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TRACES OF CERTAIN CLASSES OF HOLOMORPHIC FUNCTIONS ON FINITE UNIONS OF CARLESON SEQUENCES

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Abstract. We give a method allowing the generalization of the description of trace spaces of certain classes of holomorphic functions on Carleson sequences to finite unions of Carleson sequences. We apply the result to different classes of spaces of holomorphic functions such as Hardy classes and Bergman type spaces.

0. Introduction. Let $D = \{z \in C : |z| < 1\}$ be the unit disk and Hol(D) the space of holomorphic functions on D. For a space $X \subset \mathrm{Hol}(D)$ we define the trace space

$$X|_\Lambda = \{f|_\Lambda : f \in X\} \tag{1}$$

and the sequence space

$$X(\Lambda) = \{(f(\lambda_n))_{n\geq 1} : f \in X\}. \tag{2}$$

Using a decomposition method for Λ, we shall generalize the description of $X|_\Lambda$ if Λ is a Carleson sequence to finite unions of Carleson sequences. In order to do this we will need a certain stability condition that will be introduced in the first section. There we also give the characterization of the trace space of X on a finite union of Carleson sequences under the additional condition $H^\infty X \subset X$, where $H^\infty = \{f \in \mathrm{Hol}(D) : \|f\|_\infty = \sup_{z\in D} |f(z)| < \infty\}$ is the Hardy space of bounded analytic functions on D.

In the second section we shall apply the general characterization obtained in the first section to various classes of spaces of analytic functions on D. We thus obtain a new approach to the characterization of the traces of Hardy spaces on finite unions of Carleson sequences (cf. [**6**] and [**1**]). Also we give a hitherto unknown description of the traces of Bergman type spaces on finite unions of Carleson sequences.

1. The general result. We introduce the pseudohyperbolic metric with the aid of Möbius transformations. For $\lambda \in D$ put $b_\lambda(z) = \frac{|\lambda|}{\lambda}\frac{\lambda - z}{1-\bar{\lambda}z}$, $z \in D$, and define the pseudohyperbolic distance by $\rho(\lambda, \mu) = |b_\lambda(\mu)|$ for $\lambda, \mu \in D$ (cf. e.g.[**5**]). The corresponding neighbourhood $\Omega(\lambda, \rho)$ for $\lambda \in D$ and $0 < \rho < 1$ is given by

$$\Omega(\lambda, \rho) = \{z \in D : |b_\lambda(z)| < \rho\}.$$

We say that a sequence $\Lambda=\{\lambda_n\}_{n\geq 1}\subset D$ satisfies the *Carleson condition*, if

$$\inf_{\lambda\in\Lambda}\prod_{\substack{\mu\in\Lambda\\ \mu\neq\lambda}}|b_\mu(\lambda)|=\delta_\Lambda>0.$$

In this case we write $\Lambda\in(C)$ and call Λ a *Carleson sequence*. In what follows we will refer to δ_Λ as the *Carleson constant*. It is well known (cf.[**3**]) that $H^\infty|_\Lambda=l^\infty_\Lambda$ if and only if $\Lambda\in(C)$. Here $l_\Lambda{}^\infty$ is the space of bounded functions on Λ.

In this paper we are concerned with finite unions of Carleson sequences and for these we have the following result (cf.[**14**] and [**6**]).

PROPOSITION 1.1. *If $\Lambda=\bigcup_{i=1}^N\Lambda_i$, where $\Lambda_i\in(C)$, then for every $0<\eta<1$ there exists a partition $\Lambda=\bigcup_{n\geq 1}\sigma_n$ with the following properties.*

(i) $\sup_{n\geq 1}|\sigma_n|\leq N$, *where $|E|$ denotes the cardinal of a set E.*
(ii) $|b_\lambda(\mu)|<\eta$ *if $\lambda,\mu\in\sigma_n$, $n\geq 1$.*
(iii) *There exists $\delta>0$ such that for every choice $\Lambda_0=\{\lambda_n^0\}_{n\geq 1}$* with an arbitrary *$\lambda_n^0\in\sigma_n$, $n\geq 1$, we have $\Lambda_0\in(C)$ and $\delta_{\Lambda_0}\geq\delta$.*
(iv) *There exists a sequence $(D_n)_{n\geq 1}\subset H^\infty$ such that*

$$D_n(\lambda)=\begin{cases}1 & \textit{if}\quad \lambda\in\sigma_n,\\ 0 & \textit{if}\quad \lambda\in\Lambda\backslash\sigma_n,\end{cases}\tag{1}$$

$$\sum_{n\geq 1}|D_n(z)|\leq M,\quad z\in D.\tag{2}$$

REMARK. The first three statements have been proved by V.I. Vasyunin. (See [**14**]; see also [**6**], where a different proof of this fact was given.)

The construction of the family $(D_n)_{n\geq 1}$ was given in [**6**]. We mention that they generalize P. Beurling's functions which allow the construction of a linear operator of interpolation in the case of a single Carleson sequence.

