Can. J. Math., Vol. XXXIII, No. 4, 1981, pp. 929–936

OPEN AND PROPER MAPS CHARACTERIZED BY CONTINUOUS SETVALUED MAPS

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In the first part of the paper, given a continuous map f from a Hausdorff topological space X onto a Hausdorff topological space Y, we consider the reciprocal map f^* from Y into the collection $\mathscr{P}(X)$ of closed subsets of X, which maps $y \in Y$ to $f^{-1}(y) \in \mathscr{P}(X)$. $\mathscr{P}(X)$ is endowed with the pseudotopological structure of convergence of closed sets. We will use the filter description of this convergence, as defined by Choquet and Gähler [**2**], [**5**], which is equivalent to the "topological convergence" of sets as introduced by Frolík and Mrówka [**4**], [**10**]. These notions in fact generalize the convergence of sequences of sets defined by Hausdorff [**6**]. We show that the continuity of f^* is equivalent to the openness of f. On $f^*(Y)$, the set of fibers of f, we consider the pseudotopological structure induced by the closed convergence on $\mathscr{P}(X)$. On the other hand $f^*(Y)$ being the quotient set for the relation $R(f)$ associated with f, we can endow it with the quotient topology. We show that the quotient topology and the closed convergence on $f^*(Y)$ coincide if and only if $R(f)$ is open. We establish conditions on X, Y and f such that $f^*(Y)$ is an open or a closed subset of $\mathscr{P}(X)$. Finally we investigate the continuity of the extension of f^* to all closed sets of Y.

In the second part f is a closed map from a Hausdorff topological space X onto a Hausdorff topological space Y. It has an extension $\tilde{f}:\mathscr{P}(X) \to \mathscr{P}(Y)$ which maps $E \in \mathscr{P}(X)$ to $f(E) \in \mathscr{P}(Y)$. It is shown that the continuity of $\tilde{f}$ is equivalent to the properness of f. An even stronger result is obtained. The properness of f implies the properness of $\tilde{f}$.

1. Preliminaries. For notational conventions we refer to [**1**]. For notions about pseudotopological spaces, we refer to [**2**] and [**3**]. If X is a set and A is a subset, let $[A]$ be the filter on X generated by A. If $x \in X$ we put $[\{x\}] = \dot{x}$.

We recall briefly the definition of closed convergence. Let X be a Hausdorff topological space and $\mathscr{P}(X)$ the collection of its closed subsets. If χ is a filter on $\mathscr{P}(X)$ then its supremum, $\sup \chi$ is the set of points $p \in X$ such that for each neighborhood V of p and for each $\mathscr{A} \in \chi$ there exists an $A \in \mathscr{A}$ such that $A \cap V \neq \emptyset$ [**2**, p. 87]. Using the notion of a grill of a filter which is the collection of all subsets intersecting all

Received February 14, 1980 and in revised form July 22, 1980.

https://doi.org/10.4153/CJM-1981-073-6 Published online by Cambridge University Press

elements of the filter, we have the following equivalent formulation [**5**, p. 174]: $p \in \sup \chi$ if and only if for each neighborhood V of p there exists a $\mathscr{B} \in \operatorname{grill} \chi$ such that for every $B \in \mathscr{B}$, $B \cap V \neq \emptyset$. The following formulation of sup will also be useful. If $\mathscr{A} \in \chi$ we define $E_{\mathscr{A}} = \cup \{A | A \in \mathscr{A}\}$. In [**2**, p. 61] it is shown that $\sup \chi = \cap \{\bar{E}_{\mathscr{A}} | \mathscr{A} \in \chi\}$. If χ is not generated by $\{\emptyset\}$ then $\{E_{\mathscr{A}} | \mathscr{A} \in \chi\}$ is a filterbase on X. Let $\mathscr{F}(\chi)$ be the filter generated. Then $\sup \chi$ is the adherence of $\mathscr{F}(\chi)$, which is denoted by $\alpha \mathscr{F}(\chi)$ [**8**], [**9**]. The infimum, $\inf \chi$ is the set of points $p \in X$ with the property that for each neighborhood V of p there exists an $\mathscr{A} \in \chi$ such that for each $A \in \mathscr{A}$, $A \cap V \neq \emptyset$ [**2**], [**5**]. For any filter χ on $\mathscr{P}(X)$ we have $\inf \chi \subset \sup \chi$. If χ is an ultrafilter then $\inf \chi = \sup \chi$. A filter χ is said to converge to some $A \in \mathscr{P}(X)$ if and only if $\sup \chi = \inf \chi = A$. This convergence defines a pseudotopology on $\mathscr{P}(X)$, which is called the *closed* convergence. This means that for $E \in \mathscr{P}(X)$ we have

