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SHARP RELIABILITY BOUNDS FOR THE $\mathcal{L}$-LIKE DISTRIBUTIONS, A CLASS IN WHICH AGEING IS BASED ON LAPLACE TRANSFORM INEQUALITIES

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Abstract

We establish Klar's (2002) conjecture about sharp reliability bounds for life distributions in the $\mathcal{L}_\alpha$-class in reliability theory. The key idea is to construct a set of two-point distributions whose support points satisfy a certain system of equalities and inequalities.

Keywords: $\mathcal{L}$-class; $\mathcal{L}_\alpha$-class; reliability bound

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1. Introduction and main result

Consider two nonnegative random variables X and Y with respective life distributions F and G. We say that F is smaller than G in the Laplace transform order, written $F \leq_{\mathrm{L}} G$, if the following relation holds for all $s \geq 0$: $L_F(s) := \mathrm{E}(\mathrm{e}^{-sX}) \leq \mathrm{E}(\mathrm{e}^{-sY}) =: L_G(s)$.

Klefsjö (1983) introduced the useful $\mathcal{L}$-class of life distributions in reliability theory. This consists of all life distributions F satisfying $F \leq_{\mathrm{L}} \mathcal{E}_F$, where $\mathcal{E}_F$ stands for the exponential distribution having the same mean as F. In this paper we consider the life distributions, F, smaller than a gamma distribution, G, in the Laplace transform order. For convenience, we denote by $G_{\alpha,\beta}$ the gamma distribution with density function $g_{\alpha,\beta}(x) = x^{\alpha-1}\mathrm{e}^{-x/\beta}/\Gamma(\alpha)\beta^\alpha$, $x \geq 0$, where $\alpha, \beta > 0$. Also, we denote by $G_{\alpha,0}$ the degenerate distribution at $x = 0$. Recall that the mean of $G_{\alpha,\beta}$ is $\alpha\beta$. Then, for each $\alpha > 0$, we define the $\mathcal{L}_\alpha$-class of life distributions by

$$\begin{aligned}\mathcal{L}_\alpha &= \bigcup_{\beta\geq 0}\{F\colon \mu(F) = \alpha\beta,\ F \leq_{\mathrm{L}} G_{\alpha,\beta}\}\\ &= \bigcup_{\beta\geq 0}\{F\colon \mu(F) = \alpha\beta,\ L_F(s) \leq (1+\beta s)^{-\alpha} \text{ for } s \geq 0\},\end{aligned}$$

in which $\mu(F)$ stands for the mean of F. In particular, $\mathcal{L}_1 = \mathcal{L}$. For properties of the $\mathcal{L}_\alpha$-class, see, for example, Lin (1998), Lin and Hu (2000), and Mohan and Ravi (2002).

Sengupta (1995) investigated the implicit reliability bounds for life distributions F and G satisfying $F \leq_{\mathrm{L}} G$, where G has the same finite mean as F. Recently, Klar (2002) elaborated on Sengupta's results and derived the following explicit bounds for $F \in \mathcal{L}_\alpha$, where we write $\overline{F}(x) = 1 - F(x)$.

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Theorem 1. (Klar (2002).) *Let $\alpha > 0$ and $F \in \mathcal{L}_\alpha$ with mean $\mu > 0$. The following bounds then hold:*

(i) $\overline{F}(x) \geq 1 - [\alpha(x/\mu - 1)^2 + 1]^{-1}$ *if* $x \leq \mu$,

(ii) $\overline{F}(x) \leq [\alpha(x/\mu - 1)^2 + 1]^{-1}$ *if* $x > \mu$.

Concerning the sharpness of reliability bounds, Klar (2002) first proved that if $\alpha = 1$, the two bounds in Theorem 1 are sharp for $x/\mu \in [2 - \sqrt{2}, 1] \cup [2 + \sqrt{2}, \infty)$, and then posed the following conjecture for general values of $\alpha > 0$.

Conjecture 1. *For a general value of $\alpha > 0$, the bounds in Theorem 1, parts (i) and (ii), are sharp for $x/\mu \in [(1 + \alpha - \sqrt{1 + \alpha})/\alpha, 1]$ and $x/\mu \in [(1 + \alpha + \sqrt{1 + \alpha})/\alpha, \infty)$, respectively.*

We establish Klar's conjecture and rewrite his result as follows.

Theorem 2. *Let $\alpha > 0$ and $F \in \mathcal{L}_\alpha$ with mean $\mu > 0$. The following bounds then hold:*

(i) $\overline{F}(x) \geq 1 - [\alpha(x/\mu - 1)^2 + 1]^{-1}$ *if* $0 \leq x \leq \mu$,

(ii) $\overline{F}(x) \leq \mu/x$ *if* $\mu \leq x \leq (\alpha + 1)\mu/\alpha$,

(iii) $\overline{F}(x) \leq [\alpha(x/\mu - 1)^2 + 1]^{-1}$ *if* $x \geq (\alpha + 1)\mu/\alpha$.