Let us number $\sigma_n=\{\lambda_{n,k}\}_{k=1}^{|\sigma_n|}$ and set $\Lambda_k=\{\lambda_{n,k}\}_{n\geq 1,k\leq|\sigma_n|}$, $(k=1,\ldots,N)$. For technical reasons we will also define $\tilde{\Lambda}_i=\{\lambda_{n,i}\}_{n\geq 1}$, where we set $\lambda_{n,i}=\lambda_{n,|\sigma_n|}$ if $i\leq|\sigma_n|$. In view of statement (iii) of Proposition 1.1 these sequences are also Carleson sequences whose Carleson constant is bounded below by δ. We now define a sequence space by $l_0=X(\Lambda_1)$. This definition of l_0 depends a priori on the choice of the element $\lambda_{n,1}\in\sigma_n$, $n\geq 1$. For this reason we need the following definition of stability which is motivated by that of [**9**].

DEFINITION 1.2. Let $X\subset\mathrm{Hol}(D)$. The space X is called *(C)-stable* if for all pairs of Carleson sequences $\Lambda=\{\lambda_n\}_{n\geq 1}$ and $\tilde{\Lambda}=\{\tilde{\lambda}_n\}_{n\geq 1}$ satisfying

$$\sup_{n\geq 1}|b_{\lambda_n}(\tilde{\lambda}_n)|<1,\tag{3}$$

we have

$$X(\Lambda)=X(\tilde{\Lambda}).\tag{4}$$

A sequence $\tilde{\Lambda}$ satisfying (3) with $\eta = \sup_{n\geq 1}|b_{\lambda_n}(\tilde{\lambda}_n)|$ will be called *η-shifted with respect to* Λ. (See also [**19**].)

Remember that the sequences in $X(\Lambda)$ are indexed by the natural numbers (cf. (2) of the Introduction) and thus there will be no confusion in interpreting equality (4).

With this definition it is clear that if a space X is (C)-stable then the corresponding sequence space l_0 does not depend on the choice of $\lambda_{n,1}\in\sigma_n$.

Remarks. (1) It is easy to see that if we do not have (3) then, in general, we need not have equality (4). Take for example

$$X = H^p(D) = \{f \in \mathrm{Hol}(D)\ :\ N_p(f) = \sup_{0<r<1}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{it})|^p dt < \infty\}, 0 < p < \infty,$$

the Hardy space on the unit disk. For $1\leq p<\infty$ this is a Banach space with norm $\|f\|_p^p = N_p(f)$ and for $0<p<1$ a complete metric space. It was shown by H.S. Shapiro and A.L. Shields for $1\leq p\leq\infty$ ([**13**]) and by V. Kabaĭla (see [**4**] for this result and for exact references) that if $\Lambda\in(C)$ then

$$H^p(\Lambda) = \{a \in C^N : \sum_{n\geq 1}(1-|\lambda_n|)|a_n|^p < \infty\}. \tag{5}$$

It is easy to construct two sequences Λ, $\tilde{\Lambda}\in(C)$ such that the weights $(1-|\lambda_n|)_{n\geq 1}$ and $(1-|\tilde{\lambda}_{\mathrm{n}}|)_{n\geq 1}$ are not equivalent, and hence $X(\Lambda)\neq X(\tilde{\Lambda})$. We mention that in this case the corresponding trace spaces $X(\Lambda)$ and $X(\tilde{\Lambda})$ are isomorphic. In order to see that this is in general not true, we now give an example of a Banach space X and two Carleson sequences Λ, $\tilde{\Lambda}$ (that are not near in the sense of (3)) such that the trace spaces $X(\Lambda)$ and $X(\tilde{\Lambda})$ are not even isomorphic. We first remark that it is of no importance if we consider spaces on the unit disk or the upper half plane $\mathrm{C}_+ = \{z\in\mathrm{C}\ :\ \mathcal{J}z>0\}$. Take now $X = L_A^p$ the space of Fourier transforms on C_+ of L^p-functions: $L_A^p = \{F\in\mathrm{Hol}(\mathrm{C}_+) : F(z) = (\mathcal{F}f)(z), f\in L^p(0,\infty)\}$, where $(\mathcal{F}f)(z) = \int_0^\infty f(t)e^{itz}dt$. It was mentioned in [**16**] that for the sequences $\Lambda = \{i2^n\}_{n\geq 1}$ and $\tilde{\Lambda} = \{i+2^n\}_{n\geq 1}$, we have

$$X(\Lambda) = l^p,\quad X(\tilde{\Lambda}) = l^2,$$

which are obviously not isomorphic. Here

$$l^r = \{a \in C^N : \sum_{n\geq 1}|a_n|^r < \infty\},\qquad 1 \leq r < \infty.$$

(2) We have the following local stability property. If $\Lambda\in(C)$ with Carleson constant δ, then each sequence $\tilde{\Lambda} = \{\tilde{\lambda}_n\}_{n\geq 1}$ which is $\delta/3$-shifted with respect to Λ is a Carleson sequence (see Lemma VII.5.1 of [**5**], or [**19**]). Hence, if X is (C)-stable, then

$$X(\Lambda) = X(\tilde{\Lambda}). \tag{6}$$

Let us now introduce divided difference with respect to pseudohyperbolic metric.