(1) The filter generated by $\{E\}$ converges to E.

(2) If χ converges to E and $\chi' \supset \chi$ then χ' converges to E.

(3) If every ultrafilter finer than some filter χ converges to E then χ converges to E.

The space $\mathscr{P}(X)$ is compact Hausdorff which means that every ultrafilter converges to exactly one point.

A closure operator on $\mathscr{P}(X)$ is associated with the closed convergence in the following way. If $E \in \mathscr{P}(X)$ and $\mathscr{A} \subset \mathscr{P}(X)$ then $E \in \overline{\mathscr{A}}$ if and only if there exists a filter χ containing $\mathscr{A}$ and converging to E. $\mathscr{A}$ is said to be *dense* if $\overline{\mathscr{A}} = \mathscr{P}(X)$, *closed* if $\overline{\mathscr{A}} = \mathscr{A}$ and *open* if $\mathscr{A}^c$ is closed. It follows that $\mathscr{A}$ is open if and only if every filter converging to some point of $\mathscr{A}$ contains $\mathscr{A}$. On any subset $\mathscr{A} \subset \mathscr{P}(X)$ a pseudotopological structure of closed convergence is induced by $\mathscr{P}(X)$. A filter χ on $\mathscr{A}$ converges in $\mathscr{A}$ to $A \in \mathscr{A}$ if and only if the filter $[\chi]$ on $\mathscr{P}(X)$ generated by χ converges to A in $\mathscr{P}(X)$. We recall a few more notions in pseudotopological spaces which will be applied to $\mathscr{P}(X)$. First another notational convention has to be made. Let $f: X \to Y$ be a map and let $\mathscr{F}$ be a filter on X then $f(\mathscr{F})$ is the filter generated by $\{f(F) | F \in \mathscr{F}\}$. If $\mathscr{F}$ is a filter on Y and f is onto then $f^{-1}(\mathscr{F})$ is the filter generated by $\{f^{-1}(F) | F \in \mathscr{F}\}$. Now let f be a map from one pseudotopological space to another, then f is continuous if and only if for every (ultra) filter $\mathscr{F}$ on the domain of f converging to some point the filter $f(\mathscr{F})$ converges to the image of this point. Following [**7**] f is a *proper* map if it is continuous, onto and if whenever $\mathscr{F}$ is an ultrafilter on the image of f converging to some point each ultrafilter $\mathscr{G}$ which maps onto $\mathscr{F}$ converges to some point in the preimage of the limit point of $\mathscr{F}$. All spaces considered are Hausdorff spaces. Subsets of X and points of $\mathscr{P}(X)$ will be denoted by the same symbols. Subsets of $\mathscr{P}(X)$ are denoted by script letters, filters on $\mathscr{P}(X)$ by Greek letters.

2. Open maps. In this section f is a continuous map from X onto Y. We consider $f^*: Y \to \mathscr{P}(X)$ which maps $y \in Y$ to $f^{-1}(y) \in \mathscr{P}(X)$. Then f^* is a one to one mapping from a topological space to a pseudotopological space.

