ANALYTIC EQUIVALENCE OF ALGEBROID CURVES

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1. Introduction. Let k be an algebraically closed field and let $x_1, x_2, \ldots, x_n$ be indeterminates. Denote by R_n the ring $k[[x_1, x_2, \ldots, x_n]]$ of power series in the x_i with coefficients in the field k. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two ideals in this ring. Then $\mathfrak{A}$ and $\mathfrak{A}'$ will be said to be analytically equivalent if there is an automorphism T of R_n such that $T(\mathfrak{A}) = \mathfrak{A}'$. $\mathfrak{A}$ and $\mathfrak{A}'$ will be called analytically equivalent under T.

The above situation can be described geometrically as follows. The ideals $\mathfrak{A}$ and $\mathfrak{A}'$ can be regarded as defining algebroid varieties V and V' in $(x_1, x_2, \ldots, x_n)$-space, and these varieties will be said to be analytically equivalent under T.

The automorphism T can be expressed by means of equations of the form:

$$T(x_i) = \sum a_{ij}x_j + f_i(x)$$

where the determinant $|a_{ij}|$ is not zero and the f_i are power series of order not less than two (that is to say, containing terms of degree two or more only). If the f_i are all of order greater than or equal to r, the analytic equivalence T will be said to be of order r. Throughout this paper the only analytic equivalences which will be considered will be those in which the coefficients a_{ij} in the above equations satisfy the conditions $a_{ii} = 1$, $a_{ij} = 0$ for $i \neq j$. This will not, in fact, impose any essential restriction, for in the main theorem to be proved the analytic equivalence which appears happens in any case to be of this form.

The problem to be studied here can be formulated as follows. Suppose that $F_i(x) = 0$, $i = 1, 2, \ldots, r$ and $F_i'(x) = 0$, $i = 1, 2, \ldots, r$ are sets of equations for the varieties V and V' respectively, that is to say that F_i and the F_i' are sets of generators of the ideals $\mathfrak{A}$ and $\mathfrak{A}'$ respectively. Then if V and V' are analytically equivalent under an analytic equivalence of sufficiently high order, it is clear that the F_i' can be chosen as power series differing from the corresponding F_i only by terms of high order. The question is, to what extent is the converse of this statement true?

A partial answer to this is given by a theorem of Samuel **(4)** for the case in which $r = 1$ and the origin is an isolated singular point of the variety V. Under these conditions Samuel's theorem states that, if the order of $F_1 - F_1'$ is high enough, then the hypersurfaces V and V' are analytically equivalent. The restrictions on the singularities of V here are very strong, and are imposed by the method of proof. On the other hand it is clear that if V and V' are to be

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analytically equivalent some restrictions on the relations between their singular points must be imposed. However it seems to be rather difficult to see exactly what these conditions should be for varieties of arbitrary dimension; and so a general answer to the question formulated above seems impossible until some new methods are found.

In view of the difficulties mentioned I am restricting myself here to the attempt to answer the above question in the case of curves. The form which the answer takes in this case makes it sufficient to consider curves V and V' defined by the sets of equations $F_i = 0$ and $F_i' = 0$, respectively, i running in each case from 1 to $n - 1$. The question to be answered then becomes the following. V and V' being as just described, will they be analytically equivalent if the orders of the $F_i - F_i'$ are high enough? Samuel's theorem gives the answer yes when $n = 2$ and V is irreducible. It is clear however that in general the answer will not be affirmative without some further restriction. For if V has a multiple component, geometrically speaking a component singular on one of the $F_i = 0$ or along which two of them touch, it is intuitively obvious that modification of the equations of V, even by terms of high order, may cause such a component to split, thus making analytic equivalence impossible. But it seems reasonable to hope that components of V along which the F_i intersect simply and transversally will be carried by an analytic equivalence into similar components of V' if the orders of the $F_i - F_i'$ are high enough. This is the main result to be proved in this paper. The components of V just described correspond to the isolated prime components of the ideal $\mathfrak{A}$. The corresponding algebraic formulation of the result indicated will appear below as Theorem 2.

Since it is not impossible that the problem treated here may sometime receive an answer in the case of varieties of dimension greater than 1, some of the auxiliary results are treated with greater generality than is actually needed for the present paper, in the hope that they may be useful for a more general treatment.

The proof of the main theorem will be carried out by induction on the dimension of the ambient space, the step from $n - 1$ to n being made by means of a suitable projection. Changing the notation, denote the ideals $\mathfrak{A}$ and $\mathfrak{A}'$ by (F) and (F'), with generators F_i and F_i' respectively, $i = 1, 2, \ldots, r$. The first step is to show that the F_i and F_i' can be taken as polynomials in x_n. Having done this let H_i be the resultant, with respect to x_n, of F_i and F_r and let (H) be the ideal generated in $R_{n-1} = k[[x_1, x_2, \ldots, x_{n-1}]]$ by the H_i. Define (H') similarly by means of the F_i'. It will then be shown that, if the co-ordinates have been suitably chosen, and after a suitable adjustment of the F_i, the isolated prime components of (F) project into isolated prime components of (H). Here the projection of an ideal in R_n means its intersection with R_{n-1}. When $r = n - 1$, the induction hypothesis will then imply that the intersection (G) of the isolated primes of (H) can be carried by an analytic equivalence in R_{n-1} into the intersection (G') of certain components of (H'), provided that the orders of the $F_i - F_i'$, and so (as will be shown) of the $H_j - H_j'$, are sufficiently high. The next step is to prove that, again if the orders of the $F_i - F_i'$ are high enough, this analytic equivalence can be extended to one in R_n carrying (G, F_r) into (G', F_r'). Now the isolated primes of (F) will be components of (G, F_r) and so will be carried by the extended analytic equivalence into components of (G', F_r'). It remains to be shown that these will in fact be components of (F') if the orders of the $F_i - F_i'$ are high enough. The proof of this will be based on the fact that, in terms of a suitable topology, an analytic equivalence affects the components of (F) continuously, and that the only components of (G', F_r') which are then sufficiently near to those of (F) are already components of (F').

2. Preliminary adjustments. One of the main objects of this section is to show that the series F_i can be assumed to be polynomials in x_n. This is justified by means of the Weierstrass Preparation Theorem, which can be stated as follows:

Let F be an element of R_n and let it be of order m, and suppose a linear change of co-ordinates has, if necessary, been made so that F contains the term x_n^m with a non-zero coefficient. Then there is a power series P in R_n of order 0 (that is to say a unit of R_n) such that PF is a polynomial $x_n^m + a_1 x_n^{m-1} + \ldots + a_m$ in x_n with coefficients in R_{n-1}. Also, since PF is of order m, a_i is of order not less than i.

The classical case of this theorem applies, of course, to the case where k is the field of complex numbers, but the proof can be given entirely in terms of formal power series over any field (**1**, p. 183 ff.). If this formal algebraic proof is examined, it will be observed that the terms of various degrees of the series P are determined step by step, and that the terms up to any given degree depend only on the terms of F up to a certain degree. It follows that a complement to the above theorem can be stated, namely:

If F is as above and F' is a second series such that $F - F'$ is of sufficiently high order, then the series P and P' of order zero can be found such that $PF = x_n^m + a_1 x_n^{m-1} + \ldots + a_m$ and $P'F' = x_n^m + a_1' x_n^{m-1} + \ldots + a_m'$ where the orders of the $a_i - a_i'$ are greater than a preassigned integer.