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DEFINITION 1.3. (See [**15**] and [**1**]). Let $\sigma=\{\lambda_k\}_{k\geq 1}\subset \mathbb{D}$ and $f:\sigma\to\mathbb{C}$. Set $\lambda^{(k)}=(\lambda_1,\ldots,\lambda_k)$ and $\lambda^{(k+1)}=(\lambda^{(k)},\lambda_{k+1})$. We define

$$\Delta^0 f(\lambda^{(1)})=f(\lambda_1),$$

$$\Delta^1 f(\lambda^{(2)})=\frac{f(\lambda_2)-f(\lambda_1)}{b_{\lambda_1}(\lambda_2)}$$

and

$$\Delta^k f(\lambda^{(k+1)})=\frac{\Delta^{k-1}f(\lambda^{(k-1)},\lambda_{k+1})-\Delta^{k-1}f(\lambda^{(k-1)},\lambda_k)}{b_{\lambda_k}(\lambda_{k+1})}.$$

This enables us to define a candidate for the trace space $X|_\Lambda$ if Λ is a finite union of Carleson sequences. Set

$$l_{N,\sigma_n}=\{a\in C^\Lambda:(\sup_{k=1,\ldots,|\sigma_n|}|\Delta^{k-1}a(\lambda_n^{(k)})|)_{n\geq 1}\in l_0\},$$

where $\lambda_n^{(k)}=(\lambda_{n,1},\ldots,\lambda_{n,k})$. This space can be regarded as a discrete regularity or Sobolev-type space.

THEOREM 1.4. *Let $X\subset\mathrm{Hol}(\mathbb{D})$ be a (C)-stable vector space such that $H^\infty X\subset X$, and $\Lambda=\bigcup_{i=1}^N\Lambda_i$, $\Lambda_i\in(C)$. Then for any decomposition $\Lambda=\bigcup_{n\geq 1}\sigma_n$ satisfying* (i)–(iv) *of Proposition* 1.1 *we have*

$$X|_\Lambda=l_{N,\sigma_n}.$$

REMARK. The condition $H^\infty X\subset X$ is intimately related to the notion of free interpolation. In fact we call a sequence $\Lambda\subset\mathbb{D}$ of *free interpolation* if the space $l=X(\Lambda)$ is an ideal space; i.e. if for all $(a_n)_{n\geq 1}\in l$ and $(b_n)_{n\geq 1}\subset\mathbb{C}$ with $|b_n|\leq c|a_n|$, $n\geq 1$, for some constant $c>0$ we have $(b_n)_{n\geq 1}\in l$. In this case we write $\Lambda\in\mathrm{Int}(X)$. It is not hard to see that if $H^\infty X\subset X$ and $\Lambda\in(C)$ then Λ is of free interpolation for X (cf. [**10**] and [**6**]), and in particular the space l_0 is then ideal. We have not chosen the usual notion of free interpolating sequences that is based on the definition of a concrete target space for the restriction operator $R:X\to\mathbb{C}^\Lambda, f\mapsto f|_\Lambda$. Our terminology comes from generalized free interpolation problems. In fact Theorem 1.4 gives a generalized free interpolation result: if a sequence $a\in\mathbb{C}^\Lambda$ is interpolable by a function $f\in X$, and if for an arbitrary sequence $\mu\in l^\infty$ we set $\mu_\lambda=\mu_n$ if $\lambda\in\sigma_n$ (μ is a constant function on σ_n taking the value μ_n there), then the sequence $(\mu_\lambda a_\lambda)_{\lambda\in\Lambda}$ will also be interpolable (cf.[**10**], [**14**] and also [**6**] for generalized free interpolation). We mention that in the classical literature the target spaces that are chosen for the definition of interpolating sequences are in general ideal. We also remark that for example for Hardy and Bergman spaces these definitions are actually equivalent (for Bergman spaces, see [**17**]).

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Proof of Theorem 1.4. Let $\Lambda = \bigcup_{n\geq 1} \sigma_n$ be a decomposition satisfying (i)–(iv) of Proposition 1.1 for some $0<\eta<1$. Recall that the sequences $\tilde{\Lambda}_i$ satisfy the Carleson condition with Carleson constant δ. We shall set $\eta=\delta/3$.

Consider now the inclusion $X|_\Lambda \subset l_{N,\sigma_n}$. Let $f\in X$. We have to control the divided differences of f on σ_n.