LEMMA 2.1. *f^* is continuous if and only if for every ultrafilter $\mathscr{W}$ on Y converging to some $y \in Y$ we have $\alpha f^{-1}(\mathscr{W}) = f^{-1}(y)$.*

Proof. Suppose f^* is continuous and let $\mathscr{W}$ be an ultrafilter on Y converging to some $y \in Y$. Then $f^*(\mathscr{W})$ is an ultrafilter on $\mathscr{P}(X)$ converging to $f^{-1}(y)$. For $W \in \mathscr{W}$ we have

$$E_{f^*(W)} = \bigcup_{z \in W} f^{-1}(z) = f^{-1}(W).$$

It follows that $\mathscr{F}(f^*(\mathscr{W})) = f^{-1}(\mathscr{W})$ and so

$$\alpha f^{-1}(\mathscr{W}) = \sup f^*(\mathscr{W}) = f^{-1}(y).$$

For the converse suppose the condition above is fulfilled. Since $\mathscr{P}(X)$ is pseudotopological it suffices to consider ultrafilters. If $\mathscr{W}$ converges to $y \in Y$ then $f^*(\mathscr{W})$ converges to $f^{-1}(y) \in \mathscr{P}(X)$ since $\mathscr{F}(f^*(\mathscr{W})) = f^{-1}(\mathscr{W})$.

LEMMA 2.2. *The following properties are equivalent:*

(1) *For any $G \subset Y$ we have $\overline{f^{-1}(G)} = f^{-1}(\bar{G})$.*

(2) *f is an identification map and if $A \subset X$ is saturated so is $\bar{A}$.*

(3) *f is an identification map and if $A \subset X$ is saturated so is $\mathring{A}$.*

(4) *f is open.*

Proof. (1) $\Rightarrow$ (2). From (1) it is clear that a subset G of Y is closed if and only if $f^{-1}(G)$ is closed in X. So f is an identification map. If $A \subset X$ is saturated, $A = f^{-1}(G)$ then $\bar{A} = \overline{f^{-1}(G)} = f^{-1}(\bar{G})$ which is again saturated.

(2) $\Rightarrow$ (3). By complementation.

(3) $\Rightarrow$ (4). See [**1**, Proposition 6, § 5].

(4) $\Rightarrow$ (1). See [**1**, Proposition 7, § 5].

THEOREM 2.3. *f^* is continuous if and only if f is open.*

Proof. Suppose f^* is continuous, let G be any subset of Y and let $x \in f^{-1}(\bar{G})$. Let $\mathscr{W}$ be an ultrafilter on Y containing G and converging to $f(x)$. From (2.1) it follows that $\alpha f^{-1}(\mathscr{W}) = f^{-1}(f(x))$. Since $f^{-1}(G) \in f^{-1}(\mathscr{W})$ we have $x \in \overline{f^{-1}(G)}$. So $\overline{f^{-1}(G)} \supset f^{-1}(\bar{G})$ and the other inclusion follows from the continuity of f. Using (2.2) we have that f is open. For the converse suppose f is open and let $\mathscr{W}$ be any ultrafilter on Y converging to some $y \in Y$. Then we have

$$\alpha f^{-1}(\mathscr{W}) = \bigcap_{W \in \mathscr{W}} \overline{f^{-1}(W)} = \bigcap_{W \in \mathscr{W}} f^{-1}(\bar{W}) = f^{-1}\left(\bigcap_{W \in \mathscr{W}} \bar{W}\right) = f^{-1}(y).$$

From 2.1 it follows that f^* is continuous.

https://doi.org/10.4153/CJM-1981-073-6 Published online by Cambridge University Press

Now we consider the set of fibers of f, which is also the image of f^*. We have $f^*(Y) = \{f^{-1}(y) | y \in Y\}$. It is a subset of $\mathscr{P}(X)$ and therefore has a pseudotopological structure induced by the closed convergence. On the other hand $f^*(Y)$ being the quotient set for the relation associated with f, it can be endowed with the quotient topology. In the following theorems a comparison of these two structures is made. Let φ be the quotient map from X to $f^*(Y)$.

THEOREM 2.4. *The quotient topology on $f^*(Y)$ is coarser than the closed convergence.*

Proof. Let χ be an ultrafilter on $f^*(Y)$ converging for the closed convergence to $f^{-1}(y)$ for some $y \in Y$. We have

$$\sup \chi = \inf \chi = f^{-1}(y).$$

Let $\mathscr{O} \subset f^*(Y)$ be open in the quotient topology such that $f^{-1}(y) \in \mathscr{O}$. