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GENERALISATION OF A CORRECTED SIMPSON'S FORMULA

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Abstract

The results obtained by A. J. Roberts and N. Ujević in a recent paper are generalised. A number of inequalities for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or R-integrable functions are derived. Also, some error estimates for the derived formulae are obtained.

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1. Introduction

In [7], N. Ujević and A. J. Roberts derived a three-point quadrature formula of closed type that improves on Simpson's rule. Their results are encapsulated in the following theorem.

THEOREM 1.1. *For $f \in C^\infty[a, b]$, we have*

$$\int_a^b f(x)\,dx = \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right] - \frac{(b-a)^2}{60}[f'(b) - f'(a)] + R(f), \tag{1.1}$$

where the error term is, to a leading order estimate,

$$R(f) \approx \frac{(b-a)^6}{302400}[f^{(5)}(b) - f^{(5)}(a)]. \tag{1.2}$$

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In [3], the following two identities, named the extended Euler formulae, have been proved. For $n \geq 1$,

$$f(x) = \int_0^1 f(t)\,\mathrm{d}t + T_n(x) + R_n^1(x) \tag{1.3}$$

and

$$f(x) = \int_0^1 f(t)\,\mathrm{d}t + T_{n-1}(x) + R_n^2(x), \tag{1.4}$$

where

$$T_m(x) = \sum_{k=1}^{m} \frac{B_k(x)}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right], \tag{1.5}$$

for $1 \leq m \leq n$, $T_0(x) = 0$ and

$$R_n^1(x) = -\frac{1}{n!}\int_0^1 B_n^*(x-t)\,\mathrm{d}f^{(n-1)}(t)$$

and

$$R_n^2(x) = -\frac{1}{n!}\int_0^1 \left[B_n^*(x-t) - B_n(x)\right]\mathrm{d}f^{(n-1)}(t).$$

Here, as in the rest of the paper, we write $\int_0^1 g(t)\,d\varphi(t)$ to denote the Riemann-Stieltjes integral with respect to a function $\varphi : [0,1] \to \mathbb{R}$ of bounded variation, and $\int_0^1 g(t)\,dt$ for the Riemann integral. The identities (1.3) and (1.4) extend the well-known formula for the expansion of an arbitrary function in Bernoulli polynomials [5, page 17]. They hold for every function $f : [0,1] \to \mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geq 1$ and for every $x \in [0,1]$. The functions $B_k(t)$ are the Bernoulli polynomials, $B_k = B_k(0)$ are the Bernoulli numbers, and $B_k^*(t)$, $k \geq 0$, are periodic functions of period 1, related to the Bernoulli polynomials by

$$B_k^*(t) = B_k(t), \quad 0 \leq t < 1, \quad \text{and} \quad B_k^*(t+1) = B_k^*(t), \quad t \in \mathbb{R}.$$

The Bernoulli polynomials $B_k(t)$, $k \geq 0$, are uniquely determined by the following identities:

$$B_k'(t) = kB_{k-1}(t), \quad k \geq 1; \quad B_0(t) = 1 \tag{1.6}$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0. \tag{1.7}$$

For some further details on the Bernoulli polynomials and the Bernoulli numbers see, for example, [1] or [2]. We have

$$\begin{aligned} &B_0(t) = 1, && B_2(t) = t^2 - t + \frac{1}{6}, && B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}, \\ &B_1(t) = t - \frac{1}{2}, && B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, && B_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t, \end{aligned} \tag{1.8}$$

so that $B_0^*(t) = 1$ and $B_1^*(t)$ is a discontinuous function with a jump of -1 at each integer. From (1.7) it follows that $B_k(1) = B_k(0) = B_k$ for $k \geq 2$, so that $B_k^*(t)$ are continuous functions for $k \geq 2$. Moreover, using (1.6) we get

$$B_k^{*\prime}(t) = kB_{k-1}^*(t), \tag{1.9}$$

for every $t \in \mathbb{R}$ when $k \geq 3$, and for every $t \in \mathbb{R} \setminus \mathbb{Z}$ when $k = 1, 2$.

The aim of this paper is to establish generalisations of formula (1.1) and other results from [7] together with various error estimates for the quadrature rules based on such generalisations.

We will use the extended Euler formulae (1.3) and (1.4) to obtain two new integral identities, and then prove a number of inequalities related to those formulae for functions whose derivatives are either functions of bounded variation, Lipschitzian functions or R-integrable functions.

2. Main results

For $k \geq 1$, we define functions $G_k(t)$ and $F_k(t)$ such that

$$\begin{aligned} G_k(t) &= 7B_k^*(0 - t) + 16B_k^*(1/2 - t) + 7B_k^*(1 - t) \\ &= 14B_k^*(1 - t) + 16B_k^*(1/2 - t), \quad t \in \mathbb{R} \end{aligned}$$

and

$$F_k(t) = G_k(t) - \tilde{B}_k, \quad t \in \mathbb{R}, \quad k \geq 1,$$

where $\tilde{B}_k = 7B_k(0) + 16B_k(1/2) + 7B_k(1)$, $k \geq 1$.

Using (1.8) we get $\tilde{B}_1 = \tilde{B}_3 = \tilde{B}_4 = \tilde{B}_5 = 0$ and $\tilde{B}_2 = 1$. Also, for $k \geq 2$, we have $\tilde{B}_k = G_k(0)$, that is,

$$F_k(t) = G_k(t) - G_k(0), \quad k \geq 2, \quad \text{and} \quad F_1(t) = G_1(t), \quad t \in \mathbb{R}.$$

Obviously, $G_k(t)$ and $F_k(t)$ are periodic functions of period 1 and continuous for $k \geq 2$.

