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ALMOST COMPLEX STRUCTURES ON FOUR-DIMENSIONAL COMPLETE INTERSECTIONS

HOWARD HILLER*

Suppose X is a 4-dimensional complete intersection in $\mathbb{C}P^{r+4}$ of multidegree $d_1, \ldots, d_r$. We show that X supports infinitely many almost complex structures for exactly 8 possible multidegrees. In particular, a hypersurface of degree d in $\mathbb{C}P^5$ admits infinitely many almost complex structures if and only if $d = 2$ or 6. This generalizes a result of E. Thomas [4] for $\mathbb{C}P^4$. We give also some tables of possible Todd genera and a result for complex surfaces.

If M is a $2n$-dimensional differentiable manifold, then an *almost complex structure (acs)* on M is a complex vector bundle ω satisfying $\omega_{\mathbb{R}} \cong TM$, where TM denotes the real tangent bundle of M. Hirzebruch [2] has posed the general problem of determining the possible total Chern classes of acss on M.

E. Thomas [4] showed that $M = \mathbb{C}P^4$ admits precisely 6 acss and wrote down their Chern classes. If we view $\mathbb{C}P^4$ as a degree 1 hypersurface $X_4(1)$ in $\mathbb{C}P^5$, it is natural to consider Hirzebruch's question for $X_4(d)$, $d > 1$. If we define $N(d) = (d-2)(d-6)(5d^2 - 8d + 8)$, the answer can be described in the following fashion:

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THEOREM. *(a) If $d \neq 2,6$, then $X_4(d)$ admits only finitely many acss each uniquely determined by their first Chern class. If d is odd (resp. even) then the possible first Chern classes are the (resp. even) divisors of $N(d)$ (resp. $N(d)/8$).*

(b) For all d, if ω is an acs over $X_4(d)$ and $c_1(\omega) = c_1 x \in H^2(X_4(d))$, c_1 a non-zero integer, then the total Chern class of ω is given by:

$$(*) \quad \begin{cases} c(\omega) = 1 + c_1 x + (\frac{1}{2}(c_1^2+d^2)-3)x^2 + \\ \quad (\frac{1}{8}(c_1^3 + \frac{N(d)}{c_1} + 2c_1(d^2-6))x^3 + \frac{\chi}{d}x^4 \end{cases}$$

where χ denotes the Euler characteristic of $X_4(d)$ and x is a generator of $H^2(X_4(d))$.

(c) If $d = 2,6$ $X_4(d)$ admit an acs with total Chern class (*), c_1 any non-zero integer. Furthermore for any integer k, $X_4(2)$ has an acs with Chern class $1 - x^2 + kx^3 + 3x^4$ and $X_4(6)$ has an acs with Chern class $1 - 15x^2 + kx^3 + 435x^4$.

There is a similar result for hypersurfaces in $\mathbb{C}P^3$ involving the middle-dimensional pairing that we describe below and a result for four-dimensional complete intersections.

Elsewhere we will consider Hirzebruch's problem for odd-dimensional complete intersections.

1. Hypersurfaces in $\mathbb{C}P^5$.

Up to diffeomorphism there is unique hypersurface $X_n(d)$ of degree d in $\mathbb{C}P^{n+1}$ defined by a single homogeneous polynomial of degree d. The topology of $X_n(d)$ is well-known and is described in Kulkarni-Wood [3]. We restrict our attention to $n = 4$ but the facts we quote apply in complete generality. If $\alpha \in H^2(\mathbb{C}P^5)$ denotes the first Chern class of the hyperplane bundle H, then $x = i^*\alpha$ generates $H^2(X_4(d))$. The classes

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x^3 is d times a generator of $H^6(X_4(d))$ and x^4 evaluated on the fundamental class is d. The odd-dimensional cohomology groups are zero and $H^2(X_4(d))$ is a free abelian group of rank $\beta_2 = \frac{1}{d}\{(d-1)^6 - 1\} + 2$, so that the Euler characteristic $\chi = \beta_2 + 4$. Finally we will need the characteristic classes of $X_4(d)$ that are easily computable from the basic relation

$$T_{\mathbb{C}}(X_4(d)) \oplus H^{\otimes d} = i^* T_{\mathbb{C}}(\mathbb{C}P^5).$$

One gets:

(1.0a) $$c(X_4(d)) = \frac{(1+x)^6}{1+dx}$$

so that

(1.0b) $$w(X_4(d)) = \begin{cases} (1+x)^6 & \text{if } d \equiv 0(2) \\ (1+x)^5 & \text{if } d \equiv 1(2) . \end{cases}$$

Similarly:

(1.0c) $$p(X_4(d)) = 1 + (6-d^2)x^2 + (d^4-6d^2+15)x^4 .$$

If ω is any acs over a $2n$-manifold M, then (1) $c_n(\omega) = \chi(M)$ and (2) $c_{2i}(\omega \oplus \bar{\omega}) = p_i(M)$, $1 \leqslant i \leqslant n/2$. Hence if M is 8-dimensional, we get

