Canad. Math. Bull. Vol. **25** (4), 1982

THE BEST INTERPOLATING APPROXIMATION IS A LIMIT OF BEST WEIGHTED APPROXIMATIONS

BY

LEE L. KEENER*

ABSTRACT. Under appropriate conditions it is shown that the best interpolating approximation to a given function in the uniform norm is a limit of best unconstrained approximations with respect to a certain sequence of discontinuous weight functions.

We consider a problem recently posed by Dunham [1, problem 69] which we interpret as follows: Let $f \in C[a, b]$ and let $X \subset C[a, b]$ be of finite dimension n. Let $Z = \{z_1, z_2, \ldots, z_k\}$, be a subset of $[a, b]$. Let $\langle(w_1(i), w_2(i), \ldots, w_k(i))\rangle_{i=1}^{\infty}$ be a sequence of positive vectors, where for each j, $1 \le j \le k$, $\lim_{i\to\infty} w_j(i) = \infty$. Define W_i on $[a, b]$ by

$$W_i(x) = \begin{cases} 1 & \text{if } \; x \notin Z \\ w_j(i) & \text{if } \; x = z_j \text{ for some } j,\ 1 \le j \le k \end{cases}$$

Let p_i be the best uniform approximation to f with weight function W_i. That is, denoting the uniform norm on $[a, b]$ by $\|\cdot\|$, $p_i \in X$ and $\|W_i(f-p_i)\| \le \|W_i(f-p)\|$ for all $p \in X$. Such best approximations clearly exist. Suppose that $p^* \in X$ is a best uniform approximation to f interpolating on Z. That is, suppose $p^*(x) = f(x)$ for all $x \in Z$ and $\|p^* - f\| \le \|p - f\|$ for all $p \in K \equiv \{p \in X : p(x) = f(x)\ \forall x \in Z\}$. The existence of such a p^* is also clear provided $K \ne \phi$. Does $p_i \to p^*$ (uniformly)? We answer this question in the affirmative in the case p^* is unique. We need the following lemma.

LEMMA. $\lim_{i\to\infty} \|p_i - f\| = \|p^* - f\|$, *assuming* $K \ne \phi$.

Proof. First note that $\|p_i - f\| \le \|W_i(p_i - f)\| \le \|W_i(p^* - f)\| = \|p^* - f\|$. Suppose there is a subsequence of $\langle p_i \rangle$, call it $\langle p_{i(m)} \rangle$ and a $\delta > 0$ such that $\|p_{i(m)} - f\| \le \|p^* - f\| - \delta$ for all m. Since $\langle p_{i(m)} \rangle$ is a bounded sequence from a finite dimensional space, we may assume $p_{i(m)} \to \bar{p}$ for some $\bar{p} \in X$. $\|\bar{p} - f\| \le \|p^* - f\| - \delta$ and $\bar{p}(x) = f(x)$ for all $x \in Z$ since for all j, $\lim_{i\to\infty} w_j(i) = +\infty$. This implies $\bar{p}$ is a better interpolating approximation than p^*, a contradiction.

Received by the editors March 5, 1981.

AMS Subject Classification: 41A29

* Supported by NSERC of Canada Grant No. 8755.

https://doi.org/10.4153/CMB-1982-075-8 Published online by Cambridge University Press

THEOREM. *If f has a unique best interpolating approximation from X, then $p_i \to p^*$.*

Proof. Let $\langle p_{i(s)} \rangle$ be a convergent subsequence of $\langle p_i \rangle$, converging to $\hat{p} \in X$. It is a consequence of the lemma that $\|\hat{p} - f\| = \|p^* - f\|$. As before, $\hat{p}$ must interpolate f on Z. By the uniqueness hypothesis, $\hat{p} = p^*$. Suppose that $p_i \not\to p^*$. Then there is a second convergent subsequence $\langle p_{i(t)} \rangle$ with $p_{i(t)} \to \bar{p} \in X$, $\bar{p} \neq \hat{p}$. But applying the above argument to $\bar{p}$, we have $\bar{p} = p^* = \hat{p}$, a contradiction.

COROLLARY. *If X is a Tchebycheff space, then $p_i \to p^*$, provided $k \leq n$.*

Proof. By [2], if $k \leq n$, the best interpolating approximation from a Tchebycheff space exists and is unique.

REFERENCES

1. C. B. Dunham, *Problems in best approximation*, Technical Report 62, Department of Computer Science, University of Western Ontario, 1981.

2. H. L. Loeb, D. G. Moursund, L. L. Schumaker and G. D. Taylor, *Uniform generalized weight function polynomial approximation with interpolation*, SIAM J. Numer. Anal. **6** (1969), 283–293.

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTING SCIENCE,
DALHOUSIE UNIVERSITY, HALIFAX,
NOVA SCOTIA, CANADA, B3H 4H8

https://doi.org/10.4153/CMB-1982-075-8 Published online by Cambridge University Press