Let $f : [0, 1] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0, 1]$ for some $n \geq 1$. We introduce the following notation:

$$D(0, 1) = \frac{1}{30}\left[7f(0) + 16f(1/2) + 7f(1)\right].$$

Further, we define $\tilde{T}_0(0, 1) = 0$ and, for $1 \le m \le n$,

$$\tilde{T}_m(0, 1) = \frac{1}{30}\left[7T_m(0) + 16T_m(1/2) + 7T_m(1)\right],$$

where $T_m(x)$ is given by (1.5). It is easy to see that $\tilde{T}_1(0, 1) = 0$ and $\tilde{T}_2(0, 1) = \tilde{T}_3(0, 1) = \tilde{T}_4(0, 1) = \tilde{T}_5(0, 1) = [f'(1) - f'(0)]/60$ and for $m \ge 6$,

$$\begin{aligned}\tilde{T}_m(0, 1) &= \frac{1}{30}\sum_{k=2}^{m}\frac{\tilde{B}_k}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right]\\ &= \frac{1}{60}[f'(1) - f'(0)] + \frac{1}{30}\sum_{k=6}^{m}\frac{\tilde{B}_k}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right]\\ &= \frac{1}{60}[f'(1) - f'(0)] + \frac{1}{30}\sum_{k=3}^{[m/2]}\frac{\tilde{B}_{2k}}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right], \qquad (2.1)\end{aligned}$$

where $[m/2]$ is the greatest integer less than or equal to $m/2$.

In the next theorem we establish two formulae which play a key role in this paper.

THEOREM 2.1. *Let* $f : [0, 1] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is a continuous function of bounded variation on* $[0, 1]$, *for some* $n \ge 1$. *Then*

$$\int_0^1 f(t)\,\mathrm{d}t = D(0, 1) - \tilde{T}_n(0, 1) + \tilde{R}_n^1(f) \qquad (2.2)$$

and

$$\int_0^1 f(t)\,\mathrm{d}t = D(0, 1) - \tilde{T}_{n-1}(0, 1) + \tilde{R}_n^2(f), \qquad (2.3)$$

where

$$\tilde{R}_n^1(f) = \frac{1}{30n!}\int_0^1 G_n(t)\,\mathrm{d}f^{(n-1)}(t) \quad \text{and} \quad \tilde{R}_n^2(f) = \frac{1}{30n!}\int_0^1 F_n(t)\,\mathrm{d}f^{(n-1)}(t).$$

PROOF. Put $x = 0, 1/2, 1$ in (1.3) to get three new formulae. Next, we multiply these new formulae by $7/30$, $16/30$, $7/30$, respectively, and add. The result is (2.2). Formula (2.3) is obtained from (1.4) analogously. □

REMARK 1. The interval $[0, 1]$ is used for simplicity and involves no loss in generality. In what follows, Theorem 2.1 and others will be applied, without comment, to any interval that is convenient.

So, it is easy to prove that if $f : [a, b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a, b]$, for some $n \ge 1$, then

$$\int_a^b f(t)dt = D(a, b) - \tilde{T}_n(a, b) + \frac{(b-a)^n}{30n!}\int_a^b G_n\left(\frac{t-a}{b-a}\right) df^{(n-1)}(t) \qquad (2.4)$$

and

$$\int_a^b f(t)\,dt = D(a,b) - \tilde{T}_{n-1}(a,b) + \frac{(b-a)^n}{30n!}\int_a^b F_n\left(\frac{t-a}{b-a}\right) df^{(n-1)}(t), \quad (2.5)$$

where

$$D(a,b) = \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right]$$

and

$$\tilde{T}_m(a,b) = \frac{1}{30}\sum_{k=1}^{m}\frac{(b-a)^k}{k!}\tilde{B}_k\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right].$$

REMARK 2. Suppose that $f : [0,1] \to \mathbb{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \geq 1$. In this case (2.2) holds with

$$\tilde{R}_n^1(f) = \frac{1}{30n!}\int_0^1 G_n(t) f^{(n)}(t)\,dt,$$

while (2.3) holds with

$$\tilde{R}_n^2(f) = \frac{1}{30n!}\int_0^1 F_n(t) f^{(n)}(t)\,dt.$$

Direct calculation shows that

$$F_1(t) = G_1(t) = \begin{cases} -7, & t = 0, \\ -30t + 7, & 0 < t \leq 1/2, \\ -30t + 23, & 1/2 < t \leq 1, \end{cases} \quad (2.6)$$

$$G_2(t) = \begin{cases} 30t^2 - 14t + 1, & 0 \leq t \leq 1/2, \\ 30t^2 - 46t + 17, & 1/2 < t \leq 1, \end{cases} \quad (2.7)$$

$$F_2(t) = G_2(t) - 1 = \begin{cases} 30t^2 - 14t, & 0 \leq t \leq 1/2, \\ 30t^2 - 46t + 16, & 1/2 < t \leq 1, \end{cases} \quad (2.8)$$

$$F_3(t) = G_3(t) = \begin{cases} -30t^3 + 21t^2 - 3t, & 0 \leq t \leq 1/2, \\ -30t^3 + 69t^2 - 51t + 12, & 1/2 < t \leq 1, \end{cases} \quad (2.9)$$

$$F_4(t) = G_4(t) = \begin{cases} 30t^4 - 28t^3 + 6t^2, & 0 \leq t \leq 1/2, \\ 30t^4 - 92t^3 + 102t^2 - 48t + 8, & 1/2 < t \leq 1 \end{cases} \quad (2.10)$$

and

$$F_5(t) = G_5(t) = \begin{cases} -30t^5 + 35t^4 - 10t^3, & 0 \leq t \leq 1/2, \\ -30t^5 + 115t^4 - 170t^3 + 120t^2 - 40t + 5, & 1/2 < t \leq 1. \end{cases} \quad (2.11)$$

Applying (2.2) for $n = 1$ we get

$$\int_0^1 f(t)\,\mathrm{d}t - D(0,1) = \frac{1}{30}\int_0^1 G_1(t)\,\mathrm{d}f(t)$$

and for $n = 2, 3, 4, 5$

$$\begin{aligned}\int_0^1 f(t)\,\mathrm{d}t - D(0,1) &+ \frac{f'(1)-f'(0)}{60} \\ &= \frac{1}{60}\int_0^1 G_2(t)\,\mathrm{d}f^{(1)}(t) = \frac{1}{180}\int_0^1 G_3(t)\,\mathrm{d}f^{(2)}(t) \\ &= \frac{1}{720}\int_0^1 G_4(t)\,\mathrm{d}f^{(3)}(t) = \frac{1}{3600}\int_0^1 G_5(t)\,\mathrm{d}f^{(4)}(t).\end{aligned}$$

