphi(x + \bar{\varphi}) - (1 + \alpha\varphi)(y - \bar{\varphi}) + \alpha(z + \bar{\varphi}) = 0$$

through E with the normal vector $\vec{r}$, and

(6) the segment $E'F'$ is contained in the plane through E' with the normal vector $\vec{r}$; we denote this plane by $\hat{\Omega}$.

So, the planes (Γ and $\hat{\Gamma}$), (Λ and $\hat{\Lambda}$) and (Ω and $\hat{\Omega}$) are pairwise parallel. Since $\vec{p} \cdot \vec{q} = \vec{q} \cdot \vec{r} = \vec{r} \cdot \vec{p} = 0$, the planes Γ, Λ, Ω are mutually perpendicular. Hence, the six planes Γ, $\hat{\Gamma}$, Λ, $\hat{\Lambda}$, Ω, $\hat{\Omega}$ bound a rectangular box. Again, by (*), we have the following:

$$d(\Gamma, \hat{\Gamma}) = d(\Gamma, A') = \frac{2\varphi}{\sqrt{\alpha^2 + \varphi^2}} = \frac{2\varphi}{1 + \alpha\varphi},$$

$$d(\Lambda, \hat{\Lambda}) = d(\Lambda, D') = \frac{|\varphi \cdot 2\bar{\varphi} + 2\bar{\varphi}(1 + \alpha\varphi) + 2\alpha\bar{\varphi}|}{\sqrt{\alpha^2 + (1 + \alpha\varphi)^2 + \varphi^2}} = \frac{2\varphi\sqrt{2}}{\sqrt{2}(1 + \alpha\varphi)} = \frac{2\varphi}{(1 + \alpha\varphi)}$$

and

$$d(\Omega, \hat{\Omega}) = d(\Omega, F') = \frac{|\varphi \cdot 2\bar{\varphi} + 2\bar{\varphi}(1 + \alpha\varphi) + 2\alpha\bar{\varphi}|}{\sqrt{\alpha^2 + (1 + \alpha\varphi)^2 + \varphi^2}} = \frac{2\varphi}{(1 + \alpha\varphi)}.$$

This shows that the rectangular box bounded by six planes Γ, $\hat{\Gamma}$, Λ, $\hat{\Lambda}$, Ω, $\hat{\Omega}$ forms a $\frac{2\varphi}{1+\alpha\varphi} \times \frac{2\varphi}{1+\alpha\varphi} \times \frac{2\varphi}{1+\alpha\varphi}$ cubic box. From the construction, all points of $K(\varphi)$ are contained in the inside or on the surface of the $\frac{2\varphi}{1+\alpha\varphi} \times \frac{2\varphi}{1+\alpha\varphi} \times \frac{2\varphi}{1+\alpha\varphi}$ cubic box. This proves the theorem.

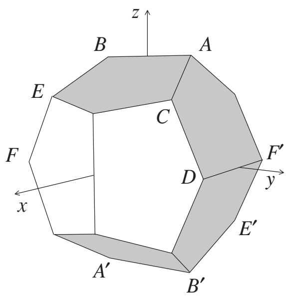

FIGURE 4a:
Labelling for Theorem 4.1

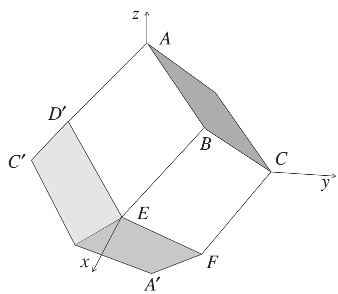

FIGURE 4b:
Labelling for Theorem 4.2

Since $K(1)$ is contained in $K(\varphi)$, we can see the rhombic dodecahedron can be inscribed in a cube smaller than a $2 \times 2 \times 2$ cube. But we can do better.

Theorem 4: A $\varphi \times \varphi \times \varphi$ cube can contain the rhombic dodecahedron $K(1)$.

Proof: The idea of this proof is similar to the proof of Theorem 3.
Let $A = (0,0,1), B = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right); A' = (0, 0, -1), B = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$.
Let $C = (0,1,0), D = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right); C' = (0, -1, 0), D' = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$,
and $E = (1,0,0), F = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right); E' = (-1,0,0), F' = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.
These are vertices of $K(1)$. See Figure 4b.