Now let (F) be the ideal generated as in the introduction by $F_1, F_2, \ldots, F_r$. Co-ordinates are to be chosen, if necessary after a linear change of variables, so that the following conditions hold:

(1) The Weierstrass Preparation Theorem can be applied to all the F_i, which can therefore be assumed to be replaced by polynomials, without changing the ideal (F).

(2) If $\mathfrak{p}$ is an isolated $(n - r)$-dimensional prime component of (F) then none of the series $\partial F_i / \partial x_n$ $(i = 1, 2, \ldots, r)$, is in $\mathfrak{p}$.

(3) No two $(n-r)$-dimensional components of (F) have the same projection in R_{n-1}. This is equivalent to the geometrical statement that no two $(n-r)$-dimensional components of the algebroid variety V have the same projection on $x_n = 0$.

It will now be checked that the co-ordinates can be chosen so that these conditions are satisfied. For these verifications it should be assumed that k has an infinite number of elements, or, if not, that the linear changes of variables made have generic coefficients, that is to say, independent indeterminates over k.

To make condition (1) hold it is sufficient to change the variables so that each F_i contains among its terms of lowest degree a power of x_n with a non-zero coefficient.

To show that condition (2) can be made to hold, let $\mathfrak{o}$ be the quotient ring of R_n with respect to $\mathfrak{p}$, that is to say, the neighbourhood ring of the algebroid variety whose ideal is $\mathfrak{p}$. Then $\mathfrak{o}$ is a regular local ring of dimension r (**2**, p. 33) with maximal ideal $\mathfrak{o}\mathfrak{p}$. This ideal is generated by the r elements F_i, which therefore form a system of parameters. It follows (**2**, p. 34) that the Jacobian matrix $(\partial F_i/\partial x_j)$ is of rank r mod $\mathfrak{p}$. In particular, for any given i, all the $\partial F_i/\partial x_h$ cannot be zero. Making the change of variables $x_i = \sum a_{ij}\bar{x}_j$, it follows that $\partial F_i/\partial \bar{x}_n = \sum \partial F_i/\partial x_h a_{hn}$ is not zero for suitably chosen a_{hn}. Thus for suitably chosen a_{ij}, $\partial F_i/\partial \bar{x}_n \not\equiv 0$ mod $\mathfrak{p}$ for each i. Assume the a_{ij} chosen in this way, make the appropriate co-ordinate change, and drop the bars over the new co-ordinates.

The verification of condition (3) is equivalent to proving a special case of the following theorem: *If a number of varieties are given in n-space, then co-ordinates can be chosen so that no two of them have the same projections on* $x_n = 0$. To indicate the method of proof it will be sufficient to consider the case of two distinct varieties V_1 and V_2 in n-space. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be the corresponding ideals in R_n. Let $\mathfrak{A}_2'$ be the ideal in $k[[y_1, y_2, \ldots, y_n]]$ obtained from $\mathfrak{A}_2$ by substituting the y_i for the corresponding x_i. Then the ideal $(\mathfrak{A}_1, \mathfrak{A}_2')$ in $k[[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]]$ defines the product variety $V_1 \times V_2$ in $2n$-space. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be independent indeterminates and write $\phi_i = (x_n - y_n)/\lambda_n - (x_i - y_i)/\lambda_i$ for $i = 1, 2, \ldots, n-1$. Then the ideal $(\mathfrak{A}_1, \mathfrak{A}_2', \phi_1, \ldots, \phi_{n-1})$ defines a subvariety of $V_1 \times V_2$ whose points are pairs (p_1, p_2) with $p_i \in V_i$ such that p_1 and p_2 project on the same point of $x_n = 0$ along the "direction" $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Calculating the Jacobian matrix of a set of generators of $(\mathfrak{A}_1, \mathfrak{A}_2', \phi_1, \ldots, \phi_{n-1})$ and using Proposition 2, (**2**, p. 34), it follows easily that the dimension of each component of this subvariety is less than the common dimension of V_1 and V_2 (it is, of course, sufficient to consider the case where the dimensions of V_1 and V_2 are the same). A suitable change of co-ordinates then gives the required result.

The three conditions above have been considered separately, but it is easy to see that they can be made to hold simultaneously.

The co-ordinates having been chosen as above, it is now necessary to make certain adjustments to the generators of the ideal (F). Let the u_{ij}, $(i, j = 1$ $2, \ldots, r)$, be independent indeterminates and write

$$\hat{F}_i = \sum_{j=1}^{r} u_{ij} F_j, \qquad i = 1, 2, \ldots, r-1,$$
$$\hat{F}_r = F_r$$

and let $\hat{k}$ be the algebraic closure of $k(u)$, the field obtained from k by adjoining the u_{ij}. Let H_i be the resultant of $\hat{F}_i$ and $\hat{F}_r$ with respect to x_n, and let (H) denote the ideal in $\hat{k}[[x_1, x_2, \ldots, x_{n-1}]]$ generated by the $H_i (i = 1, 2, \ldots, r-1)$.

A component of (H) will be said to be independent of (u) if it has generators in $k[[x_1, x_2, \ldots, x_n]]$. For example, it is not hard to see that a component of the ideal in $\hat{k}[[x_1, x_2, \ldots, x_n]]$ generated by the $\hat{F}_i$, and so by the F_i, will project into a component of (H) independent of (u). What is important for the present purpose is essentially the converse of this result. Namely:

LEMMA 1. *If* $\mathfrak{p}$ *is a prime component of* (H) *independent of* (u), *then, for any* i, *a common root of the equations* $\hat{F}_i = 0$ *and* $\hat{F}_r = 0$, *both regarded as polynomial equations in* x_n *with the coefficients reduced mod* $\mathfrak{p}$, *is necessarily a common root of all the* $\hat{F}_j = 0$ *with coefficients reduced mod* $\mathfrak{p}$.

Proof. For it can be assumed that the u_{ij} are independent indeterminates over the field of fractions of $k[[x_1, x_2, \ldots, x_{n-1}]]/\mathfrak{p}$. Any root of $\hat{F}_r = F_r = 0$ reduced mod $\mathfrak{p}$ is algebraic over this field, and so, if it is a root of $\hat{F}_i = 0$, reduced mod $\mathfrak{p}$, it must be a root of all the equations $F_j = 0$, reduced mod $\mathfrak{p}$, and this is equivalent to the result stated above.

At this stage the notation will be changed. $\hat{F}_i$ will simply be written as F_i, and $\hat{k}$ as k. But it is to be understood that, in the new notation, k contains a subfield and $r(r-1)$ indeterminates u_{ij}, so that the phrase "independent of (u)" retains its meaning.

3. Projection of an isolated prime of (F). Assume that the co-ordinates have been chosen as indicated in §2, so that in particular the F_i are polynomials in x_n with highest coefficient unity and the other coefficients in R_{n-1}.