The same identities are obtained from (2.3) for $n = 1, 3, 4, 5$, since $F_k(t) = G_k(t)$ for $k = 1, 3, 4, 5$, while for $n = 2$ we obtain

$$\int_0^1 f(t)\,\mathrm{d}t - D(0,1) = \frac{1}{60}\int_0^1 (G_2(t)-1)\,\mathrm{d}f^{(1)}(t)$$

since $F_2(t) = G_2(t) - 1$. For $n = 6$, (2.3) yields an identity

$$\int_0^1 f(t)\,\mathrm{d}t - D(0,1) + \frac{f'(1)-f'(0)}{60} = \frac{1}{21600}\int_0^1 F_6(t)\,\mathrm{d}f^{(5)}(t).$$

Next, we use formulae derived in Theorem 2.1 to prove a number of inequalities for various classes of functions. First, we need some properties of the functions $G_k(t)$ and $F_k(t)$ defined earlier (see, for example, [5]).

The Bernoulli polynomials are symmetric with respect to $1/2$, that is,

$$B_k(1-t) = (-1)^k B_k(t), \quad \forall\, t \in \mathbb{R},\ k \ge 1. \tag{2.12}$$

Also, we have $B_k(1) = B_k(0) = B_k$, $k \ge 2$, $B_1(1) = -B_1(0) = 1/2$ and $B_{2j-1} = 0$, $j \ge 2$. Using $B_n(1/2) = -(1-2^{1-n})B_n$, $j \ge 1$, we get

$$\tilde{B}_{2j-1} = 0, \quad j \ge 1 \tag{2.13}$$

and

$$\tilde{B}_{2j} = 14B_{2j} + 16B_{2j}(1/2) = -(2 - 16\cdot 2^{1-2j})B_{2j}, \quad j \ge 1. \tag{2.14}$$

Now, (2.13) implies that

$$F_{2j-1}(t) = G_{2j-1}(t), \qquad j \ge 1, \tag{2.15}$$

and (2.14) implies

$$F_{2j}(t) = G_{2j}(t) - \tilde{B}_{2j} = G_{2j}(t) + (2 - 16 \cdot 2^{1-2j})B_{2j}, \qquad j \geq 1. \tag{2.16}$$

Further, the points 0 and 1 are zeros of $F_k(t) = G_k(t) - G_k(0)$, $k \geq 2$, that is, $F_k(0) = F_k(1) = 0$, $k \geq 2$. As we shall see below, 0 and 1 are the only zeros of $F_{2j}(t)$ for $j \geq 3$. Next, setting $t = 1/2$ in (2.12) we get

$$B_k(1/2) = (-1)^k B_k(1/2), \quad k \geq 1,$$

which implies that

$$B_{2j-1}(1/2) = 0, \quad j \geq 1.$$

Using the above formulae, we get $F_{2j-1}(1/2) = G_{2j-1}(1/2) = 0$, $j \geq 1$. We shall see that 0, 1/2 and 1 are the only zeros of $F_{2j-1}(t) = G_{2j-1}(t)$, for $j \geq 3$. Also, note that

$$G_{2j}(1/2) = 14B_{2j}(1/2) + 16B_{2j} = (2 + 14 \cdot 2^{1-2j})B_{2j}, \quad j \geq 1,$$

and

$$F_{2j}(1/2) = G_{2j}(1/2) - \tilde{B}_{2j} = (4 - 2^{2-2j})B_{2j}, \quad j \geq 1. \tag{2.17}$$

LEMMA 2.2. *For $k \geq 2$ we have*

$$G_k(1-t) = (-1)^k G_k(t), \quad 0 \leq t \leq 1,$$

and

$$F_k(1-t) = (-1)^k F_k(t), \quad 0 \leq t \leq 1.$$

PROOF. As we noted in the introduction, the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geq 2$. Therefore, for $k \geq 2$ and $0 \leq t \leq 1$ we have

$$\begin{aligned} G_k(1-t) &= 14B_k^*(t) + 16B_k^*(t - 1/2) \\ &= \begin{cases} 14B_k(t) + 16B_k(t + 1/2), & 0 \leq t \leq 1/2, \\ 14B_k(t) + 16B_k(t - 1/2), & 1/2 \leq t \leq 1 \end{cases} \\ &= (-1)^k \times \begin{cases} 14B_k(1-t) + 16B_k(1/2 - t), & 0 \leq t \leq 1/2, \\ 14B_k(1-t) + 16B_k(3/2 - t), & 1/2 \leq t \leq 1 \end{cases} \\ &= (-1)^k G_k(t), \end{aligned}$$

which proves the first identity. Further, we have $F_k(t) = G_k(t) - G_k(0)$ and $(-1)^k G_k(0) = G_k(0)$, since $G_{2j+1}(0) = 0$, so that we have

$$F_k(1-t) = G_k(1-t) - G_k(0) = (-1)^k[G_k(t) - G_k(0)] = (-1)^k F_k(t),$$

which proves the second identity. □

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Note that the identities established in Lemma 2.2 are valid for $k = 1$, too, except at the points 0, 1/2, 1 of discontinuity of $F_1(t) = G_1(t)$.