Moreover, the bound in part (i) is sharp for $x \in [(1 + \alpha - \sqrt{1 + \alpha})\mu/\alpha, \mu]$, and the bound in part (iii) is sharp for $x \in [(1 + \alpha + \sqrt{1 + \alpha})\mu/\alpha, \infty)$.

Note that

$$\frac{\mu}{x} < \left[\alpha\left(\frac{x}{\mu} - 1\right)^2 + 1\right]^{-1} \quad \text{if } x \in \left(\mu, \frac{\alpha + 1}{\alpha}\mu\right)$$

and, hence, the bound in Theorem 2(iii) (the same one as in Theorem 1(ii)) is not sharp for $x \in (\mu, (\alpha + 1)\mu/\alpha)$. On the other hand, recall the functional lower bound

$$\overline{F}(x) \geq 1 - \left(\frac{x}{\mu}\mathrm{e}^{1 - x/\mu}\right)^\alpha, \qquad x \in [0, \mu],$$

where $F \in \mathcal{L}_\alpha$ with mean $\mu > 0$ (see Lin and Hu (2000)). This lower bound is greater than that in Theorem 2(i) for small x. Therefore, the bound in Theorem 2(i) (the same one as in Theorem 1(i)) is not sharp for small x.

2. Proof and lemmas

Theorem 2(ii) follows immediately from Markov's inequality. In order to prove the conjecture about the sharpness of the bounds, we require the following lemmas.

Lemma 1. *Let $0 < \mu < \sqrt{\gamma}$ and let p_1, p_2, x_1, and x_2 be positive real numbers satisfying*

$$p_1 + p_2 = 1, \tag{1}$$

$$p_1 x_1 + p_2 x_2 = \mu, \tag{2}$$

$$p_1 x_1^2 + p_2 x_2^2 \leq \gamma, \tag{3}$$

$$0 < x_1 < \mu < x_2. \tag{4}$$

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Then

$$\frac{(x_2-\mu)^2}{(x_2-\mu)^2+\gamma-\mu^2} \le p_1 \le \frac{\gamma-\mu^2}{(x_1-\mu)^2+\gamma-\mu^2}, \tag{5}$$

$$\frac{(x_1-\mu)^2}{(x_1-\mu)^2+\gamma-\mu^2} \le p_2 \le \frac{\gamma-\mu^2}{(x_2-\mu)^2+\gamma-\mu^2}. \tag{6}$$

Strict equalities hold in (5) and (6) if $p_1x_1^2+p_2x_2^2=\gamma$.

Proof. By conditions (1) and (2), we have $p_2=1-p_1$ and $x_2=(\mu-p_1x_1)/(1-p_1)$. Substituting p_2 and x_2 into (3) and solving the inequality yields

$$p_1 \le \frac{\gamma-\mu^2}{(x_1-\mu)^2+\gamma-\mu^2},$$

which is exactly the second inequality in (5). We recover the second inequality in (6) similarly. That $p_1+p_2=1$ then completes the proofs of inequalities (5) and (6). As a consequence, all equalities in (5) and (6) hold if $p_1x_1^2+p_2x_2^2=\gamma$.

Lemma 2. *Let L be the Laplace transform of the gamma distribution $G_{\alpha,\beta}$ with mean $\mu=\alpha\beta>0$, and let p_1, p_2, x_1, and x_2 be positive real numbers satisfying conditions (1) and (2). If*

$$L''(s)+(x_1+x_2)L'(s)+x_1x_2L(s)\ge 0 \quad \textit{for } s\ge 0, \tag{7}$$

where a prime denotes differentiation, then $L(s)\ge p_1\mathrm{e}^{-x_1s}+p_2\mathrm{e}^{-x_2s}$ *for* $s\ge 0$.