LEMMA 2. *Let* $\mathfrak{p}$ *be an isolated prime component of* (F) *and let* $\bar{\mathfrak{p}} = R_{n-1} \cap \mathfrak{p}$ *be its projection. Then* $\bar{\mathfrak{p}}$ *is an isolated prime component of* (H).

Proof. $\bar{\mathfrak{p}}$ is obviously a prime ideal in R_{n-1} containing (H); the essential point is to show that it occurs as an isolated component in a primary decomposition of (H).

Let $\mathfrak{o}$ be the quotient ring of R_n with respect to $\mathfrak{p}$, and $\bar{\mathfrak{o}}$ that of R_{n-1} with respect to $\bar{\mathfrak{p}}$. Let K be the residue class field $\bar{\mathfrak{o}}/\overline{\mathfrak{o}\mathfrak{p}}$. This field can clearly be identified with a subfield of the residue class field $\mathfrak{o}/\mathfrak{o}\mathfrak{p}$. Also if $\xi_1, \xi_2, \ldots, \xi_n$ are the residue classes mod $\mathfrak{p}$ of $x_1, x_2, \ldots, x_n$, $\mathfrak{o}/\mathfrak{o}\mathfrak{p}$ is the field of fractions of the power series ring $k[[\xi_1, \ldots, \xi_n]]$, while a similar statement holds for K, the element ξ_n being omitted. On the other hand ξ_n is algebraic over K, since it satisfies each of the equations $F_i = 0$, reduced mod $\mathfrak{p}$. And since the coefficients of the powers of x_n in each of the F_i are all of positive order, it follows that in the minimal equation of ξ_n over K all the coefficients, except the highest which is 1, will be power series in $\xi_1, \ldots, \xi_{n-1}$ of positive order. It follows that in any power series in $k[[\xi_1, \ldots, \xi_n]]$, a power of ξ_n greater than the degree of ξ_n over K can be replaced by lower powers without lowering the degree of the term in which it occurs. Each such power series can therefore be written as a polynomial in ξ_n over K. That is to say, the field $\mathfrak{o}/\mathfrak{o}\mathfrak{p}$ can be written as $K(\xi_n)$.

$\bar{\mathfrak{o}}$ and $\mathfrak{o}$ are both neighbourhood rings of irreducible algebroid varieties, and so are regular local rings (**2**, p. 33). Their completions $\bar{\mathfrak{o}}^*$ and $\mathfrak{o}^*$ are therefore also regular, and so (**3**, p. 88) are isomorphic to power series rings in the appropriate number of indeterminates over their respective residue class fields, namely K and $K(\xi_n)$. For the present purpose it will be convenient to identify K and $K(\xi_n)$ with subfields of $\bar{\mathfrak{o}}^*$ and $\mathfrak{o}^*$, respectively, writing, in particular, $\bar{\mathfrak{o}}^* = K[[y_1, y_2, \ldots, y_{n-r}]]$. In this notation the maximal ideal $\bar{\mathfrak{o}}^*\mathfrak{p}$ of $\bar{\mathfrak{o}}^*$ is that generated by the y_i.

Now it has already been noted that ξ_n is a root of each of the equations $F_i = 0$, regarded as a polynomial equation in x_n with the coefficients reduced mod $\bar{\mathfrak{p}}$. But F_i, written as a polynomial in x_n, has coefficients in R_{n-1}, which is a subring of $\bar{\mathfrak{o}}^* = K[[y_1, y_2, \ldots, y_{n-r}]]$. Thus F_i is a polynomial in x_n with coefficients which are power series in the y_i over K such that, when the y_i are set equal to zero (this is equivalent to reducing mod $\bar{\mathfrak{p}}$) the resulting polynomial has ξ_n as a root. Also $\partial F_i/\partial x_n \neq 0$ for $x_n = \xi_n$ and all the y_j set equal to zero; for otherwise $\partial F_i/\partial x_n$ would be equal to zero mod $\mathfrak{p}$, contrary to the condition (2) made to hold in §2. It follows at once by the implicit function theorem for polynomial equations with power series coefficients that there is a power series ϕ_i in the y_j with coefficients in $K(\xi_n)$ and with constant term ξ_n such that $F_i(x_1, x_2, \ldots, x_{n-1}, \phi_i) = 0$.

The polynomials F_i in x_n have coefficients in $K[[y_1, y_2, \ldots, y_{n-r}]] = K[[y]]$. Let an algebraic extension of the field of fractions of this ring be made so that the F_i factorize completely into linear factors. Write

$$(1) \qquad F_i = \prod (x_n - \phi_{ij})$$

where, in particular $\phi_{i1} = \phi_i$ for each i. Then by the theory of the resultant of a pair of polynomials (**5**)

$$(2) \qquad H_i = \prod (\phi_{ih} - \phi_{rj}) \qquad (i = 1, 2, \ldots, r-1).$$

Now the prime ideal $\bar{\mathfrak{p}}$ is independent of (u) (cf. §2) and $(H) \subset \bar{\mathfrak{p}}$. And so, by Lemma 1, the only common zeros of F_i and F_r for any i, these equations being reduced mod $\bar{\mathfrak{p}}$, must be common to all the F_i mod $\bar{\mathfrak{p}}$. But, since it has been arranged that only one component $\mathfrak{p}$ of (F) projects on $\bar{\mathfrak{p}}$ it follows at once that the only common zero of F_i and F_r reduced mod $\bar{\mathfrak{p}}$, for any i, is

ξ_n. And since ξ_n is a simple root of each F_i reduced mod $\bar{\mathfrak{p}}$ (by condition (2) of §2) the only factor of H_i which is congruent to zero modulo the appropriate extension of $\bar{\mathfrak{p}}$ is $\phi_i - \phi_r$. Thus (2) can be rewritten as

$$H_i = (\phi_i - \phi_r)K_i,$$

where K_i is not zero modulo a suitable extension of $\bar{\mathfrak{p}}$.

Now H_i is rational in the coefficients of F_i and F_r and so is in $K[[y]]$. On the other hand the ϕ_i are power series in the y_j with coefficients in $K(\xi_n)$, and so the same is true of the K_i. Writing for brevity $K' = K(\xi_n)$, this means that the K_i are in the ring $K'[[y]] = K'[[y_1, y_2, y_3, \ldots, y_{n-r}]]$ and are not zero modulo the maximal ideal $K'[[y]]\bar{\mathfrak{p}}$ of this ring. That is to say the K_i are units of $K'[[y]]$, and so

(3) $$K'[[y]](H) = K'[[y]](\phi_1 - \phi_r, \phi_2 - \phi_r, \ldots, \phi_{r-1} - \phi_r).$$

Going back to equation (1), multiply out the factors on the right for which $j \neq 1$, and arrange the result as a power series in the y_i with coefficients in $K'[x_n]$. In particular write $A_i(x_n)$ for the term independent of the y_j. Thus (1) becomes:

(4) $$F_i = (x_n - \phi_i)(A_i(x_n) + \ldots)$$

where the dots represent terms containing the y_j.