Lemma 1.5. *Let $f\in \mathrm{Hol}(D)$, $\sigma = \{\lambda_k\}_{k=1}^N \subset D$ and $0<\varepsilon<1$. Then there exist $u_i\in \partial\Omega(\lambda_i,\varepsilon)$, $(i=1,\ldots,N)$, such that for all $z \in \bigcup_{i=1}^N \Omega(\lambda_i,\varepsilon)$*

$$|\Delta^{k-1} f(\lambda^{(k-1)}, z)| \leq \left(\frac{2}{\varepsilon}\right)^{k-1} \sum_{i=1}^N |f(u_i)| \qquad (k=1,\ldots,N). \tag{7}$$

Proof. Let $z \in \bigcup_{i=1}^N \Omega(\lambda_i,\varepsilon)$. Take $u_i\in\partial\Omega(\lambda_i,\varepsilon)$ with $|f(u_i)| = \sup_{\zeta\in\Omega(\lambda_i,\varepsilon)} |f(\zeta)|$. The assertion (7) becomes trivial for the case $k=1$. Consider the case in which $2\leq k\leq N$. Suppose that (7) holds true for $k\leq N-1$. Define

$$g(z) = \Delta^k f(\lambda^{(k)}, z).$$

If $|b_{\lambda k}(z)|\geq\varepsilon$, then in view of (7) we have

$$|g(z)| = \left|\frac{\Delta^{k-1} f(\lambda^{(k-1)}, z) - \Delta^{k-1} f(\lambda^{(k)})}{b_{\lambda_k}(z)}\right| \leq \frac{2}{\varepsilon}\left(\frac{2}{\varepsilon}\right)^{k-1} \sum_{i=1}^N |f(u_i)|. \tag{8}$$

If $|b_{\lambda_k}(z)|\geq\varepsilon$, then, by the maximum modulus principle, the function $|g(z)|$ is bounded by $\sup_{\zeta\in\partial\Omega(\lambda_k,\varepsilon)}|g(\zeta)|$. This allows us to apply (8) again, and we get the assertion also in the case $|b_{\lambda_k}(z)|<\varepsilon$.

As $\lambda_{n,k} \in \bigcup_{i=1}^{\sigma_n} \Omega(\lambda_{n,i},\eta)$ we may apply the lemma to get the existence of $u_{n,i} \in \partial\Omega(\lambda_{n,i},\eta)$, $n\geq 1$, $i=1,\ldots,|\sigma_n|$, such that

$$\sup_{k=1,\ldots,|\sigma_n|} |\Delta^{k-1} f(\lambda_n^{(k)})| \leq c \sum_{i=1}^{|\sigma_n|} |f(u_{n,i})|, \quad n\geq 1. \tag{9}$$

Take $u_{n,i}=\lambda_{n,|\sigma_n|}$ for $i>|\sigma_n|$. As $f\in X$ and $U_i = \{u_{n,i}\}_{n\geq 1}$, $i=1,\ldots,N$, is η-shifted with respect to $\tilde{\Lambda}_i$, we have $(f(u_{n,i}))_{n\geq 1}\in l_0$. Therefore $(\sum_{i=1}^N |f(u_{n,i})|)_{n\geq 1} \in l_0$ and hence $X|_\Lambda\subset l_{N,\sigma_n}$. Observe that we have used the ideal property of l_0 and the stability property of X.

Consider now the reverse inclusion $l_{N,\sigma_n}\subset X|\Lambda$. For each sequence $a\in l_{N,\sigma_n}$, we will construct a function $f\in X$ interpolating the sequence a on Λ. Unfortunately we neither know if this operation is linear nor if it is continuous. In order to construct the function f we use the sequence $(D_n)_{n\geq 1}\subset H^\infty$ of Proposition 1.1. We shall show that there exist N functions $f_l\in X$, $l=1,\ldots,N$, such that

$$f(z) = \sum_{n\geq 1} D_n(z) \sum_{l=1}^{|\sigma_n|} \prod_{k=1}^{l-1} b_{\lambda_{n,k}}(z) f_l(z)$$

is in X and interpolates a on Λ.

REMARK. As $f_l \in \mathrm{Hol}(\mathrm{D})$ and in view of the property (ii) of the sequence $(D_n)_{n\geq 1}$ the convergence of this series is uniform on compact subsets D. In particular we may rearrange the series in the following way:

$$f(z) = \sum_{l=1}^{N} f_l(z) \underbrace{\sum_{n\geq 1, l\leq|\sigma_n|} D_n(z) \prod_{k=1}^{l-1} b_{\lambda_{n,k}}(z)}_{G_l(z)}$$

where $G_l \in H^\infty$. The property $f \in X$ will therefore be an easy consequence of $f_l \in X$, $l=1,\ldots,N$, and $H^\infty X \subset X$; (recall that X is a vector space).

Take now for $n\geq 1$

$$\gamma(\lambda_{n,k}) = \frac{a(\lambda_{n,k}) - \sum_{l=1}^{k-1} \prod_{j=1}^{l-1} b_{\lambda_{n,j}}(\lambda_{n,k}) f_l(\lambda_{n,k})}{\prod_{j=1}^{k-1} b_{\lambda_{n,j}}(\lambda_{n,k})}, \qquad k = 1, \ldots, |\sigma_n|, \tag{10}$$
$$\gamma(\lambda_{n,k}) = 0, k > |\sigma_n|.$$

(For $k=1$, $\gamma(\lambda_{n,k})$ reduces to $a(\lambda_{n,1})$.)