(1.0d) $$\begin{cases} (i) & c_4(\omega) = \chi(M) \\ (ii) & p_1 = c_1(\omega)^2 - 2c_2(\omega) \\ (iii) & p_2 \equiv c_2(\omega)^2 - 2c_1(\omega)c_3(\omega) + 2c_4(\omega) . \end{cases}$$

Substituting (i) and (ii) in (iii) and rearranging we obtain as in [4, Theorem 1.8]:

LEMMA 1.1. *If* ω *is an acs over* M^8, $c_1 = c_1(\omega)$, $c_3 = c_3(\omega)$ *then*

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in $H^8(M;\mathbb{Z})$

$$8\chi(M) + p_1(M)^2 - 4p_2(M) = 8c_1c_3 - c_1^4 + 2p_1(M)c_1^2 .$$

It is now easy to compute the left-hand side of this identity for $M = X_4(d)$.

LEMMA 1.2. *In* $H^8(X_4(d))$:

$$8\chi + p_1^2 - 4p_2 = N(d)x^4$$

where $N(d) = (d-2)(d-6)(5d^2-8d+8)$, *as above.*

Proof. From (1.0a) or the formula for the second Betti number above, one can compute $\chi = c_4(X_4(d)) = (d^4-6d^3+15d^2-20d+15)x^4$. Combined with (1.0c) we get:

$$8\chi + p_1^2 - 4p_2 = 5d^4 - 48d^3 + 132d^2 - 160d + 96 = N(d)$$

after some easy computing.

COROLLARY 1.3. *If* ω *is an acs over* $X_4(d)$ *and* $c_1(\omega) = ax$, $c_3(\omega) = bx^3$, $a,b \in \mathbb{Z}$, *then:*

(1.3.1) $$a(8b - a^3 + 2a(6-d^2)) = N(d).$$

Proof. This is immediate from (1.1) and (1.2).

It remains to prove the converse of (1.3). According to Thomas [4, Theorem 3.1] it suffices to check

(1.3.2) $$Sq^2(c_1(\omega)^3 + c_1(\omega)^2w_2 + c_1(\omega)w_4 + c_3(\omega)) = 0$$

in $H^8(X_4(d);\mathbb{Z}/2)$. For $d \equiv 0(2)$ this is immediate as $x^3 \equiv 0(2)$. If $d \equiv 1(2)$, we use (1.0b) to observe that $w_2 \equiv 1(2)$ and $w_4 \equiv 0(2)$, so that the condition reduces to $b \equiv 0(2)$. Hence it suffices to show:

LEMMA 1.4. *If* $a,d \equiv 1(2)$, *then* $a^4 + N(d) + 2a^2(d^2-6) \equiv 0(16)$.

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Proof. From the proof of (1.2) we know

$$N(d) \equiv 5d^4 + 4d^2 \equiv 9(16)$$

since $d^2 \equiv 1(8)$ and $d^4 \equiv 1(16)$. Similarly $a^4 \equiv 1(16)$, $2a^2d^2 \equiv 2(16)$ and $-12a^2 \equiv 4(16)$. Hence $a^4 + N_d + 2a^2(d^2-6) \equiv 1 + 9 + 2 + 4 \equiv 0(16)$.

Proof of Theorem. For d odd, (a) follows from (1.3), (1.4) and Thomas' criterion (1.3.2). If d is even, a is also even, so letting $a = 2k$ in (1.3.1) provides the result. (b) follows by computing c_2, c_3 explicitly from (1.0d). Finally (c) follows by observing that $d = 2,6$ are roots of $N(d)$, so if $c_1 = 0$, c_3 is arbitrary, and if $c_1 \neq 0$, it determines c_3 uniquely from (1.3.2).

REMARKS. 1. The case $d = 1$ of the theorem agrees with Thomas' result for $\mathbb{C}P^4$ [4, Theorem 3.2].

2. It is easy to compute now the precise number of acss on $X_4(d)$. For example, if d is odd then it is $2d(N(d))$ where $d(\ .\ )$ is the usual divisor function. (The 2 comes from allowing negative divisors.) See Table 1 for low values of d.

3. The formula

$$\text{Todd}(\omega) = \frac{1}{720}(-c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4)$$

allows one to compute the possible Todd genera of acss on $X_4(d)$. See Table 2 for low values of d. If $d = 2,6$ one can easily show:

COROLLARY 1.5. *(a) A non-zero integer t occurs as the Todd genus of an acs on the Klein quadric $X_4(2)$ if and only if $t = \frac{1}{12}(k^4-k^2)$, for some $k \in \mathbb{Z}$.*
(b) A non-zero integer t occurs as the Todd genus of an acs on $X_4(6)$ if and only if $t = \frac{1}{4}(k^4 + 15k^2 + 8)$, for some $k \in \mathbb{Z}$.