LEMMA 2.3. *For $k \geq 3$ the function $G_{2k-1}(t)$ has no zeros in the interval $(0, 1/2)$. The sign of this function is determined by*

$$(-1)^k G_{2k-1}(t) > 0, \quad 0 < t < 1/2.$$

PROOF. For $k = 3$, $G_5(t)$ is given by (2.11) and it is easy to see that

$$G_5(t) < 0, \quad 0 < t < 1/2.$$

Thus our assertion is true for $k = 3$. Now, assume $k \geq 4$ so $2k - 1 \geq 7$. $G_{2k-1}(t)$ is continuous and an at least twice differentiable function. Using (1.9) we get

$$G'_{2k-1}(t) = -(2k-1)G_{2k-2}(t)$$

and

$$G''_{2k-1}(t) = (2k-1)(2k-2)G_{2k-3}(t).$$

Let us suppose that G_{2k-3} has no zeros in the interval $(0, 1/2)$. We know that 0 and 1/2 are zeros of $G_{2k-1}(t)$ but let us suppose that some $\alpha, 0 < \alpha < 1/2$, is also a zero of $G_{2k-1}(t)$. Then inside each of the intervals $(0, \alpha)$ and $(\alpha, 1/2)$ the derivative $G'_{2k-1}(t)$ must have at least one zero, say β_1, $0 < \beta_1 < \alpha$ and β_2, $\alpha < \beta_2 < 1/2$. Therefore the second derivative $G''_{2k-1}(t)$ must have at least one zero inside the interval (β_1, β_2). Thus from the assumption that $G_{2k-1}(t)$ has a zero inside the interval $(0, 1/2)$, it follows that $(2k-1)(2k-2)G_{2k-3}(t)$ also has a zero inside this interval which is not true. Therefore, $G_{2k-1}(t)$ cannot have a zero inside the interval $(0, 1/2)$. To determine the sign of $G_{2k-1}(t)$, note that $G_{2k-1}(1/4) = 2B_{2k-1}(1/4)$. We have (see, for example, [1])

$$(-1)^k B_{2k-1}(t) > 0, \quad 0 < t < 1/2,$$

which implies

$$(-1)^k G_{2k-1}(1/4) = 2 \cdot (-1)^k B_{2k-1}(1/4) > 0.$$

Consequently, we have $(-1)^k G_{2k-1}(t) > 0$, for $0 < t < 1/2$. □

COROLLARY 2.4. *For $k \geq 3$, the functions $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly increasing on the interval $(0, 1/2)$ and strictly decreasing on the interval $(1/2, 1)$. Further, for $k \geq 3$, we have*

$$\max_{t\in[0,1]} |F_{2k}(t)| = 4(1 - 2^{-2k})|B_{2k}|$$

and

$$\max_{t\in[0,1]} |G_{2k}(t)| = (2 + 14 \cdot 2^{1-2k})|B_{2k}|.$$

PROOF. Using (1.9) we get

$$[(-1)^{k-1}F_{2k}(t)]' = [(-1)^{k-1}G_{2k}(t)]' = (-1)^k \cdot 2k \cdot G_{2k-1}(t).$$

From Lemma 2.3 we conclude that $[(-1)^{k-1}F_{2k}(t)]' > 0$ for $0 < t < 1/2$. Thus $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly increasing on the interval $(0, 1/2)$. Also, by Lemma 2.2, we have $F_{2k}(1-t) = F_{2k}(t)$ and $G_{2k}(1-t) = G_{2k}(t)$ for $0 \le t \le 1$, which implies that $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly decreasing on the interval $(1/2, 1)$. Further, $F_{2k}(0) = F_{2k}(1) = 0$, which implies that $|F_{2k}(t)|$ achieves its maximum at $t = 1/2$, that is,

$$\max_{t\in[0,1]} |F_{2k}(t)| = |F_{2k}(1/2)| = (4 - 2^{2-2k})|B_{2k}|.$$

Also,

$$\max_{t\in[0,1]} |G_{2k}(t)| = \max\left\{|G_{2k}(0)|, |G_{2k}(1/2)|\right\}$$
$$= |G_{2k}(1/2)| = (2 + 14 \cdot 2^{1-2k})|B_{2k}|,$$

which completes the proof. □

COROLLARY 2.5. *For $k \ge 3$, we have*

$$\int_0^1 |F_{2k-1}(t)|\,dt = \int_0^1 |G_{2k-1}(t)|\,dt = \frac{4}{k}(1 - 2^{-2k})|B_{2k}|.$$

Also, we have

$$\int_0^1 |F_{2k}(t)|\,dt = |\tilde{B}_{2k}| = (2 - 16 \cdot 2^{1-2k})|B_{2k}|$$

and

$$\int_0^1 |G_{2k}(t)|\,dt \le 2|\tilde{B}_{2k}| = (4 - 16 \cdot 2^{2-2k})|B_{2k}|.$$

PROOF. Using (1.9) it is easy to see that

$$G'_m(t) = -mG_{m-1}(t), \quad m \ge 3. \tag{2.18}$$

Now, using Lemmas 2.2–2.3 and (2.18) we get

$$\int_0^1 |G_{2k-1}(t)|\,dt = 2\left|\int_0^{1/2} G_{2k-1}(t)\,dt\right| = \frac{1}{k}\left|G_{2k}(1/2) - G_{2k}(0)\right|$$
$$= \frac{4}{k}(1 - 2^{-2k})|B_{2k}|,$$

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which proves the first assertion. Since $F_{2k}(0) = F_{2k}(1) = 0$, from Corollary 2.4 we conclude that $F_{2k}(t)$ does not change sign on $(0, 1)$. Therefore, using (2.16) and (2.18), we get

$$\int_0^1 |F_{2k}(t)|\,dt = \left|\int_0^1 F_{2k}(t)dt\right| = \left|\int_0^1 \left(G_{2k}(t) - \tilde{B}_{2k}\right) dt\right|$$
$$= \left|-\frac{1}{2k+1} G_{2k+1}(t)\Big|_0^1 - \tilde{B}_{2k}\right| = |\tilde{B}_{2k}|,$$

which proves the second assertion. Finally, we use (2.16) together with the triangle inequality to obtain

$$\int_0^1 |G_{2k}(t)|\,dt = \int_0^1 \left|F_{2k}(t) + \tilde{B}_{2k}\right| dt \le \int_0^1 |F_{2k}(t)|\,dt + |\tilde{B}_{2k}| = 2|\tilde{B}_{2k}|,$$

which proves the third assertion. □

THEOREM 2.6. *Let $f : [0, 1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0, 1]$ for some $n \ge 1$. Then*

$$\left|\int_0^1 f(t)\,dt - D(0, 1) + \tilde{T}_{n-1}(0, 1)\right| \le \frac{L}{30n!} \int_0^1 |F_n(t)|\,dt \tag{2.19}$$

and

$$\left|\int_0^1 f(t)dt - D(0, 1) + \tilde{T}_n(0, 1)\right| \le \frac{L}{30n!} \int_0^1 |G_n(t)|\,dt. \tag{2.20}$$