We show that there exist $f_l \in X$, $l=1,\ldots,N$, such that f_l interpolates $(\gamma(\lambda_{n,l}))_{n\geq 1}$ on Λ_l. In fact, if this is the case, then for $\lambda_{n,k} \in \sigma_n$, $n\geq 1$, $k=1,\ldots,|\sigma_n|$, we obtain the interpolation property of f:

$$\begin{aligned} f(\lambda_{n,k}) &= \sum_{m\geq 1} D_m(\lambda_{n,k}) \sum_{l=1}^{|\sigma_m|} \prod_{j=1}^{l-1} b_{\lambda_{m,j}}(\lambda_{n,k}) f_l(\lambda_{n,k}) \\ &= \sum_{l=1}^{k-1} \prod_{j=1}^{l-1} b_{\lambda_{n,j}}(\lambda_{n,k}) f_l(\lambda_{n,k}) + \prod_{j=1}^{k-1} b_{\lambda_{n,j}}(\lambda_{n,k}) \underbrace{f_k(\lambda_{n,k})}_{\gamma(\lambda_{n,k})} \\ &= a(\lambda_{n,k}). \end{aligned}$$

We shall need the following lemma which gives us a more convenient relation between f_k and $f_1,\ldots,f_{k-1}$, $(k=1,\ldots,N)$. The proof is essentially based on Newton type interpolation with the usual zero divisor $z-\lambda$ replaced by b_λ. This is very natural in view of the definition of l_{N,σ_n} and the divided differences. The purely algebraic proof will be given at the end of this section.

LEMMA 1.6. *Let* $\sigma=\{\lambda_1,\ldots,\lambda_k\}\subset \mathrm{D}$, $a\in \mathrm{C}^\sigma$. *If functions* $f_l : \sigma \mapsto C, (l = 1, \ldots, k)$ *exist such that* $f_l(\lambda_l)=\gamma(\lambda_l)$, $1\leq l\leq k$, *then*

$$\gamma(\lambda_l) = \Delta^{l-1} a(\lambda^{(l)}) - \sum_{i=1}^{l-1} \Delta^{l-i} f_i(\lambda_i, \ldots, \lambda_l) \quad (1 \leq l \leq k). \tag{11}$$

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Resume the proof of the theorem. It is clear from the definition of l_0 that there exists a function f_1 interpolating $(\gamma_{n,1})_{n\geq 1}$ on Λ_1. We shall now construct inductively the functions f_l, $l=2,\ldots,N$. We will show that $(\gamma(\lambda_{n,k}))_{n\geq 1}\in l_0, (k=1,\ldots,N)$. To this end we suppose that for $2\leq l\leq N$ there exist functions $f_1,\ldots,f_{l-1}\in X$ such that $f_j(\lambda_{n,j})=\gamma(\lambda_{n,j}), (n\geq 1,\ \ j=1,\ldots,l-1)$. Owing to Lemma 1.5, there will exist $u_{n,k,j}\in\partial\Omega(\lambda_{n,k},\eta)$ for $n\geq 1$, $k=1,\ldots,|\sigma_n|$ and $j=1,\ldots,\min(|\sigma_n|,l-1)$ such that

$$|\Delta^{l-j}f_j(\lambda_{n,j},\ldots,\lambda_{n,l})|\leq c\sum_{i=1}^{|\sigma_n|}|f_j(u_{n,i,j})|.$$

Applying Lemma 1.6 to $f_j|_{\sigma_n}$, we get the following estimate for $n\geq 1$:

$$|\gamma(\lambda_{n,l})|\leq\sup_{k=1,\ldots,\min(l,|\sigma_n|)}\left(|\Delta^{k-1}a(\lambda_n^{(k)})|+c\sum_{j=1}^{l-1}\sum_{i=1}^{|\sigma_n|}|f_j(u_{n,i,j})|\right)\quad (l\leq|\sigma_n|),$$

$$|\gamma(\lambda_{n,l})|=0\quad (l>|\sigma_n|).$$

Take $u_{n,k,i}=\lambda_{n,|\sigma_n|}$ for $k>|\sigma_n|$, $(i=1,\ldots,N)$ and define $U_{k,i}=(u_{n,k,i})_{n\geq 1}$. The sequences $U_{k,i}$, $(i=1,\ldots,N)$, are η-shifted with respect to $\tilde{\Lambda}_k$, $(k=1,\ldots,N)$. Therefore $(|f_i(u_{n,k,i})|)_{n\geq 1}\in l_0$ for $k,i=1,\ldots,N$. By definition $(\sup_{k=1,\ldots,|\sigma_n|}|\Delta^{k-1}a(\lambda_n^{(k)})|)_{n\geq 1}\in l_0$ and therefore $(\gamma(\lambda_{n,l}))_{n\geq 1}\in l_0=X(\tilde{\Lambda}_l)$. (Observe that we had set $\lambda_{n,l}=\lambda_{n,|\sigma_n|}$ for $l>|\sigma_n|$). Hence there exists $f_l\in X$ such that $f_l\ (\lambda_{n,l})=\gamma(\lambda_{n,l}),\ n\geq 1,\ l=1,\ldots,|\sigma_n|$.