Proof. First, recall that $L(s)=(1+\beta s)^{-\alpha}$, $s\ge 0$. By (7), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{x_2s}[L'(s)+x_1L(s)])=\mathrm{e}^{x_2s}\{L''(s)+(x_1+x_2)L'(s)+x_1x_2L(s)\}\ge 0 \quad \text{for } s\ge 0,$$

and, hence,

$$L'(s)+x_1L(s)\ge \mathrm{e}^{-x_2s}[L'(0)+x_1L(0)]=\mathrm{e}^{-x_2s}(-\mu+x_1), \qquad s\ge 0.$$

This in turn implies that

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{x_1s}L(s))=\mathrm{e}^{x_1s}[L'(s)+x_1L(s)]\ge \mathrm{e}^{-(x_2-x_1)s}(-\mu+x_1), \qquad s\ge 0.$$

Integrating both sides of the above inequality yields

$$\mathrm{e}^{x_1s}L(s)-1\ge \frac{-\mu+x_1}{x_2-x_1}[1-\mathrm{e}^{-(x_2-x_1)s}], \qquad s\ge 0.$$

Therefore,

$$\begin{aligned} L(s) &\ge \left(1-\frac{\mu-x_1}{x_2-x_1}\right)\mathrm{e}^{-x_1s}+\frac{\mu-x_1}{x_2-x_1}\mathrm{e}^{-x_2s} \\ &= p_1\mathrm{e}^{-x_1s}+p_2\mathrm{e}^{-x_2s}, \qquad s\ge 0. \end{aligned}$$

The last equality follows from conditions (1) and (2). The proof is thus complete.

Lemma 3. *Let L be the Laplace transform of the gamma distribution $G_{\alpha,\beta}$ with mean $\mu = \alpha\beta > 0$, and let p_1, p_2, x_1, and x_2 be positive real numbers satisfying conditions (1)–(4) with $\gamma = (\alpha+1)\mu^2/\alpha$. If $x_1x_2 \geq (\alpha+1)\mu^2/\alpha$ then (7) holds.*

Proof. It follows from conditions (1)–(3), with $\gamma = (\alpha+1)\mu^2/\alpha$, that

$$\begin{aligned}
&\frac{\alpha+1}{\alpha}\mu^2 - \mu(x_1+x_2) + x_1x_2 \\
&\qquad \geq (p_1x_1^2 + p_2x_2^2) - (x_1+x_2)(p_1x_1+p_2x_2) + x_1x_2(p_1+p_2) \\
&\qquad = 0
\end{aligned}$$

or, equivalently, $(\alpha+1)\mu^2/\alpha + x_1x_2 \geq \mu(x_1+x_2)$. Now set $x_s = 1+\beta s \geq 1$, $s \geq 0$. Then, for $s \geq 0$, we have

$$\begin{aligned}
L''(s) + (x_1+x_2)L'(s) + x_1x_2L(s) &= \frac{1}{x_s^{\alpha+2}}\left(x_1x_2x_s^2 - \mu(x_1+x_2)x_s + \frac{\alpha+1}{\alpha}\mu^2\right) \\
&\geq \frac{1}{x_s^{\alpha+2}}\left[x_1x_2x_s^2 - \left(x_1x_2 + \frac{\alpha+1}{\alpha}\mu^2\right)x_s + \frac{\alpha+1}{\alpha}\mu^2\right] \\
&= \frac{1}{x_s^{\alpha+2}}\left(x_1x_2x_s - \frac{\alpha+1}{\alpha}\mu^2\right)(x_s-1) \\
&\geq 0, \qquad x_s \geq 1.
\end{aligned}$$

The last inequality is due to the assumption that $x_1x_2 \geq (\alpha+1)\mu^2/\alpha$. The proof is thus complete.

Lemma 4. *Let $\alpha > 0$ and let p_1, p_2, x_1, and x_2 be positive real numbers satisfying*

$$p_1 + p_2 = 1, \tag{8}$$

$$p_1x_1 + p_2x_2 = \mu, \tag{9}$$

$$p_1x_1^2 + p_2x_2^2 = \frac{\alpha+1}{\alpha}\mu^2, \tag{10}$$

$$x_1x_2 \geq \frac{\alpha+1}{\alpha}\mu^2, \tag{11}$$

$$0 < x_1 < \mu < x_2. \tag{12}$$

Then $p_1 = [\alpha(x_1/\mu - 1)^2 + 1]^{-1}$, $p_2 = [\alpha(x_2/\mu-1)^2+1]^{-1}$, and the solution set for (x_1, x_2) is $\mathcal{S} := \{(x_1,x_2)\colon (x_1-\mu)(x_2-\mu) = -\mu^2/\alpha,\ x_1 \in I_1,\ x_2 \in I_2\}$, where

$$I_1 = \left[\frac{1+\alpha-\sqrt{1+\alpha}}{\alpha}\mu, \mu\right) \quad \textit{and} \quad I_2 = \left[\frac{1+\alpha+\sqrt{1+\alpha}}{\alpha}\mu, \infty\right).$$