PROOF. For any integrable function $\Phi : [0, 1] \to \mathbb{R}$ we have

$$\left|\int_0^1 \Phi(t)df^{(n-1)}(t)\right| \le L \int_0^1 |\Phi(t)|\,dt, \tag{2.21}$$

since $f^{(n-1)}$ is an L-Lipschitzian function. If we take $\Phi(t) = F_n(t)$, we'll get

$$\left|\frac{1}{30n!} \int_0^1 F_n(t)\,df^{(n-1)}(t)\right| \le \frac{L}{30n!} \int_0^1 |F_n(t)|\,dt.$$

Inequality (2.19) is obtained from identity (2.3) after applying the above inequality. Similarly, we apply (2.21) for $\Phi(t) = G_n(t)$ and then use (2.2) to obtain inequality (2.20). □

COROLLARY 2.7. *Let $f : [0, 1] \to \mathbb{R}$. If f is L-Lipschitzian on $[0, 1]$, then*

$$\left|\int_0^1 f(t)dt - D(0, 1)\right| \le \frac{113}{900} L.$$

If f' is L-Lipschitzian on $[0, 1]$, *then*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{697}{40500} L.$$

PROOF. From (2.6) and (2.8) we get

$$\int_0^1 |F_1(t)|\,\mathrm{d}t = \frac{113}{30} \quad \text{and} \quad \int_0^1 |F_2(t)|\,\mathrm{d}t = \frac{697}{675}.$$

Applying (2.19) for $n = 1, 2$, we get the above inequalities. □

Using (2.13) and (2.14), for $m \ge 6$ from (2.1) we get

$$\tilde{T}_m(0,1) = \frac{1}{60}\left[f'(1) - f'(0)\right] + \frac{1}{30}\sum_{k=3}^{[m/2]} \frac{B_{2k}}{(2k)!}(16 \cdot 2^{1-2k} - 2)\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]. \tag{2.22}$$

COROLLARY 2.8. *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0, 1]$ for some $n \ge 5$. For any integer r such that $1 \le r \le n/2$ define*

$$D_r(f) := \frac{1}{30}\sum_{i=1}^{r} \frac{B_{2i}}{(2i)!}(16 \cdot 2^{1-2i} - 2)\left[f^{(2i-1)}(1) - f^{(2i-1)}(0)\right]. \tag{2.23}$$

If $n = 2k - 1$, $k \ge 3$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + D_{k-1}(f)\right| \le \frac{4L}{15(2k)!}\left(1 - 2^{-2k}\right)|B_{2k}|.$$

If $n = 2k$, $k \ge 3$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + D_{k-1}(f)\right| \le \frac{L}{30(2k)!}(2 - 16 \cdot 2^{1-2k})|B_{2k}|$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + D_k(f)\right| \le \frac{L}{15(2k)!}(2 - 16 \cdot 2^{1-2k})|B_{2k}|.$$

PROOF. For $n = 2k - 1$, by (2.22) we have $\tilde{T}_{n-1}(0,1) = D_{k-1}(f)$. Thus the first inequality follows from Corollary 2.5 and (2.19). For $n = 2k$, by (2.22) we have $\tilde{T}_{n-1}(0,1) = D_{k-1}(f)$ and $\tilde{T}_n(0,1) = D_k(f)$. Now, the second inequality follows from Corollary 2.5 and (2.19), while the third follows from Corollary 2.5 and (2.20). □

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REMARK 3. Suppose that $f : [0,1] \to \mathbb{R}$ is such that $f^{(n)}$ exists and is bounded on $[0,1]$, for some $n \geq 1$. Therefore the inequalities established in Theorem 2.6 hold with $L = \|f^{(n)}\|_\infty$.

THEOREM 2.9. *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n \geq 1$. Then*

$$\left|\int_0^1 f(t)\,dt - D(0,1) + \tilde{T}_{n-1}(0,1)\right| \leq \frac{1}{30n!}\max_{t\in[0,1]}|F_n(t)|\, V_0^1(f^{(n-1)}) \tag{2.24}$$

and

$$\left|\int_0^1 f(t)\,dt - D(0,1) + \tilde{T}_n(0,1)\right| \leq \frac{1}{30n!}\max_{t\in[0,1]}|G_n(t)|\, V_0^1(f^{(n-1)}), \tag{2.25}$$

where $V_0^1(f^{(n-1)})$ is the total variation of $f^{(n-1)}$ on $[0,1]$.

PROOF. If $\Phi : [0,1] \to \mathbb{R}$ is bounded on $[0,1]$ and the Riemann-Stieltjes integral $\int_0^1 \Phi(t)\,df^{(n-1)}(t)$ exists, then

$$\left|\int_0^1 \Phi(t)\mathrm{d}f^{(n-1)}(t)\right| \leq \max_{t\in[0,1]}|\Phi(t)|\, V_0^1(f^{(n-1)}). \tag{2.26}$$

Now, (2.24) is obtained from identity (2.3) after applying the above inequality for $\Phi(t) = F_n(t)$. Analogously, we derive inequality (2.25) from identity (2.2) by applying (2.26) to $\Phi(t) = G_n(t)$. □

COROLLARY 2.10. *Let $f : [0,1] \to \mathbb{R}$. If f is a continuous function of bounded variation on $[0,1]$, then*

$$\left|\int_0^1 f(t)\,dt - D(0,1)\right| \leq \frac{4}{15}V_0^1(f).$$

If f' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\,dt - D(0,1)\right| \leq \frac{49}{1800}V_0^1(f').$$

PROOF. From the explicit expressions (2.6) and (2.8), we get

$$\max_{t\in[0,1]}|F_1(t)| = -F_1\left(\frac{1}{2}\right) = 8 \quad\text{and}\quad \max_{t\in[0,1]}|F_2(t)| = F_2\left(\frac{7}{30}\right) = \frac{49}{30}.$$