It has already been observed that ξ_n is a simple root of each of the F_i reduced mod $\bar{\mathfrak{p}}$, and so $A_i(\xi_n) \not\equiv 0$ for each i. Now $\bar{\mathfrak{o}}^* \subset \mathfrak{o}^*$, and K' is the residue class field of $\mathfrak{o}^*$, and is identified with a subfield of this ring; finally $x_n \in \mathfrak{o}^*$. It follows that both factors on the right of (4) are in $\mathfrak{o}^*$. On the other hand, all the y_j are in $\bar{\mathfrak{p}}$ and so in $\mathfrak{o}^*\mathfrak{p}$, and $x_n - \xi_n \in \mathfrak{o}^*\mathfrak{p}$, whence the second factor on the right of (4) is congruent to $A_i(\xi_n)$ mod $\mathfrak{o}^*\mathfrak{p}$. Thus the second factor on the right of (4) is not zero mod $\mathfrak{o}^*\mathfrak{p}$, and so is a unit of $\mathfrak{o}^*$. It follows at once that

(5) $$\begin{aligned}\mathfrak{o}^*(F) &= \mathfrak{o}^*(x_n - \phi_1, x_n - \phi_2, \ldots, x_n - \phi_r)\\ &= \mathfrak{o}^*(\phi_1 - \phi_r, \phi_2 - \phi_r, \ldots, x_n - \phi_r).\end{aligned}$$

Now compare equations (3) and (5). Since $\mathfrak{p}$ is an isolated prime component of (F), $\mathfrak{o}^*(F) = \mathfrak{o}^*\mathfrak{p}$, and so, by (5)

$$\mathfrak{o}^*\mathfrak{p} = \mathfrak{o}^*(\phi_1 - \phi_r, \phi_2 - \phi_r, \ldots, x_n - \phi_r).$$

A straightforward verification shows that the intersection of the ideal on the right of the last equation with $K'[[y]]$ is obtained simply by dropping the last generator. And so, applying equation (3), it follows that

(6) $$\mathfrak{o}^*\mathfrak{p} \cap K'[[y]] = K'[[y]](H).$$

An element of $K'[[y]]$ is a power series in the y_j and is congruent mod $\mathfrak{o}^*\mathfrak{p}$ to its constant term, an element of K', namely the residue class field of the local ring $\mathfrak{o}^*$. It follows at once that $\mathfrak{o}^*\mathfrak{p} \cap K'[[y]] \subset K'[[y]]\bar{\mathfrak{p}}$ .The reverse inclusion relation is obvious. Hence (6) is equivalent to

(7) $$K'[[y]](H) = K'[[y]]\bar{\mathfrak{p}}.$$

The next step is to form the intersection of each side of the last equation with $K[[y]]$. Noting that K' is a finite algebraic extension of K, and writing the elements of K' in terms of a linear basis (including the element 1) over K it follows at once from (7) that $K[[y]](H) = K[[y]]\bar{\mathfrak{p}}$. Since $K[[y]] = \bar{\mathfrak{o}}^*$ this implies that $\bar{\mathfrak{p}}$ is an isolated prime component of (H) as required.

4. Equations over a field with a valuation. Let K be a field complete with respect to a valuation v, the value group being written additively. Thus $v(ab) = v(a) + v(b)$ and $v(a + b) \geqslant \min(v(a), v(b))$, where a and b are any elements of K. Extend v to the algebraic closure $\bar{K}$ of K; this can be done in a unique manner **(5)**. The extended valuation will still be denoted by v.

For each α in the value group of v let N_α be the set of elements of $\bar{K}$ with values greater than α. Then the collection of sets of the type N_α can be taken as the basis of the neighbourhoods of 0 defining on $\bar{K}$ the structure of a topological group under addition. With this topology $\bar{K}$ is actually a topological field; that is to say, the operation of multiplication is also continuous.

LEMMA 3. *Let $F(z)$ and $F'(z)$ be two polynomials in z of the same degree, both with highest coefficient* 1 *and with all other coefficients in K having non-negative values under v. Let V be a preassigned neighbourhood of* 0 *in $\bar{K}$. Then if the coefficients of F' are sufficiently near those of F, in the sense of the topology just defined, each root of F' will differ from some root of F by an element of V.*

Proof. According to the definition of extended valuations (cf. van der Waerden **(5)** the conditions on the coefficients of F' imply that $v(\zeta_i') \geqslant 0$ for each root ζ_i' of F'. Let c be the smallest of the values under v of the differences of corresponding coefficients of F and F'. Then, for each root of F', the definition of a valuation along with the condition $v(\zeta_i') \geqslant 0$ implies that $v(F(\zeta_i') - F'(\zeta_i')) \geqslant c$. That is to say $v(F(\zeta_i')) \geqslant c$. But if $\zeta_1, \zeta_2, \ldots, \zeta_m$ are the roots of F,

$$F(\zeta_i') = (\zeta_i' - \zeta_1)(\zeta_i' - \zeta_2) \ldots (\zeta_i' - \zeta_m)$$

and so the last inequality implies that, for some j, $v(\zeta_i' - \zeta_j) \geqslant c/m$. If the preassigned neighbourhood V of 0 is assumed to be the set of elements of $\bar{K}$ for which $v(a) \geqslant c/m$, the last inequality establishes the lemma.

5. Lifting theorem for analytic equivalence. Let (F) and (F') be ideals in $k[[x_1, x_2, \ldots, x_n]]$ with sets of generators $F_1, F_2, \ldots, F_r$ and $F_1', F_2', \ldots, F_r'$, respectively. The object here is to show that, under certain conditions, if the orders of the $F_i - F_i'$ are high enough there exists an analytic equivalence between the isolated prime components of (F) and certain components of (F'). Clearly the final result is not going to be affected if the various adjustments described in §2 are made in advance. In particular the F_i can be assumed to have been replaced by polynomials in x_n. The complementary remark to the Weierstrass Preparation Theorem in §2 implies that the F_i' can also be replaced by polynomials in x_n, and that F_i' will be of the same degree as F_i, the differences of corresponding coefficients being of high order if the order of $F_i' - F_i$ is high enough. It will also be assumed that the F_i and F_i' have been replaced in advance by generic linear combinations involving indeterminates u_{ij} as in §2, so that the results of §3 can be applied; but as at the end of §2, the presence of the u_{ij} will not be indicated in the notation.

Let $H_1, H_2, \ldots, H_{r-1}$ be the set of resultants of F_r with $F_1, F_2, \ldots, F_{r-1}$, respectively, with respect to x_n, and let $H_1', H_2', \ldots, H_{r-1}'$ be calculated in the same way from the F_i'. Since the resultant of two polynomials is rational in the coefficients in the polynomials, it follows that the orders of the $H_i' - H_i$ will be arbitrarily high if those of the $F_i' - F_i$ are high enough.

Assume that the following condition, to be referred to later as condition **A**, holds; it will be shown later that this is certainly so for the case in which the variety defined by (F) is a curve:

A. Given an integer m, there exists an integer m' such that, if the orders of the $F_i - F_i' > m'$, then there is an analytic equivalence S in R_{n-1} carrying the isolated prime components of (H) into components of (H'), and also an extension of this to an analytic equivalence T in R_n carrying (G, F_r) into (G', F_r'), where (G) is the intersection of the isolated primes of (H) and (G') is its image under S, and S and T are both of order $> m$.