Proof. For a given pair (x_1, x_2), by (10) the values of p_1 and p_2 follow immediately from Lemma 1. Equalities (8)–(10) together imply that

$$\begin{aligned}
&\frac{\alpha+1}{\alpha}\mu^2 - \mu(x_1+x_2) + x_1x_2 \\
&\qquad = (p_1x_1^2 + p_2x_2^2) - (x_1+x_2)(p_1x_1+p_2x_2) + x_1x_2(p_1+p_2) \\
&\qquad = 0
\end{aligned}$$

or, equivalently,

$$(x_1 - \mu)(x_2 - \mu) = -\mu^2/\alpha. \tag{13}$$

The set of pairs (x_1, x_2) satisfying conditions (8)–(12) is exactly the intersection of the curve (13) with regions (11) and (12). Note that, under condition (12), the intersection of the curve (13) with the boundary of region (11) consists of the point

$$\left(\frac{1+\alpha-\sqrt{1+\alpha}}{\alpha}\mu,\ \frac{1+\alpha+\sqrt{1+\alpha}}{\alpha}\mu\right).$$

The proof is thus complete.

We are now ready to prove the main result.

Proof of Theorem 2. It remains to prove the sharpness of the bounds in parts (i) and (iii) of Theorem 2.

(a) If $x = \mu = \alpha\beta$, let F_μ be the degenerate distribution at $x = \mu$. It can be shown that $F_\mu \leq_{\mathrm{L}} G_{\alpha,\beta}$ (or, equivalently, $F_\mu \in \mathcal{L}_\alpha$). Moreover, $\overline{F}_\mu(\mu) = 0$, meaning that the bound in part (i) is attained. This proves the sharpness of the bound in part (i) for the case $x = \mu$. If $x \in I_1$, defined in Lemma 4, then set $x_1 = x$. Then choose an $x_2 \in I_2$ such that $(x_1, x_2) \in \mathcal{S}$ in Lemma 4. In addition, let $p_1 = [\alpha(x_1/\mu - 1)^2 + 1]^{-1}$ and $p_2 = [\alpha(x_2/\mu - 1)^2 + 1]^{-1}$. Then p_1, p_2, x_1, and x_2 together satisfy conditions (8)–(12). Define the random variable Z so as to satisfy $\mathrm{P}(Z = x_1) = p_1$ and $\mathrm{P}(Z = x_2) = p_2$, and denote its two-point distribution by H. Then, by Lemmas 2 and 3, $H \leq_{\mathrm{L}} G_{\alpha,\beta}$ or, equivalently, $H \in \mathcal{L}_\alpha$. Moreover,

$$\overline{H}(x) = p_2 = 1 - p_1 = 1 - [\alpha(x/\mu - 1)^2 + 1]^{-1},$$

meaning that the bound in part (i) is attained. This proves the sharpness of the bound in part (i) for $x \in [(1 + \alpha - \sqrt{1+\alpha})\mu/\alpha, \mu)$.

(b) For $x \in I_2$, let $x_2 > x$ and choose an $x_1 \in I_1$ such that $(x_1, x_2) \in \mathcal{S}$ in Lemma 4. Also, let $p_1 = [\alpha(x_1/\mu - 1)^2 + 1]^{-1}$ and $p_2 = [\alpha(x_2/\mu - 1)^2 + 1]^{-1}$. Consequently, p_1, p_2, x_1, and x_2 together satisfy conditions (8)–(12). Now define the random variable Z^* so as to satisfy $\mathrm{P}(Z^* = x_1) = p_1$ and $\mathrm{P}(Z^* = x_2) = p_2$. Its distribution then belongs to the class $\mathcal{L}_\alpha$, by Lemmas 2 and 3, and $\mathrm{P}(Z^* > x) = p_2 \to [\alpha(x/\mu - 1)^2 + 1]^{-1}$ as $x_2 \to x^+$. Therefore, the bound in part (iii) is sharp for $x \in [(1 + \alpha + \sqrt{1+\alpha})\mu/\alpha, \infty)$. The proof is thus complete.

References

Klar, B. (2002). A note on the $\mathcal{L}$-class of life distributions. *J. Appl. Prob.* **39,** 11–19.

Klefsjö, B. (1983). A useful ageing property based on the Laplace transform. *J. Appl. Prob.* **20,** 615–626.

Lin, G. D. (1998). On weak convergence within the $\mathcal{L}$-like classes of life distributions. *Sankhyā A* **60,** 176–183.

Lin, G. D. and Hu, C.-Y. (2000). A note on the $\mathcal{L}$-class of life distributions. *Sankhyā A* **62,** 267–272.

Mohan, N. R. and Ravi, S. (2002). Some properties of two families of classes of life distributions. *J. Appl. Prob.* **39,** 581–589.

Sengupta, D. (1995). Reliability bounds for the $\mathcal{L}$-class and Laplace order. *J. Appl. Prob.* **32,** 832–835.

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