We get the above inequalities from (2.24) for $n = 1$ and $n = 2$. □

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COROLLARY 2.11. *Let $f : [0, 1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0, 1]$ for some $n \geq 5$. Define $D_r(f)$, $r \geq 1$ as in Corollary 2.8. If $n = 2k - 1$, $k \geq 3$, then*

$$\left| \int_0^1 f(t)\,\mathrm{d}t - D(0,1) + D_{k-1}(f) \right| \leq \frac{1}{30(2k-1)!} \max_{t \in [0,1]} |F_{2k-1}(t)| V_0^1(f^{(2k-2)}).$$

If $n = 2k$, $k \geq 3$, then

$$\left| \int_0^1 f(t)\,\mathrm{d}t - D(0,1) + D_{k-1}(f) \right| \leq \frac{2}{15(2k)!}(1 - 2^{-2k})|B_{2k}| V_0^1(f^{(2k-1)})$$

and

$$\left| \int_0^1 f(t)\,\mathrm{d}t - D(0,1) + D_k(f) \right| \leq \frac{1}{15(2k)!}(1 + 7 \cdot 2^{1-2k})|B_{2k}| V_0^1(f^{(2k-1)}).$$

PROOF. The argument is similar to that used in the proof of Corollary 2.8. We apply Theorem 2.9 and use the formulae established in Corollary 2.4. □

REMARK 4. Suppose that $f^{(n)} : [0, 1] \to \mathbb{R}$ is an R-integrable function for some $n \geq 1$. In this case $f^{(n-1)}$ is a continuous function of bounded variation on $[0, 1]$ and we have $V_0^1(f^{(n-1)}) = \int_0^1 |f^{(n)}(t)|\,\mathrm{d}t = \|f^{(n)}\|_1$. Therefore the inequalities established in Theorem 2.9 hold with $\|f^{(n)}\|_1$ in place of $V_0^1(f^{(n-1)})$. A similar observation can be made for the results of Corollaries 2.10 and 2.11.

THEOREM 2.12. *Assume (p, q) is a pair of conjugate exponents, that is, $1 < p, q < \infty$, $1/p + 1/q = 1$ or $p = \infty$, $q = 1$. Let $|f^{(n)}|^p : [0, 1] \to \mathbb{R}$ be an R-integrable function for some $n \geq 1$. Then we have*

$$\left| \int_0^1 f(t)\,\mathrm{d}t - D(0,1) + \tilde{T}_{n-1}(0,1) \right| \leq K(n,p)\|f^{(n)}\|_p \tag{2.27}$$

and

$$\left| \int_0^1 f(t)\,\mathrm{d}t - D(0,1) + \tilde{T}_n(0,1) \right| \leq K^*(n,p)\|f^{(n)}\|_p, \tag{2.28}$$

where

$$K(n,p) = \frac{1}{30n!} \left[\int_0^1 |F_n(t)|^q\,\mathrm{d}t \right]^{1/q}$$

and

$$K^*(n,p) = \frac{1}{30n!} \left[\int_0^1 |G_n(t)|^q\,\mathrm{d}t \right]^{1/q}.$$

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PROOF. Applying the Hölder inequality we have

$$\left|\frac{1}{30n!}\int_0^1 F_n(t)f^{(n)}(t)\,dt\right| \le \frac{1}{30n!}\left[\int_0^1 |F_n(t)|^q\,dt\right]^{1/q} \left\|f^{(n)}\right\|_p = K(n,p)\|f^{(n)}\|_p.$$

Bearing Remark 2 in mind, from the above inequality and (2.3), we get estimate (2.27). Similarly, from (2.2) we obtain estimate (2.28). □

REMARK 5. For $p = \infty$ we have

$$K(n,\infty) = \frac{1}{30n!}\int_0^1 |F_n(t)|\,dt \quad \text{and} \quad K^*(n,\infty) = \frac{1}{30n!}\int_0^1 |G_n(t)|\,dt.$$

The results established in Theorem 2.12 for $p = \infty$ coincide with the results of Theorem 2.6 with $L = \|f^{(n)}\|_\infty$. Moreover, by Remark 3 and Corollary 2.7, we have

$$\left|\int_0^1 f(t)\,dt - D(0,1)\right| \le K(n,\infty)\|f^{(n)}\|_\infty, \quad n = 1, 2,$$

where

$$K(1,\infty) = \frac{113}{900} \quad \text{and} \quad K(2,\infty) = \frac{697}{40500}.$$

REMARK 6. Let us define for $p = 1$

$$K(n,1) = \frac{1}{30n!}\max_{t\in[0,1]}|F_n(t)| \quad \text{and} \quad K^*(n,1) = \frac{1}{30n!}\max_{t\in[0,1]}|G_n(t)|.$$

Then, using Remark 4 and Theorem 2.9, we can extend the results established in Theorem 2.12 to the pair $p = 1$, $q = \infty$. Also, by Remark 4 and Corollary 2.10, we have

$$\left|\int_0^1 f(t)\,dt - D(0,1)\right| \le K(n,1)\|f^{(n)}\|_1, \quad n = 1, 2,$$

where $K(1,1) = 4/15$ and $K(2,1) = 49/1800$.

REMARK 7. Note that $K^*(1,p) = K(1,p)$, for $1 < p \le \infty$, since $G_1(t) = F_1(t)$. Also, for $1 < p \le \infty$, we can easily calculate $K(1,p)$:

$$K(1,p) = \frac{1}{30}\left[\frac{7^{q+1}+8^{q+1}}{15(q+1)}\right]^{1/q}, \quad 1 < p \le \infty.$$

In the limit case when $p \to 1$, that is, when $q \to \infty$, we have

$$\lim_{q\to\infty}\frac{1}{30}\left[\frac{7^{q+1}+8^{q+1}}{15(q+1)}\right]^{1/q} = \frac{4}{15} = K(1,1).$$

Now we use (2.2) to obtain a Grüss-type inequality related to the formulae derived in Theorem 2.1. To do this we need the following two technical lemmas. The first one was proved in [4] and the second one is the key result from [6].