THEOREM 1. *If the integers m and m' of condition* **A** *are big enough, then the analytic equivalence T carries the isolated prime components of (F) into components of (F').*

Proof. Let the generators of (G) be denoted by G_i, $i = 1, 2, \ldots, q$, those of (G') by G_i', $i = 1, 2, \ldots, q$. In the following proof (x) will stand for the set $(x_1, x_2, \ldots, x_{n-1})$, and $S^{-1}(x)$ for the set of series $S^{-1}(x_i)$ obtained by applying the inverse S^{-1} of S to the x_i. $S^{-1}(\xi)$ will denote the result of replacing the x_i in $S^{-1}(x)$ by their residue classes mod $\mathfrak{p}$, where $\mathfrak{p}$ is a given isolated prime component of (F). Condition **A** states that T carries (G, F_r) into (G', F_r'), and, of course, T^{-1} effects the reverse transformation. Hence there are elements A_i and B of R_n such that

$$T^{-1}F_r'(x, x_n) = \sum A_i G_i(x) + BF_r(x, x_n),$$

or, what is the same thing,

$$F_r'(S^{-1}(x), T^{-1}(x_n)) = \sum A_i G_i(x) + BF_r(x, x_n). \tag{8}$$

By Lemma 2, $\bar{\mathfrak{p}} = R_{n-1} \cap \mathfrak{p}$ is an isolated prime component of (H) and so of (G); from which it follows that $G_i(\xi) = 0$ for each i. Substituting the ξ_i for the x_i, $i = 1, 2, \ldots, n-1$, in (8), and writing $(\xi') = S^{-1}(\xi)$:

$$F_r'(\xi', T^{-1}(x_n)) = B(\xi, x_n) F_r(\xi, x_n).$$

Since, however, ξ_n is a root of $F_r(\xi, x_n) = 0$, the last equation implies that $\xi_n' = T^{-1}(\xi_n)$ is a root of $F_r'(\xi', x_n) = 0$. The main task of this proof is to show that, under the conditions of the theorem, ξ_n' is a root of each of the equations $F_i'(\xi', x_n) = 0$. This will be done now by applying Lemma 3.

As in §3 take K as the residue class field of the local ring $\bar{\mathfrak{o}}$. A valuation is to be introduced on K in such a way that the ξ_i, $i = 1, 2, \ldots, n-1$, all have values greater than zero. This can be done as follows. Making, if necessary, a suitable linear change of variables (it will be assumed that this has already been done in advance along with the other adjustments in §2) it can be arranged that no element of $k[[x_1, x_2, \ldots, x_{n-r}]]$ vanishes when reduced mod $\bar{\mathfrak{p}}$, while for each i from 1 to $r-1$, ξ_{n-r+i} is integral over $k[[\xi_1, \xi_2, \ldots, \xi_{n-r+i-1}]]$. In addition, the coefficients of the minimal equation of ξ_{n-r+i} over the ring $k[[\xi_1, \xi_2, \ldots, \xi_{n-r+i-1}]]$, except the first which is 1, are all power series of positive order in the ξ_j. The assertions just made follow from repeated applications of the Weierstrass Preparation Theorem. Define the valuation v on the field of fractions of $k[[\xi_1, \xi_2, \ldots, \xi_{n-r}]]$ by setting $v(\alpha)$, for a power series α in the ξ_i, equal to the order of α. This valuation v can then be extended in the usual way, step by step, to K, and the nature of the coefficients of the minimal equations of the ξ_i ensures that $v(\xi_i) > 0$ for all i. The field of fractions of $k[[\xi_1, \xi_2, \ldots, \xi_{n-r}]]$ is complete with respect to v, and so, since K is obtained by a finite algebraic extension, K is complete with respect to the extended valuation **(5)**. As in §4 let $\bar{K}$ be the algebraic closure of K and let v be extended to $\bar{K}$.

The equations $F_i(\xi, x_n) = 0$ and $F_i'(\xi', x_n) = 0$ are now to be compared. Note that, since $(\xi') = S^{-1}(\xi)$, K is the field of fractions of $k[[\xi_1', \xi_2', \ldots, \xi_{n-1}']]$ and so the equations to be examined both have their coefficients in K. If the order of $F_i(x, x_n) - F_i'(x, x_n)$ and that of the analytic equivalence S are high enough it is clear that $F_i(\xi, x_n)$ and $F_i'(\xi', x_n)$ will be of the same degree in x_n and that the differences of corresponding coefficients will have arbitarily high values under v. In particular the highest coefficients will both be 1, and all the other coefficients will have non-negative values. Conditions are therefore suitable for the application of Lemma 3.

By Lemma 3, if the integers m and m' of condition **A** are large enough, each root of $F_i'(\xi', x_n) = 0$ is arbitrarily near some root of $F_i(\xi, x_n) = 0$. Choose the neighbourhood V of Lemma 3 so that, among the set of all the roots of $F_i(\xi, x_n) = 0$ and $F_r(\xi, x_n) = 0$, the common root ξ_n being counted just once, no two differ by an element of $V + V$. Then if m and m' are big enough, each root of $F_i'(\xi', x_n) = 0$ is in a V-neighbourhood of some root of $F_i(\xi, x_n) = 0$; a similar statement holds for $F_r'(\xi', x_n) = 0$ in relation to $F_r(\xi, x_n) = 0$. It follows that no root of $F_i'(\xi', x_n) = 0$ can coincide with a root of $F_r'(\xi', x_n) = 0$ except possibly roots of these equations lying in a V-neighbourhood of ξ_n.

But, by hypothesis, S carries $\bar{\mathfrak{p}}$ into some component of (H'). And so, since $(\xi') = S^{-1}(\xi)$, (ξ') is a zero of (H'). In particular the resultant of F_i' and F_r' with respect to x_n vanishes when (x) is replaced by (ξ'), whence the equations $F_i'(\xi', x_n) = 0$ and $F_r'(\xi', x_n) = 0$ must have at least one common root. It has just been shown that this common root must be in a V-neighbourhood of ξ_n; if it is proved that $F_r'(\xi', x_n) = 0$ has only one root near ξ_n when the coefficients of $F_r'(\xi', x_n)$ are near those of $F_r(\xi, x_n)$ then this root must be ξ_n', which, being equal to $T^{-1}(\xi_n)$, is certainly near ξ_n if m' is large. The fact that ξ_n' is a root of $F_i'(\xi', x_n) = 0$ would then be established as required.

To check this last point note that if, on the contrary, arbitrary neighbourhoods of ξ_n contain two roots of $F_r'(\xi', x_n) = 0$ for large values of m and m', then in the limit as m, m' tend to infinity, it would turn out that ξ_n would be a double root of $F_r(\xi, x_n) = 0$. This is not so, and thus the proof that ξ_n' is a root of each $F_i'(\xi', x_n) = 0$ is completed.

The result just obtained implies that $(\xi_1', \xi_2', \ldots, \xi_n')$ is a zero of the ideal (F'), and so of some prime component $\mathfrak{p}'$ of (F'). Now if $\phi \in \mathfrak{p}'$, $\phi(\xi', \xi_n') = \phi(S^{-1}(\xi), T^{-1}(\xi_n)) = 0$; hence $T^{-1}\phi \in \mathfrak{p}$ and so $T^{-1}\mathfrak{p}' \subset \mathfrak{p}$. A similar argument shows that $T\mathfrak{p} \subset \mathfrak{p}'$. Hence $\mathfrak{p}' = T\mathfrak{p}$. That is to say, it has been shown that T carries $\mathfrak{p}$ into a component of (F'). Since this holds for any isolated prime component of (F) the proof of the theorem is completed.