The segments (AB and $A'B'$), (CD and $C'D'$) and (EF and $E'F'$) are pairwise parallel, and symmetric with respect to the origin. We have $\overrightarrow{AB} = \frac{1}{2}\vec{u}$, $\overrightarrow{CD} = -\frac{1}{2}\vec{v}$ and $\overrightarrow{EF} = \frac{1}{2}\vec{w}$, where $\vec{u} = \langle 1,1,-1\rangle$, $\vec{v} = \langle 1,1,1\rangle$ and $\vec{w} = \langle -1,1,-1\rangle$. Let $\vec{p} = \langle 1, \varphi, \varphi^2\rangle$, $\vec{q} = \langle 1, -\varphi, -\bar{\varphi}\rangle$ and $\vec{r} = \langle 1, \bar{\varphi}^2, \bar{\varphi}\rangle$. Then we can verify the following dot products:

$$\vec{u}\cdot\vec{p} = \vec{v}\cdot\vec{q} = \vec{w}\cdot\vec{r} = \vec{p}\cdot\vec{q} = \vec{q}\cdot\vec{r} = \vec{r}\cdot\vec{p} = 0.$$

The first part of three dot products: $\vec{u}\cdot\vec{p} = \vec{v}\cdot\vec{q} = \vec{w}\cdot\vec{r} = 0$ shows the following:

(1) the segment AB is contained in the plane $\Gamma : x + \varphi y + \varphi^2(z - 1) = 0$ that passes through A with the normal vector $\vec{p}$,

(2) the segment $A'B'$ is contained in the plane, call it $\hat{\Gamma}$, through A' with the normal vector $\vec{p}$,

(3) the segment CD is contained in the plane $\Lambda : x - \varphi(y - 1) - \bar{\varphi}z = 0$ through C with the normal vector $\vec{q}$,

(4) the segment $C'D'$ is contained in the plane, call it $\hat{\Lambda}$, through C' with the normal vector $\vec{q}$,

(5) the segment EF is contained in the plane $\Omega : (x - 1) + \bar{\varphi}^2 y + \bar{\varphi}z = 0$ through E with the normal vector $\vec{r}$, and

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(6) the segment $E'F'$ is contained in the plane $\hat{\Omega}$ through E' with the normal vector $\vec{r}$.

So the six planes $(\Gamma, \hat{\Gamma})$, $(\Lambda, \hat{\Lambda})$, $(\Omega, \hat{\Omega})$ are pairwise parallel.

Since $\vec{p}\cdot\vec{q} = \vec{q}\cdot\vec{r} = \vec{r}\cdot\vec{p} = 0$, three planes Γ, Λ, Ω are mutually perpendicular. Hence, planes Γ, $\hat{\Gamma}$, Λ, $\hat{\Lambda}$, Ω, $\hat{\Omega}$ bound a rectangular box. Moreover, we have

$$d\left(\Gamma, \hat{\Gamma}\right) = d\left(\Gamma, A'\right) = \frac{\left|\varphi^2(-1 - 1)\right|}{\sqrt{1 + \varphi^2 + \varphi^4}} = \frac{2\varphi^2}{\sqrt{4\varphi^2}} = \varphi,$$

$$d\left(\Lambda, \hat{\Lambda}\right) = d\left(\Lambda, C'\right) = \frac{|2\varphi|}{\sqrt{1 + \varphi^2 + \bar{\varphi}^2}} = \frac{2\varphi}{\sqrt{4}} = \varphi, \text{ and}$$

$$d\left(\Omega, \hat{\Omega}\right) = d\left(\Omega, E'\right) = \frac{|-2|}{\sqrt{1 + \bar{\varphi}^2 + \bar{\varphi}^4}} = \frac{|-2|}{\sqrt{(2\bar{\varphi})^2}} = \left|\frac{-2}{2}(-\varphi)\right| = |\varphi| = \varphi.$$

So, these six planes Γ, $\hat{\Gamma}$, Λ, $\hat{\Lambda}$, Ω, $\hat{\Omega}$ bound a $\varphi \times \varphi \times \varphi$ cube. From the construction, all vertices of $K(1)$ lie on or inside of the $\varphi \times \varphi \times \varphi$ cubic box. Therefore, the theorem follows.

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107.25 A refinement of Griffiths' formula for the sums of the powers of an arithmetic progression

Consider the sum of k-th powers of the terms of an arithmetic progression with first term a and common difference d

$$S_k^{a,d}(n) = a^k + (a + d)^k + (a + 2d)^k + \ldots + (a + (n - 1)d)^k,$$

where k, a, d and n are assumed to be integer variables with $k, a \geqslant 0$ and $d, n \geqslant 1$. In [1], Griffiths derived the following polynomial formula:

$$S_k^{a,d}(n) = \sum_{m=1}^{k+1} \sum_{r=m}^{k+1} \frac{1}{r} d^{k-m} \left\{ {k \atop r - 1} \right\} \overline{\left[{r \atop m} \right]} \left((a + nd)^m - a^m \right), \qquad (1)$$