LEMMA 2.13. *Let $k \geq 1$ and $\gamma \in \mathbb{R}$. Then $\int_0^1 B_k^*(\gamma - t)\,dt = 0$.*

LEMMA 2.14. *Let $F, G : [0, 1] \to \mathbb{R}$ be two integrable functions. If*

$$m \leq F(t) \leq M, \quad 0 \leq t \leq 1,$$

and $\int_0^1 G(t)\,dt = 0$, then

$$\left|\int_0^1 F(t)G(t)\,dt\right| \leq \frac{M-m}{2}\int_0^1 |G(t)|\,dt. \tag{2.29}$$

THEOREM 2.15. *Let $f : [0, 1] \to \mathbb{R}$ be such that $f^{(n)}$ exists and is integrable on $[0, 1]$, for some $n \geq 1$. Suppose $m_n \leq f^{(n)}(t) \leq M_n$, $0 \leq t \leq 1$, for some constants m_n and M_n. Then*

$$\left|\int_0^1 f(t)\,dt - D(0,1) + \tilde{T}_n(0,1)\right|$$
$$\leq \begin{cases} 113(M_1 - m_1)/1800 & \text{for } n = 1, \\ 19\sqrt{19}(M_2 - m_2)/20250 & \text{for } n = 2, \\ 253(M_3 - m_3)/720000 & \text{for } n = 3, \\ (M_4 - m_4)/29160 & \text{for } n = 4, \\ (2 - 2^{1-2k})(M_{2k-1} - m_{2k-1})\dfrac{|B_{2k}|}{15(2k)!} & \text{for } n = 2k-1,\ k \geq 3, \\ (2 - 16 \cdot 2^{1-2k})(M_{2k} - m_{2k})\dfrac{|B_{2k}|}{30(2k)!} & \text{for } n = 2k,\ k \geq 3. \end{cases} \tag{2.30}$$

PROOF. By Remark 2 we can rewrite $\tilde{R}_n^1(f)$ as

$$\tilde{R}_n^1(f) = \frac{1}{30n!}\int_0^1 F(t)G(t)\,dt,$$

where $F(t) = f^{(n)}(t)$ and $G(t) = G_n(t)$, $0 \leq t \leq 1$. Using Lemma 2.13 we get

$$\int_0^1 G(t)\,dt = 0.$$

Also, using Corollary 2.5 for $n \geq 5$ we get

$$\int_0^1 |G_n(t)|\,dt \begin{cases} = (4/k)(1 - 2^{-2k})|B_{2k}| & \text{for } n = 2k-1, \\ \leq (4 - 16 \cdot 2^{2-2k})|B_{2k}| & \text{for } n = 2k. \end{cases}$$

For $n = 1, 2, 3, 4$ we have

$$\int_0^1 |G_1(t)|\,dt = \frac{113}{30}, \quad \int_0^1 |G_2(t)|\,dt = \frac{76\sqrt{19}}{675},$$
$$\int_0^1 |G_3(t)|\,dt = \frac{253}{2000}, \quad \int_0^1 |G_4(t)|\,dt = \frac{4}{81}.$$

We apply inequality (2.29) to obtain the estimate

$$|\tilde{R}_n^1(f)| \le \frac{1}{30n!}\frac{M_n - m_n}{2}\int_0^1 |G_n(t)|\,dt$$
$$\begin{cases} = 113(M_1 - m_1)/1800 & \text{for } n = 1, \\ = 19\sqrt{19}(M_2 - m_2)/20250 & \text{for } n = 2, \\ = 253(M_3 - m_3)/720000 & \text{for } n = 3, \\ = (M_4 - m_4)/29160 & \text{for } n = 4, \\ = (2 - 2^{1-2k})(M_{2k-1} - m_{2k-1})\dfrac{|B_{2k}|}{15(2k)!} & \text{for } n = 2k - 1,\ k \ge 3, \\ \le (2 - 16 \cdot 2^{1-2k})(M_{2k} - m_{2k})\dfrac{|B_{2k}|}{30(2k)!} & \text{for } n = 2k,\ k \ge 3, \end{cases}$$

which proves our assertion. □

REMARK 8. Results from Theorem 2.15 for $n = 2, \dots, 6$ were obtained in [7].

In the following discussion we assume that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative of order n, for some $n \ge 1$. In this case we can use (2.3) and the second formula from Remark 2 to obtain, for $n = 2k$,

$$\tilde{R}_{2k}^2(f) = \frac{1}{30(2k)!}\int_0^1 F_{2k}(s) f^{(2k)}(s)\,ds. \tag{2.31}$$

THEOREM 2.16. *If $f : [0, 1] \to \mathbb{R}$ is such that $f^{(2k)}$ is a continuous function on $[0, 1]$, for some $k \ge 3$, then there exists a point $\eta \in [0, 1]$ such that*

$$\tilde{R}_{2k}^2(f) = \frac{1}{30(2k)!}(2 - 16 \cdot 2^{1-2k}) B_{2k} f^{(2k)}(\eta). \tag{2.32}$$

PROOF. Using (2.31) we can rewrite $\tilde{R}_{2k}^2(f)$ as

$$\tilde{R}_{2k}^2(f) = \frac{(-1)^{k-1}}{30(2k)!} J_k, \tag{2.33}$$

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where

$$J_k = \int_0^1 (-1)^{k-1} F_{2k}(s) f^{(2k)}(s)\,ds. \tag{2.34}$$

From Corollary 2.4 it follows that $(-1)^{k-1}F_{2k}(s) \geq 0$, for $0 \leq s \leq 1$, and the claim follows from the mean value theorem for integrals and Corollary 2.5. □

REMARK 9. For $k = 3$, formula (2.32) reduces to

$$\tilde{R}_6^2(f) = \frac{1}{604800} f^{(6)}(\eta).$$

The same approximation was obtained in [7].