6. Algebroid curves. Attention will now be restricted to the case $r = n - 1$, the components of $(F) = (F_1, F_2, \ldots, F_{n-1})$ being all one-dimensional, and the result indicated in the introduction will be proved. As pointed out there, the proof will be by induction on n, the case $n = 2$ being established by means of the following theorem of Samuel (stated here only in the case of two variables):

LEMMA 4. *If F and F' are power series in x and y over a field k and if $F - F'$ is in the ideal $(x, y)(\partial F/\partial x, \partial F/\partial y)^2$, then there is an analytic equivalence of $k[[x, y]]$ carrying F into F'.*

Proof. See Samuel **(4)**.

Now if F has no multiple factors, F, $\partial F/\partial x$ and $\partial F/\partial y$ have an isolated common zero at the origin, and so a power of the ideal (x, y) is in $(\partial F/\partial x, \partial F/\partial y)$ mod F. It follows easily **(4)** that:

LEMMA 5. *If $F - F'$ is of sufficiently high order and F is free of multiple factors then the principal ideals (F) and (F') are analytically equivalent.*

In the case where F does possibly have double factors, let G be the product of the simple factors of F. Thus G is a product of simple factors. In order that Lemma 5 can be applied to this situation, it must be shown that F' has a factor G' differing from G by terms of high degree, provided that the order of $F - F'$ is high enough. This will be done by means of the following modification of Hensel's Lemma.

LEMMA 6. *Let F, G, H be polynomials in y with coefficients which are power series in x over a field k, and let $F = GH$. Also suppose that G and H have no common factor. Then if F' is a polynomial in y of the same degree as F, also with* *power series in x as coefficients, and if the differences of corresponding coefficients are of sufficiently high order, in particular the coefficients of the highest powers of y in F, F', G, and H all being 1, F' will have a factorisation $G'H'$ where G' and H' are polynomials in y of the same degrees as G and H respectively, and such that $G - G'$ and $H - H'$, written as series in x, are of arbitrarily high order. The highest coefficients of G' and H' will be* 1.

Proof. If the orders of the differences of corresponding coefficients of F and F' are all greater than r, it will be convenient to use the notation $F \equiv F'(x^r)$. This notation will be used throughout this proof.

By hypothesis, the highest common factor of G and H, regarded as polynomials in y over the field of fractions of the power series ring $k[[x]]$, is 1. Remembering that an element of this field of fractions can be written as a series of positive powers of x divided by a power of x, it follows at once that there is an integer h, and polynomials A and B in y with coefficients in $k[[x]]$ such that the degrees of A and B in y are less than those of H and G, respectively, and

$$AG + BH = x^h. \tag{9}$$

Now let s be any integer greater than $2h$, and suppose that $F \equiv F'(x^s)$. The lemma will be proved if it can be shown that, for each $q \geqslant s$, there are polynomials G_q and H_q in y with coefficients in $k[[x]]$, the highest coefficients being in each case 1, of the same degrees in y as G and H respectively, and satisfying the conditions

$$\left.\begin{aligned} F' &\equiv G_q H_q(x^q) \\ G_q &\equiv G(x^{s-h}) \\ H_q &\equiv H(x^{s-h}) \end{aligned}\right\}. \tag{10}$$

For then G' and H' can be taken as the limits of G_q and H_q as q tends to ∞. This result will be proved by induction on q; it is clearly true for $q = s$, taking $G_s = G$ and $H_s = H$. Suppose that G_q and H_q have already been found satisfying (10). It will now be shown that, setting $G_{q+1} = G_q + ux^{q-h}$ and $H_{q+1} = H_q + vx^{q-h}$, u and v can be determined as polynomials in y over $k[[x]]$ of degrees less than those of G and H respectively in such a way that conditions (10) hold with q replaced by $q + 1$.

By the first of the conditions (10) $F' = G_q H_q + wx^q$, where w is a polynomial in y over $k[[x]]$, the degree in y being less than that of F. And so

$$F' - G_{q+1}H_{q+1} = x^{q-h}(wx^h - vG_q - uH_q) - uvx^{2q-2h}. \tag{11}$$

Multiply (9) by w, obtaining $AwG + BwH = wx^h$. Here the right-hand side is of degree in y less than that of F, and so the standard adjustment, using the long division algorithm, can be made to Aw and Bw, replacing them by polynomials A' and B' in y over $k[[x]]$ of degrees less than those of H and G respectively. Thus

$$A'G + B'H = wx^h.$$

Applying the second and third conditions of (10)

$$A'G_q + B'H_q \equiv wx^h(x^{s-h}). \tag{12}$$

Now in the definitions of G_{q+1} and H_{q+1} take $u = A'$ and $v = B'$, and (11) along with (12) gives at once the result $F' - G_{q+1}H_{q+1} \equiv 0 \ (x^{q+1})$. The other two conditions corresponding to (10) with q replaced by $q + 1$ are clearly satisfied and so the induction is completed, and with it the proof of this lemma.

The above lemmas can now be combined to give the following result:

LEMMA 7. *Let F and F' be power series in x and y over k. Then if the order of $F - F'$ is sufficiently high there is an analytic equivalence in $k[[x, y]]$ which carries the isolated prime components of the ideal (F) into components of (F').*

Proof. It may be assumed that co-ordinates have been changed so that F and F' can be replaced by polynomials in y over $k[[x]]$ of the same degree in y and having highest coefficient 1. Then the isolated prime components of (F) are the components of (G) where G is the product of the simple factors of F. By Lemma 6 there is a factor G' of F' such that the order of $G - G'$ is high if that of $F - F'$ is high enough. Applying Lemma 5 to G and G', the result follows at once.

The main result of this paper can now be stated:

THEOREM 2. *Let an algebroid curve in n-space be defined by the ideal $(F) = (F_1, F_2, \ldots, F_{n-1})$ in $k[[x_1, x_2, \ldots, x_n]]$. Then if the orders of the series $F_i - F_i'$ are high enough, there is an analytic equivalence of arbitrarily high order carrying the isolated prime components of (F) into components of the ideal (F').*

Proof. As usual the preliminary adjustments to the co-ordinates and to the generators of (F) described in §2 will be assumed to have been carried out in advance, so that the results of §§3, 5 can be applied here. The proof of the present theorem will be carried out by induction on n, the case $n = 2$ having been already established in Lemma 7. Clearly the proof will be completed if it is shown that, on the basis of the induction hypothesis that the present theorem holds for algebroid curves in $(n - 1)$-space, condition **A** of §5 must hold; for then Theorem 1 can be applied to give the transition from $n - 1$ to n. But the holding of condition **A** under this induction hypothesis was proved as Lemma 6.1 in **(6)**. Admittedly I was dealing in that paper with algebroid curves defined over the real field, but the proof of the quoted lemma was entirely algebraic in character, and so applies equally well to the present situation. The inductive proof of Theorem 2 is thus completed.

