A REMARKABLE CLASS OF MANNHEIM-CURVES

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Introduction. It is well known that the determination of a (non-isotropic) curve in the euclidean 3-space with given curvature $\kappa(s)$ and torsion $\tau(s)$, where s represents the arc-length, depends upon the integration of a Riccati equation; and that this can be performed only if a particular integral of the equation is known.

The following paper is based on an observation made in connection with this Riccati equation written in a somewhat modified form (see the usual one e.g. in [1] p. 36).

The Riccati equation: let

$$\vec{t}(\xi_1, \xi_2, \xi_3), \ \vec{p}(\eta_1, \eta_2, \eta_3), \ \vec{b}(\zeta_1, \zeta_2, \zeta_3)$$

be the tangent, principal normal and binormal unit vectors respectively, each being, of course, functions of the arc-length s. We have

$$\xi_m^2 + \eta_m^2 + \zeta_m^2 = 1, \quad (m = 1, 2, 3). \tag{1}$$

If we put

$$u = \frac{\xi_m + i\zeta_m}{1 - \eta_m} = \frac{1 + \eta_m}{\xi_m - i\zeta_m}; \ -\frac{1}{v} = \frac{\xi_m - i\zeta_m}{1 - \eta_m} = \frac{1 + \eta_m}{\xi_m + i\zeta_m}, \tag{2}$$

differentiate with respect to s and take into account the Frenet equations

$$\xi'_m = \kappa\eta_m,$$

$$\eta'_m = -\kappa\xi_m + \tau\zeta_m, \tag{3}$$

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$$\zeta_m' = -\tau\eta_m ,$$

we find that both u and v satisfy the same Riccati equation:

$$\text{(4)} \qquad u' = -\frac{u^2}{2}(\kappa + i\tau) - \frac{1}{2}(\kappa - i\tau).$$

Observation: This form of the Riccati equation suggests the choice: $\kappa = \omega \cos \beta s$, $\tau = \omega \sin \beta s$ with ω and β positive constants. In this case the above equation can be written

$$\text{(5)} \qquad u' = -\frac{\omega}{2}(u^2 e^{i\beta s} + e^{-i\beta s}).$$

It is immediately obvious that there exist two particular integrals of (5) of the form $Ae^{-i\beta s}$. If we substitute $u = Ae^{-i\beta s}$ into (5) and cancel $e^{-i\beta s}$ we get the quadratic equation

$$\omega A^2 - 2i\beta A + \omega = 0,$$

whose roots, in terms of $\alpha = +\sqrt{\omega^2 + \beta^2}$, are

$$A_1 = i\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}, \quad A_2 = -i\sqrt{\frac{\alpha-\beta}{\alpha+\beta}} .$$

If we choose $u = A_1 e^{-i\beta s}$, $v = A_2 e^{-i\beta s}$ and substitute into

$$\eta_m = \frac{u+v}{u-v}$$

we get

$$\eta_m = \frac{A_1 + A_2}{A_1 - A_2} = \frac{\beta}{\alpha} .$$

One of the components of the principal normal $\vec{p}$ can, therefore, be chosen to be constant and we have the

THEOREM 1. A curve (C) in euclidean 3-space defined by $\kappa = \omega \cos \beta s$, $\tau = \omega \sin \beta s$, where ω and β are positive constants, has the property that its principal normal makes a constant angle with a (suitably chosen) fixed direction.

Note: The curves (C) belong to the larger class of curves defined by

$$\kappa^2 + \tau^2 = \omega^2 = \text{constant},$$

which appear to have been considered first in 1878 by A. Mannheim (see [2], also [3] and [4]). We shall call them Mannheim-curves and denote them by (M).

The parametric equations of a curve (C): Since equation (5) admits two known particular integrals, its general integral can be obtained by one additional quadrature. Equations (2) will then enable us to determine $\vec{t}(s)$ and a further quadrature will yield $\vec{r}(s)$, the finite parametric equations of (C).

Theorem 1 suggests an equivalent (and simpler) way to solve the problem of curves (C): Since one of the components of $\vec{p}$ can be chosen to be constant $= \frac{\beta}{\alpha}$, consider the linear differential equation satisfied by $\vec{p}$. If we start from the Frenet equations written in their vectorial form:

$$\text{(6)} \qquad \begin{aligned} \vec{t}\,' &= \kappa \vec{p}, \\ \vec{p}\,' &= -\kappa \vec{t} + \tau \vec{b}, \\ \vec{b}\,' &= -\tau \vec{p}, \end{aligned}$$

differentiate the second equation twice and write for $\vec{t}\,'$ and $\vec{b}\,'$ their expressions given by the first and third, we obtain as the linear differential equation for $\vec{p}$:

$$\text{(7)} \qquad \vec{p}\,''' + \alpha^2 \vec{p}\,' = 0.$$

The general integral of this differential equation subject to the conditions $\vec{p}^{\,2} = 1$, $\vec{p}\,'^{2} = \omega^2$, can be shown to be

$$\text{(8)} \qquad \vec{p} = \frac{\omega}{\alpha} \cos \alpha s . \vec{i} + \frac{\omega}{\alpha} \sin \alpha s . \vec{j} + \frac{\beta}{\alpha} . \vec{k},$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors of a positively oriented, fixed, orthogonal cartesian coordinate system.

The first of equations (6) with $\kappa = \omega \cos \beta s$ gives us by integration (taking into account that $\vec{t}^2 = 1$ and $\vec{t}.\vec{p} = 0$):

$$(9) \qquad \vec{t} = \frac{\omega^2}{2\alpha}\left[\frac{\sin(\alpha+\beta)s}{\alpha+\beta} + \frac{\sin(\alpha-\beta)s}{\alpha-\beta}\right]\vec{i} - \frac{\omega^2}{2\alpha}\left[\frac{\cos(\alpha+\beta)s}{\alpha+\beta} + \frac{\cos(\alpha-\beta)s}{\alpha-\beta}\right]\vec{j} + \frac{\omega}{\alpha}\sin\beta s.\vec{k}.$$

A further integration yields, finally, the position vector $\vec{r}$ of the curve (C):

$$(10) \qquad \vec{r} = -\frac{\omega^2}{2\alpha}\left[\frac{\cos(\alpha+\beta)s}{(\alpha+\beta)^2} + \frac{\cos(\alpha-\beta)s}{(\alpha-\beta)^2}\right]\vec{i} - \frac{\omega^2}{2\alpha}\left[\frac{\sin(\alpha+\beta)s}{(\alpha+\beta)^2} + \frac{\sin(\alpha-\beta)s}{(\alpha-\beta)^2}\right]\vec{j} - \frac{\omega}{\alpha\beta}\cos\beta s.\vec{k}.$$

Formula (10) provides the "canonical" form for the parametric equations of a curve (C) in the sense that an orthogonal cartesian coordinate system in the euclidean 3-space can be found and an origin and direction for measuring the arc-length s on (C) chosen such that the parametric equations of (C) are given by (10).

Properties of curves (C): Theorem 1 gives a first property of these curves which can, of course, be read directly from formula (8). Furthermore, the same formula shows that as a point moves with constant velocity on (C) the corresponding principal normal $\vec{p}$ rotates with constant angular velocity about the fixed direction $\vec{k}$. Further interesting properties can easily be obtained from (10). If we denote the coefficients of $\vec{i}$, $\vec{j}$, $\vec{k}$ by x, y, z respectively, we notice that these coordinates satisfy the equation:

$$(11) \qquad \frac{x^2+y^2}{(2\beta/\omega^2)^2} - \frac{z^2}{(2/\omega)^2} = 1.$$

We therefore have the

THEOREM 2. A curve (C) in euclidean 3-space defined by $\kappa = \omega\cos\beta s$, $\tau = \omega\sin\beta s$, where ω and β are positive constants, lies on the hyperboloid of revolution (11).

If we calculate $\vec{r}^{\,2} = x^2 + y^2 + z^2$, we obtain

$$\vec{r}^{\,2} = \frac{4\beta^2}{\omega^4} + \frac{1}{\beta^2} \cos^2 \beta s \; ;$$

$\vec{r}^{\,2}$ satisfies, therefore, the double inequality

$$\frac{4\beta^2}{\omega^4} \leq \vec{r}^{\,2} \leq \frac{4\beta^2}{\omega^4} + \frac{1}{\beta^2}$$

and we have the

THEOREM 3. A curve (C) in euclidean 3-space defined by $\kappa = \omega \cos \beta s$, $\tau = \omega \sin \beta s$, where ω and β are positive constants, is "bounded"; more precisely, it is wholly located in the space between the two concentric spheres with radii

$$\frac{2\beta}{\omega^2} \quad \text{and} \quad \sqrt{\frac{4\beta^2}{\omega^4} + \frac{1}{\beta^2}}\,, \quad \text{respectively.}$$

It follows from equation (10) that a curve (C) will be algebraic and closed if and only if the ratio $\frac{\alpha}{\beta}$ is a rational number. We have, therefore, an infinite number of closed curves (C), which are all algebraic. The simplest of these will be those for which α is an integral multiple of β. In this case the curve (C) can be considered as the intersection of the hyperboloid of revolution (11) with the cylinder parallel to the y-axis obtained by eliminating $\cos \beta s$ between x and z. We list the first two cases:

1). $\alpha = 2\beta$:

$$(C) \quad \begin{cases} x^2 + y^2 - \frac{1}{3} z^2 = \frac{4}{9}\,, \\[2ex] x = \frac{8}{9\sqrt{3}} z^3 + \frac{1}{\sqrt{3}} z \; ; \end{cases}$$

2). $\alpha = 3\beta$:

$$(C) \quad \begin{cases} 16(x^2 + y^2) - 2z^2 = 1, \\[2ex] x = \frac{27 z^4}{32} - \frac{1}{4} \;. \end{cases}$$

Boundedness considerations concerning general Mannheim-curves: For a curve (M) we can write $\kappa = \omega \cos \varphi$, $\tau = \omega \sin \varphi$, with φ a function of s, which is assumed to be at least twice differentiable, and ω a positive constant. If φ is a constant, (M) is a circular helix; if φ is a non-constant linear function of s we have the case of the curve (C) just treated. The question under what conditions a general curve (M) will be a bounded curve seems to the author to be an attractive one. The differential equation satisfied by the principal normal $\vec{p}$ reduces in this case to

$$\varphi' [\vec{p}''' + (\omega^2 + \varphi'^2) \vec{p}'] - \varphi''[\vec{p}'' + \omega^2 \vec{p}] = 0. \tag{12}$$

Certainly the stability of equation (12) is a necessary condition for the boundedness of (M). That it is not sufficient can be seen from the example of the circular helix.

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