COROLLARY 2.17. *Let $f \in C^\infty[0, 1]$ and $\lambda \in \mathbb{R}$ be such that $0 < \lambda < 2\pi$ and $|f^{(2k)}(t)| \leq \lambda^{2k}$ for $t \in [0, 1]$ and $k \geq k_0$ for some $k_0 \geq 3$. Then*

$$\int_0^1 f(t)\,dt = D(0, 1) - \frac{1}{30}\sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}(16 \cdot 2^{1-2j} - 2)\left[f^{(2j-1)}(1) - f^{(2j-1)}(0)\right]. \tag{2.35}$$

PROOF. From Theorem 2.16, when $k \geq k_0$ it follows that

$$|\tilde{R}_{2k}^2(f)| \leq \frac{2|B_{2k}|}{30(2k)!}\lambda^{2k} \approx \frac{1}{15(2k)!} \cdot 2\frac{(2k)!}{(2\pi)^{2k}}\lambda^{2k} = \frac{2}{15}\left(\frac{\lambda}{2\pi}\right)^{2k},$$

so (2.35) follows. □

THEOREM 2.18. *If $f : [0, 1] \to \mathbb{R}$ is such that $f^{(2k)}$ is a continuous function on $[0, 1]$, for some $k \geq 3$, and does not change its sign on $[0, 1]$, then there exists a point $\theta \in [0, 1]$ such that*

$$\tilde{R}_{2k}^2(f) = \theta\frac{4 - 2^{2-2k}}{30(2k)!} B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]. \tag{2.36}$$

PROOF. Suppose that $f^{(2k)}(t) \geq 0$, $0 \leq t \leq 1$. From Corollary 2.4 it follows that

$$0 \leq (-1)^{k-1}F_{2k}(s) \leq (-1)^{k-1}F_{2k}(1/2), \quad 0 \leq s \leq 1.$$

Therefore, if J_k is given by (2.34), then $0 \leq J_k \leq (-1)^{k-1}F_{2k}(1/2)\int_0^1 f^{(2k)}(s)\,ds$. Using (2.17), we get

$$0 \leq J_k \leq (-1)^{k-1}(4 - 2^{2-2k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right],$$

which means that there must exist a point $\theta \in [0, 1]$ such that

$$J_k = \theta(-1)^{k-1}(4 - 2^{2-2k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

Combining this with (2.33) we get (2.36). The argument is the same when $f^{(2k)}(t) \le 0$, $0 \le t \le 1$, since in that case we get

$$(-1)^{k-1}(4 - 2^{2-2k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right] \le J_k \le 0. \qquad \square$$

REMARK 10. The same series expansion of $\int_0^1 f(t)\,dt$ as in Corollary 2.17 can be obtained from the previous theorem under the assumption

$$\left|f^{(2k-1)}(1) - f^{(2k-1)}(0)\right| \le \lambda^{2k}$$

for every $k \ge k_0$ for some $k_0 \ge 3$ where $0 < \lambda < 2\pi$.

REMARK 11. If we approximate $\int_0^1 f(t)\,dt$ by

$$I_{2k}(f) = D(0, 1) - \frac{1}{60}[f'(1) - f'(0)] - \frac{1}{30}\sum_{j=3}^{k-1}\frac{B_{2j}}{(2j)!}(16 \cdot 2^{1-2j} - 2)\left[f^{(2j-1)}(1) - f^{(2j-1)}(0)\right],$$

then the next approximation will be $I_{2k+2}(f)$. The difference

$$\Delta_{2k}(f) = I_{2k+2}(f) - I_{2k}(f)$$

is equal to the last term in $I_{2k+2}(f)$, that is,

$$\Delta_{2k}(f) = \frac{B_{2k}}{30(2k)!}(2 - 16 \cdot 2^{1-2k})\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

We see that, under the assumptions of Theorem 2.18, $\tilde{R}^2_{2k}(f)$ and $\Delta_{2k}(f)$ are of the same sign. Moreover, we have

$$\tilde{R}^2_{2k}(f) = \theta\,\frac{2 - 2^{1-2k}}{1 - 8 \cdot 2^{1-2k}}\,\Delta_{2k}(f).$$

THEOREM 2.19. *Suppose that* $f : [0, 1] \to \mathbb{R}$ *is such that* $f^{(2k+2)}$ *is a continuous function on* $[0, 1]$ *for some* $k \ge 3$. *If*

$$f^{(2k)}(x) \ge 0 \quad \text{and} \quad f^{(2k+2)}(x) \ge 0, \quad x \in [0, 1], \tag{2.37}$$

or

$$f^{(2k)}(x) \le 0 \quad \text{and} \quad f^{(2k+2)}(x) \le 0, \quad x \in [0, 1], \tag{2.38}$$

then the remainder $\tilde{R}^2_{2k}(f)$ *has the same sign as the first neglected term* $\Delta_{2k}(f)$ *and* $|\tilde{R}^2_{2k}(f)| \le |\Delta_{2k}(f)|$.

PROOF. We have $\Delta_{2k}(f) + \tilde{R}^2_{2k+2}(f) = \tilde{R}^2_{2k}(f)$, that is,

$$\Delta_{2k}(f) = \tilde{R}^2_{2k}(f) - \tilde{R}^2_{2k+2}(f). \tag{2.39}$$

By (2.31) we have

$$\tilde{R}^2_{2k}(f) = \frac{1}{30(2k)!} \int_0^1 F_{2k}(s) f^{(2k)}(s)\, ds$$

and

$$-\tilde{R}^2_{2k+2}(f) = \frac{1}{30(2k+2)!} \int_0^1 [-F_{2k+2}(s)] f^{(2k+2)}(s)\, ds.$$

From Corollary 2.4 it follows that for all $s \in [0, 1]$

$$(-1)^{k-1} F_{2k}(s) \geq 0 \quad \text{and} \quad (-1)^{k-1}[-F_{2k+2}(s)] \geq 0.$$

We conclude that $\tilde{R}^2_{2k}(f)$ has the same sign as $-\tilde{R}^2_{2k+2}(f)$. Therefore, because of (2.39), $\Delta_{2k}(f)$ must have the same sign as $\tilde{R}^2_{2k}(f)$ and $-\tilde{R}^2_{2k+2}(f)$. Moreover, it follows that $|\tilde{R}^2_{2k}(f)| \leq |\Delta_{2k}(f)|$ and $|\tilde{R}^2_{2k+2}(f)| \leq |\Delta_{2k}(f)|$. □

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