7. Projection of analytically equivalent curves. It has already been shown that if C and C' are analytically equivalent curves in $(n - 1)$-space then they lift into analytically equivalent curves in a hypersurface in n-space, provided that the given analytic equivalence is of sufficiently high order cf **(6**, Lemma 6.1**)**. A sort of converse to this will now be obtained, namely, that algebroid curves of n-space which are analytically equivalent to a sufficiently high order will project into analytically equivalent curves of $(n-1)$-space. This result is non-trivial since a curve and its projection need not be analytically equivalent; for example the space curve $x = t^3$, $y = t^4$, $z = t^5$ is not analytically equivalent to any plane curve. And it is certainly not obvious that a given analytic equivalence can be modified in such a way that series not involving x_n are carried into series not involving x_n.

THEOREM 3. *Let C and C' be algebroid curves in n-space analytically equivalent under T, and, possibly after a sufficiently general change of co-ordinates, let $\bar{C}$ and $\bar{C}'$ be their projections into $(x_1, x_2, \ldots, x_{n-1})$-space. Then if T is of sufficiently high order, $\bar{C}$ and $\bar{C}'$ will be analytically equivalent to an arbitrarily high order.*

Proof. Let $\mathfrak{A}$ be the ideal of C, $\mathfrak{A}'$ that of C'. Of course, by the definition of algebroid varieties (see **(2)**) C is actually defined by some ideal in R_n; $\mathfrak{A}$ is to be the radical of that ideal. Thus $\mathfrak{A}$ is an intersection of primes, say $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$. A similar remark holds for $\mathfrak{A}'$. Write $\mathfrak{p}_i' = T(\mathfrak{p}_i)$, $i = 1, 2, \ldots, m$.

The first step of the proof is to find an ideal (F) in R_n having exactly $n-1$ generators and having $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$ as isolated prime components. Let $\mathfrak{o}_i$ be the quotient ring of R_n with respect to the prime ideal $\mathfrak{p}_i$. Then $\mathfrak{o}_i$ is the neighbourhood ring of a variety and so is a regular local ring **(2**, p. 33**)**. The maximal ideal of $\mathfrak{o}_i$ is $\mathfrak{o}_i\mathfrak{p}_i$. For each i, j with $i \neq j$ choose $\phi_j^{\,i}$ in $\mathfrak{p}_i$ but not in $\mathfrak{p}_j$; this is possible since no two of the $\mathfrak{p}_i$ contain one another. Then make the definition

$$\phi_j = \prod_{i \neq j} \phi_j^i$$

the product being taken over i. It is clear that ϕ_j is in each $\mathfrak{p}_i$ for $i \neq j$, but is not in $\mathfrak{p}_j$, since none of its factors is. Next define ψ_i by

$$\psi_i = \sum_{j \neq i} \phi_j$$

the summation being over j. ψ_i is in $\mathfrak{p}_i$, since each of its summands is, but $\psi_i \equiv \phi_j\ (\mathfrak{p}_j)$, and so $\psi_i \notin \mathfrak{p}_j$, for $i \neq j$. Setting $\psi_i^{\,n-1} = \psi_i$ choose co-ordinates so that the $\mathfrak{p}_i$ have distinct projections in R_{n-1}, and, proceeding as above, find an element $\psi_i^{\,n-2}$ of $\mathfrak{p}_i \cap R_{n-1}$ which is not in any $\mathfrak{p}_j \cap R_{n-1}$ for $i \neq j$. In this way, step by step, a set of elements $\psi_i^{\,1}, \psi_i^{\,2}, \ldots, \psi_i^{\,n-1}$ of $\mathfrak{p}_i$ is obtained with the property that they are not in any $\mathfrak{p}_j$ for $j \neq i$. In addition, if $\xi_1, \xi_2, \ldots, \xi_n$ denote the residue classes of $x_1, x_2, \ldots, x_n$ mod $\mathfrak{p}_i$, it is known that, possibly after a suitable linear change of co-ordinates $\xi_2, \xi_3, \ldots, \xi_n$ are separably algebraic over the field of fractions of $k[[\xi_1]]$ **(2**, p. 32**)**. It follows that, possibly after discarding superfluous factors, one can assume that

$$\partial\psi_i^j/\partial x_{j+1} \not\equiv 0(\mathfrak{p}_i). \tag{13}$$

Finally define

$$F_j = \prod_i \psi_i^j.$$

Then the ideal (F) generated by the F_i has the above asserted property, namely that the $\mathfrak{p}_i$ are all isolated prime components. For, extending to $\mathfrak{o}_i$, $\mathfrak{o}_i(F) = \mathfrak{o}_i(\psi_i^1, \psi_i^2, \ldots, \psi_i^{n-1})$, all the other factors of the F_j being units of $\mathfrak{o}_i$. The condition (13) implies that the ψ_i^j form a regular system of parameters in $\mathfrak{o}_i$ **(2**, p. 34**)** and so $\mathfrak{o}_i(F) = \mathfrak{o}_i\mathfrak{p}_i$; that is to say, $\mathfrak{p}_i$ is an isolated prime component of (F) as was to be shown.

Let the given analytic equivalence T carry F_i into F_i', and write (F') for the ideal generated by the F_i' in R_n.

If necessary making a further linear change of co-ordinates, apply the Weierstrass Preparation Theorem to the F_i and the F_i'. The ideals (F) and (F') can thus be generated by $G_1, G_2, \ldots, G_{n-1}$ and $G_1', G_2', \ldots, G_{n-1}'$ respectively, where the G_i and G_i' are all polynomials in x_n. Also if the order of T is high enough, the orders of the corresponding coefficient differences of G_i and G_i' for each i can be made arbitrarily high. The top coefficients of the G_i and G_i' are of course all equal to 1. It will in addition be assumed that the procedure of §2 has been applied to the G_i and G', introducing indeterminates u_{ij}, whose presence, however, is not indicated by the notation.

Let H_i be the resultant with respect to x_n of G_i and G_{n-1}, and let (H) denote the ideal generated in R_n by the H_i; the H_i' and (H') are to be similarly defined from the G_j'. Since the resultants of polynomials are rational in the coefficients, it follows that the order of $H_i - H_i'$, for each i, can be made arbitrarily high if the order of T is high enough.

From the last remark it follows by means of Theorem 2 that if the order of T is high enough there will be an analytic equivalence S in R_{n-1} of arbitrarily high order carrying the isolated prime components of (H) into components of (H'). In particular, for each i, $\bar{\mathfrak{p}}_i = \mathfrak{p}_i \cap R_{n-1}$ is an isolated prime component of (H), provided the co-ordinates are suitably chosen (Lemma 2). Thus $S(\bar{\mathfrak{p}}_i)$ is a component $\mathfrak{p}_i'$ of (H'). It is required to prove now that $\bar{\mathfrak{p}}_i' = \mathfrak{p}_i' \cap R_{n-1}$. If this is known for each i, then it will be known that S carries $\bar{C}$ into $\bar{C}'$ as was to be proved.