It remains to prove the algebraic lemma.

Proof of Lemma 1.6. Let us introduce some notations.

$$\begin{array}{lll} a_l=a(\lambda_l), & \alpha_l=\Delta^{l-1}a(\lambda^{(l)}), & \\ \gamma_l=\gamma(\lambda_l), & & \\ f_{i,l}=f_i(\lambda_l), & \psi_{i,l}=\Delta^{l-i}f_i(\lambda_i,\ldots,\lambda_l), & \tau_l=\sum_{i=1}^{l-1}\psi_{i,l}\,, \\ \beta_{i,l}=\prod_{j=1}^{i-1}b_{\lambda_j}(\lambda_l), & \beta_{j,l}/\beta_{i,l}=\prod_{m=1}^{j-1}b_{\lambda_m}(\lambda_l)(j\geq i). & \end{array}$$

Then (10) with $\lambda_{n,k}$ replaced by λ_l reduces to

$$\gamma_l=\frac{a_l-\sum_{i=1}^{l-1}\beta_{i,l}f_{i,l}}{\beta_{l,l}} \tag{12}$$

and assertion (11) becomes

$$\gamma_l=\alpha_l-\tau_l. \tag{13}$$

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For a function $F : \sigma \mapsto C$ we introduce the following Newton type interpolating function: (cf. [**6**] and also [**15**]). Let $j \leq l$.

$$F(\lambda_l) = \sum_{i=j}^{l} \prod_{m=j}^{i-1} b_{\lambda_m}(\lambda_l) \Delta^{i-j} F(\lambda_j, \ldots, \lambda_i). \tag{14}$$

With this, on setting $j=1$ we get $a_l = \sum_{i=1}^{l} \beta_{i,l}\alpha_i = \beta_{l,l}\alpha_l + \sum_{i=1}^{l-1} \beta_{i,l}\alpha_i$. Substituting this expression in (12) we obtain

$$\gamma_l = \alpha_l - \frac{1}{\beta_{l,l}} \sum_{i=1}^{l-1} \beta_{i,l}(f_{i,l} - \alpha_i). \tag{15}$$

Using again (14) applied to the function f_i we get

$$f_{i,l} = \sum_{j=i}^{l} (\beta_{j,l}/\beta_{i,l})\psi_{i,j} = f_{i,i} + \sum_{j=i+1}^{l} (\beta_{j,l}/\beta_{i,l})\psi_{i,j}.$$

Replacing $f_{i,l}$ in (15) and observing that $f_{i,i} = \gamma_i$ we obtain

$$\begin{aligned}
\gamma_l &= \alpha_l - \frac{1}{\beta_{l,l}} \sum_{i=1}^{l-1} \beta_{i,l}(\gamma_i - \alpha_i + \sum_{j=i+1}^{l} \frac{\beta_{j,l}}{\beta_{i,l}} \psi_{i,j}) \\
&= \alpha_l - \frac{1}{\beta_{l,l}} \left(\sum_{i=1}^{l-1} \beta_{i,l}(\gamma_i - \alpha_i) + \sum_{i=1}^{l-1} \sum_{j=i+1}^{l} \beta_{j,l}\psi_{i,j} \right) \\
&= \alpha_l - \frac{1}{\beta_{l,l}} \left(\sum_{i=1}^{l-1} \beta_{i,l}(\gamma_i - \alpha_i) + \sum_{j=2}^{l} \beta_{j,l} \sum_{i=1}^{j-1} \psi_{i,j} \right).
\end{aligned}$$

By definition $\tau_j = \sum_{i=1}^{j-1} \psi_{i,j}$. Observe also that the sum over j may be extended to $j=1$ (as $\tau_1 = 0$). Hence

$$\gamma_l = \alpha_l - \tau_l - \frac{1}{\beta_{l,l}} \sum_{i=1}^{l-1} \beta_{i,l}(\gamma_i - \alpha_i + \tau_i).$$

This is equivalent to $\sum_{i=1}^{l} \beta_{i,l}(\gamma_i - \alpha_i + \tau_i) = 0$, $l = 1, \ldots, k$. As the matrix $(\beta_{i,l})_{i,l=1}^{k}$ is invertible (as an upper triangular matrix with $\beta_{l,l} = \prod_{i=1}^{l-1} b_{\lambda_i}(\lambda_l) \neq 0$ on the diagonal) we now get the assertion (13).

2. Examples. Even if the following examples are all complete metric spaces, observe that in order to apply Theorem 1.4 it is not necessary to have this property. Thus the following list of examples is certainly not exhaustive. Note also that in the following examples we show the stability property with the aid of known descriptions of $X|_\Lambda$ or $X(\Lambda)$ where $\Lambda \in (C)$.