2. Four-dimensional complete intersections

A complete intersection $X_n = X_n(d_1, \ldots, d_n) \subseteq \mathbb{C}P^{n+r}$

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is the transverse intersection of hypersurfaces of degrees $d_1, \ldots, d_r$ in CP^{n+r}. The ordered r-tuple $d_1 < \ldots < d_r$ is called the multidegree of $X_n \subseteq CP^{n+r}$ and determines it up to diffeomorphism. The product $d_1 \ldots d_r$ is the degree of X.

The results of §1. admit generalization to 4-dimensional complete intersections. We describe these here while omitting most of the details.

We will need some terminology from the theory of symmetric functions to define the analogue of $N(d)$. Fix a positive integer r and view $d_1, \ldots, d_r$ as formal variables. If $I = (i_1, \ldots, i_r)$ is a partition of $k = i_1 + \ldots + i_r$ then

$$m_I(d_1, \ldots, d_r) = \sum d_1^{i_1} \ldots d_r^{i_r}$$

is the usual monomial symmetric function containing the indicated monomial. The complete elementary symmetric function h_k is then given by:

$$h_k = \sum_I m_I$$

where I varies over all partitions of k. The power-sum symmetric function is given by:

$$P_k = \sum_{i=1}^{r} d_i^k$$

We now introduce the following symmetric polynomial

$$\begin{aligned} N_r(d_1, \ldots, d_r) = {} & (8h_4 - 2p_4 - p_2^2) - 8(r+5)h_3 + \\ & 4(r+5)(r+4)h_2 + 2(r+5)p_2 - \\ & 8\binom{r+5}{3}h_1 + 8\binom{r+5}{4} - (r+5)^2 + 2(r+5) \end{aligned}$$

We leave it to the reader to check the analogue of (1.2), that $8X + p_1^2 - 4p_2 = N_r(d_1, \ldots, d_r)$ in $H^8(X_4(d_1, \ldots, d_r))$, in particular, $N_1 = N$. It is now easy to describe a parametrization of the acss over

$X_4(d_1, \ldots, d_r)$ as in section 1.

It remains only to determine which multidegrees support infinitely many acss or equivalently acss with zero first Chern class. These correspond precisely to the integral zeros of $N_r(d_1, \ldots, d_r)$. It is possible to check the following.

PROPOSITION 2.1. *The following multidegrees support infinitely many acss:*

$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
2	2,5	2,2,4	2,2,2,3	2,2,2,2,2
6	3,4	2,3,3		

and these are all of them.

3. Hypersurfaces in $\mathbb{C}P^3$

The main result is:

THEOREM 3.1. *The almost complex structures on* $X_2(d) \subseteq \mathbb{C}P^3$ *correspond to the elements* $\beta \in H^2(X_2(d);\mathbb{Z})$ *satisfying* $\beta \equiv dx(2)$ *and* $\beta^2 = (d-4)^2x^2$.

Proof. According to Wu [5] and Ehresmann [1] acss on a 4-manifold M are classified by elements $\beta \in H^2(M)$ satisfying $\beta \equiv w_2(M)(2)$ and $\beta^2 - 2\chi(M) = p_1(M)$. If $M = X_2(d)$, we have $p_1 = (4 - d^2)x^2$, $\chi(M) = c_2(M) = (d^2 - 4d + 6)x^2$ and $w_2 \equiv d(2)$; so the result follows.

COROLLARY 3.2. *If* X *is a smooth* K3 *surface, then acss on* X *correspond to even null vectors in* $\mathbb{Z}^{22}$ *with respect to the form* $2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. X is realized by the smooth quartic hypersurface in $\mathbb{C}P^3$.

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degree d	number of acss on $X_4(d)$
1	6
2	∞
3	8
4	8
5	12
6	∞
7	8
8	24
9	32
10	16
11	48
12	32

TABLE 1

Chern classes of almost complex structures on $X_4(d)$.

$\pm c_1$	c_2	$\pm c_3$	c_4	Todd genus
d = 3				
1	2	-10	9	0
3	6	2	9	1
29	422	3070	9	5565
87	3786	82378	9	447931
d = 4				
2	7	-8	47	1
4	13	11	47	6
14	103	376	47	441
28	397	2813	47	6566
d = 5				
1	10	-30	165	1
3	14	6	165	6
9	50	130	165	126
31	490	3870	165	12501
93	4334	100986	165	978306
279	38930	2716030	165	78934630

TABLE 2

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Mathematiches Institut
Universitat Göttingen
D-4500 Göttingen
West Germany

and

Department of Mathematics
Columbia University
New York, N.Y. 10027
U.S.A.

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