Write ξ_j for the residue class of x_j mod $\mathfrak{p}_i$, and denote $(\xi_1, \xi_2, \ldots, \xi_{n-1})$ by (ξ). Thus (ξ) is a zero of $\bar{\mathfrak{p}}_i$. In the notation employed in the proof of Theorem 1, it follows that $S^{-1}(\xi)$ is a zero of $\bar{\mathfrak{p}}_i'$. But $\bar{\mathfrak{p}}_i'$ is a component of (H'), and so, for some value of j which is to be fixed in the meantime (any value of course will do) $S^{-1}(\xi)$ is a zero of H_j'. It follows that $G_j'(S^{-1}(\xi), x_n)$ and $G_{n-1}'(S^{-1}(\xi), x_n)$, polynomials in x_n, have a common zero. Call this zero ξ_n'.

The polynomials $G_j'(S^{-1}(\xi), x_n)$ and $G_j(\xi, x_n)$ are now to be compared. Assume as in the proof of Theorem 1 a valuation has been introduced on the field of fractions of $k[[\xi_1, \xi_2, \ldots, \xi_{n-1}]]$ and extended to its algebraic closure. If the order of T, and so that of S, is sufficiently high the values of the differences of the corresponding coefficients of $G_j'(S^{-1}(\xi), x_n)$ and $G_j(\xi, x_n)$ will be arbitrarily large. Lemma 3 then shows that, if the order of T, and so of S, is large enough, the root ξ_n' of $G_j'(S^{-1}(\xi), x_n)$ will lie in a V-neighbourhood of some root of $G_j(\xi, x_n)$, where V is preassigned. Similarly, replacing j by $n - 1$, ξ_n' will lie in a V-neighbourhood of some root of $G_{n-1}(\xi, x_n)$. If V is taken so that no two of the roots of $G_j(\xi, x_n)$ and $G_{n-1}(\xi, x_n)$ differ by an element of $V + V$, this implies that ξ_n' is in a V-neighbourhood of a common root of these polynomials. In view of Lemma 1 the only common root of $G_j(\xi, x_n)$ and $G_{n-1}(\xi, x_n)$ is common to all the $G_h(\xi, x_n)$, $h = 1, 2, \ldots, n - 1$, and, since condition (3) of §2 is supposed to be satisfied here by the G_h, the only such common root is ξ_n. Thus if the order of T is sufficiently high ξ_n' is arbitrarily near ξ_n. It follows that ξ_n' is the unique common root of all the $G_{h'}(S^{-1}(\xi), x_n)$, $h = 1, 2, \ldots, n - 1$, unique since otherwise a limiting procedure would lead to a double root of $G_{n-1}(\xi, x_n)$, which is ruled out by condition (2) of §2. Thus $(S^{-1}(\xi), \xi_n')$ is a zero of a uniquely defined prime component $\mathfrak{p}_i''$ of $(F') = (G')$. The proof of the theorem will be completed by showing that $\mathfrak{p}_i'' = \mathfrak{p}_i'$.

The proof that $\mathfrak{p}_i'' = \mathfrak{p}_i'$ will involve an additional technical device which will now be described. Let $\mathfrak{p}$ be any prime ideal of R_n of dimension one. Then, as has already been remarked, the fraction field of $R_n/\mathfrak{p}$ is an algebraic extension of a field of power series in one variable. Thus it is a field with a discrete valuation and clearly contains a congruent representative of the residue class field corresponding to this valuation, namely k itself. Thus, by a known theorem of valuation theory, the field of fractions of $R_n/\mathfrak{p}$ can be identified with a subfield of the field of fractions of $k[[t]]$. This could also be expressed by saying that the algebroid curve defined by $\mathfrak{p}$ can be parametrized by means of power series in t.

Now if parametrizations are introduced as above on a number of algebroid curves, the same parameter symbol t can be used for all of them. This gives a means of comparing different curves. An immediate question which arises is: What is the relation between different parametrizations of the same curve? The answer is that they may be obtained from one another by means of an invertible power series substitution, replacing t by a power series in t having zero constant term but non-zero linear term. This is easily seen by noting that in a parametrization, the parameter can be identified with any element of minimum value.

Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two ideals of R_n and let $x_i^1(t)$ and $x_i^2(t)$, i running in each case from 1 to n, be parametrizations of the corresponding curves. Define

$$\delta(\mathfrak{p}_1, \mathfrak{p}_2) = \sup[\text{min order of } [x_i^1(t) - x_i^2(t)]]$$

the supremum being taken over all possible parametrizations of the two curves, and the minimum over i; the order of an element is to be its order as a series in t.

In the first place note that $\delta(\mathfrak{p}_1, \mathfrak{p}_2)$ is finite for $\mathfrak{p}_1 \neq \mathfrak{p}_2$. To prove this, it is sufficient in the above definition to assume that the parametrization of the first curve is kept fixed; this can always be arranged by means of a simultaneous change of parameter without changing the order of any of the $x_i^1(t) - x_i^2(t)$. But then, if the orders of the $x_i^1(t) - x_i^2(t)$ were unbounded, the parametrization of the second curve being varied, it would follow by a limiting process that the two curves would have a common parametrization, contrary to the fact that $\mathfrak{p}_1 \neq \mathfrak{p}_2$.

Returning now to the proof that $\mathfrak{p}_i'' = \mathfrak{p}_i'$, apply T^{-1} to both ideals. Thus it is to be shown that $T^{-1}\mathfrak{p}_i'' = \mathfrak{p}_i$. To do this replace the zero (ξ, ξ_n) of $\mathfrak{p}_i$ by some parametrization. The zero $(TS^{-1}(\xi), T(\xi_n'))$ of $T^{-1}\mathfrak{p}_i''$ is automatically replaced by a parametrization of the appropriate curve, simply by substitution of the power series. And the orders (in the parameter t) of the corresponding co-ordinate differences of (ξ, ξ_n) and $(TS^{-1}(\xi), T(\xi_n'))$ will be arbitrarily high if the orders of T and S are high enough (it will be remembered that the ξ_j are all of positive value, as noted in the proof of Theorem 1, and also that the value of $\xi_n' - \xi_n$ is high when the orders of S and T are high enough). It follows at once that $\delta(\mathfrak{p}_i, T^{-1}\mathfrak{p}_i'')$ can be made arbitrarily large if the order of T is high enough. Then choose T and S so that $\delta(\mathfrak{p}_i, T^{-1}\mathfrak{p}_i'') > \delta(\mathfrak{p}_i, \mathfrak{p}_j)$ for all components $\mathfrak{p}_j$ of (F). Then $T^{-1}\mathfrak{p}_i''$, which is a component of (F), can only be equal to $\mathfrak{p}_i$, as was to be proved.

References

1. S. Bochner and W. T. Martin, *Several complex variables* (Princeton, 1948).

2. C. Chevalley, *Intersections of algebraic and algebroid varieties*, Trans. Amer. Math. Soc., *57* (1945), 1–85.

3. I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc., *59* (1946), 54–106.

4. P. Samuel, *L'algébricité de certains points singuliers algébroides*, J. de Math. pures et appl., (9), *35* (1956), 1–6.

5. B. L. van der Waerden *Moderne algebra*.

6. A. H. Wallace, *Algebraic approximation of curves*, Can. J. Math., *10* (1958), 242–278.

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