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2.1. Hardy spaces. H^p, $0<p\leq\infty$. It is clear that $H^\infty H^p\subset H^p$, and therefore we have to show the stability property of H^p. The characterization of the trace space $H^p|_\Lambda$, $(0<p\leq\infty)$, where Λ is a Carleson sequence, has been given in (5). Take now two Carleson sequences $\Lambda=\{\lambda_n\}_{n\geq1}$ and $\tilde\Lambda=\{\tilde\lambda_n\}_{n\geq1}$ satisfying $|b_{\lambda_n}(\tilde\lambda_n)|<\eta$, $(n\geq1)$, for some $0<\eta<1$; then the implication (see [**19**]) for $\lambda,\ \mu\in D$

$$|b_\lambda(\mu)|<\eta\Rightarrow\frac{1-\eta}{1+\eta}\leq\frac{1-|\lambda|}{1-|\mu|}\leq\frac{1+\eta}{1-\eta},\tag{16}$$

shows the equivalence of the weights $(1-|\lambda_n|)_{n\geq1}$ and $(1-|\tilde\lambda_n|)_{n\geq1}$. We thus obtain the same characterization of $H^p|_\Lambda$ where Λ is a finite union of Carleson sequences, as in [**6**]. In contrast to those results, the method presented here does not furnish a linear continuous operator of interpolation. We remark that there is another characterization of $H^p|_\Lambda$ by means of maximal functions in [**1**]. Besides giving a very general necessary condition for a sequence to be in $H^p|_\Lambda$ they were actually able to characterize finite unions of interpolating sequences by surjectivity of the restriction operator $f\mapsto f|_\Lambda$ onto their trace space. (It would be interesting to prove directly the equivalence between the two descriptions if Λ is a finite union of Carleson sequences).

2.2. Increasing functions H^∞_ω. Set $C_{1/2}=\{z\in C:\mathcal{R}z>1/2\}$ and let $\varphi:C_{1/2}\mapsto C$ be a function satisfying

(i) φ is analytic in $C_{1/2}$,
(ii) $\varphi|_{[1,\infty)}>0$,
(iii) $\varphi|_{[1,\infty)}$ is increasing,
(iv) there exists $c>0$, such that $|\varphi(z)|\leq c\varphi(|z|)$, $(z\in C_{1/2})$.

Put $\omega(u)=\sigma(1/(1-u))$, $(u\in D)$, and consider the weight $\omega|_{[0,1)}$. Let $\Lambda=\{\lambda_n\}_{n\geq1}\subset D$. Define

$$\begin{aligned}H^\infty_\omega&=\{f\in \mathrm{Hol}(D):|f(z)|\leq c_f\,\omega(|z|),z\in D\},\\ l^\infty_{\omega,\Lambda}&=\{a\in C^N:|a_n|\leq c_a\omega(|\lambda_n|),n\geq1\}.\end{aligned}\tag{17}$$

In [**19**] it was shown that if $\Lambda\in(C)$, and if φ satisfies the conditions (i)–(iv) then we have

$$H^\infty_\omega(\Lambda)=l^\infty_{\omega,\Lambda}.\tag{18}$$

Now, let $\Lambda=\bigcup_{i=1}^N\Lambda_i$, where $\Lambda_i\in(C)$ and $l_0=l^\infty_{\omega,\Lambda_1}$. It is obvious that we always have $H^\infty H^\infty_\omega\subset H^\infty_\omega$. Moreover, due to (18) and (17), if there exists a constant $c\geq1$ such that

$$\frac{1}{c}\omega(|\lambda|)\leq\omega(|\mu|)\leq c\omega(|\lambda|),\tag{19}$$

for $\lambda,\mu\in D$ and $|b_\lambda(\mu)|<\eta$, where $0<\eta<1$ is a fixed constant, then the space H^∞_ω will be (C)-stable.

It may be shown that this condition is intimately related to the following one (cf. [7]):

$$(1-|\lambda^2|)|\omega'(\lambda)| \leq c_1|\omega(\lambda)|, \lambda \in \Gamma_\alpha,$$

for $\alpha \geq 1$, where $\Gamma_\alpha = \{z \in \mathbb{D} : |z-1| \leq \alpha(1-|z|)\}$ is a Stoltz angle centered in 1. This second form of our regularity condition suggests that the growth of the weight $\omega(t)$ towards the boundary may be at most polynomial.

A possible weight. Let $r > 0$. Take

$$\omega(t) = \left(\frac{1}{1-t}\right)^r (0 \leq t < 1),$$

and $\varphi : C_{1/2} \to C,\ z \mapsto z^r$. The conditions (i)–(iv) are obvious. The condition (19) is now a consequence of (16). In fact if $|b_\lambda(\mu)| < \eta$, for some $0 < \eta < 1$, then

$$\frac{1}{c} = \left(\frac{1-\eta}{1+\eta}\right)^r \leq \left(\frac{1-|\mu|}{1-|\lambda|}\right)^r = \left(\frac{\omega(|\lambda|)}{\omega(|\mu|)}\right) \leq \left(\frac{1+\eta}{1-\eta}\right)^r = c.$$

Hence we can apply Theorem 1.4 to these spaces. In the literature the spaces H^∞_ω associated with the weight $\omega(t) = (1/(1-t))^r$ are often denoted by A^{-r} (cf. e.g. [2]).

One may also construct weights that grow logarithmically towards the boundary (e.g. $\omega(u) = \mathrm{Log}\,(1/(1-u)) + \pi$).

An impossible weight. We now give an example of a weight ω such that the corresponding space H^∞_ω is not (C)-stable. Take $\varphi(z) = \exp(z)$. This function satisfies the conditions (i)–(iv) (in particular we have $|\exp(z)| = \exp(\mathcal{R}z) \leq \exp|z|$). Thus the function φ is admissible for the description on a Carleson sequence (cf. [19]). We obtain

$$H^\infty_\omega|_\Lambda = \{a \in C^\Lambda : |a(\lambda)| \leq c_a \exp(1/(1-|\lambda|)), \lambda \in \Lambda\}.$$

Take Λ an arbitrary infinite Carleson sequence and define $\lambda_n \in \Omega(\lambda_n, \delta/3)$ by $|\tilde{\lambda}_n| = \max\{|z| : z \in \Omega(\lambda_n, \delta/3)\}$, where δ is the Carleson constant associated with Λ. The corresponding sequence $\tilde{\Lambda}$ will be a Carleson sequence which is $\delta/3$-shifted with respect to Λ. The reader may now verify that the sequence given by $a_n = \exp(1/(1-|\tilde{\lambda}_n|))$ is in $H^\infty_\omega(\tilde{\Lambda})$ but not in $H^\infty_\omega(\Lambda)$.

Remark. Another approach to the characterization of the trace spaces $H^\infty_\omega|_\Lambda$, where Λ is a finite union of Carleson sequences and ω satisfies the conditions (i)–(iv) as well as the condition (19), has been given in [7]. There actually we have constructed a continuous linear operator of interpolation from $(l^\infty_{\omega,\Lambda_1})_{N,\sigma_n}$ to H^∞_ω, which provides a stronger result than the simple description of the trace space given in this paper.

2.3. Bergman spaces B^p_α. We introduce weighted Bergman spaces. (See e.g. [**9**].)

Definition 2.1 Let $0<p\leq\infty$ and $\alpha\geq -1/p$. The *Bergman space* B^p_α on the unit disk is defined as follows:

$$B^p_\alpha = \{f\in \mathrm{Hol}(D) : \| f\|^p_{p,\alpha} = \int_D ((1-|z|^2)^\alpha |f(z)|)^p dm < \infty\},$$

where $dm = dxdy/\pi$ is the planar Lebesgue measure on the unit disk.

It is easy to see that $H^\infty B^p_\alpha \subset B^p_\alpha$, and we therefore have to verifiy the stability property. Due to the remark after Theorem 1.4, Carleson sequences $\Lambda = \{\lambda_n\}_{n\geq 1}$ and $\tilde{\Lambda} = \{\tilde{\lambda}_n\}_{n\geq 1}$ are interpolating sequences for B^p_α, which means for a separated sequence (and Carleson sequences are separated, see e.g. [**9**] for the definitions), that

$$B^p_\alpha(\Lambda) = l^p_{2/p+\alpha},$$

where

$$l^p_\beta = \{a\in C^N : \| a\|_{p,\beta} < \infty\},$$

and

$$\| a\|^p_{p,\beta} = \sum_{n\geq 1} \{(1-|\lambda_n^2|)^\beta |a_n|\}^p,$$

$$\| a\|_{\infty,\beta} = \sup_{n\geq 1} \{(1-|\lambda_n^2|)^\beta |a_n|\}.$$

As in the case of Hardy spaces, we now get the equivalence of the weights $((1-|\lambda_n^2|)^{2+\alpha p})_{n\geq 1}$ and $((1-|\tilde{\lambda}_n^2|)^{2+\alpha p})_{n\geq 1}$ as a consequence of (16). Thus $B^p_\alpha(\Lambda) = l^p_{2/p+\alpha} = B^p_\alpha(\tilde{\Lambda})$ if $|b_{\lambda_n}(\tilde{\lambda}_n)| < \eta$, for some $0<\eta<1$. Hence we can also apply Theorem 1.4 to weighted Bergman spaces on the unit disk.

It is interesting to mention that finite unions of Carleson sequences play an important rôle in factorization theory in Bergman spaces. In [**8**] it was shown that a Blaschke product is a so-called universal divisor (i.e. $f\in B^p_0$ and $f/B\in\mathrm{Hol(D)}$ imply $f/B\in B^p_0$) if and only if its zero set is a finite union of Carleson sequences.

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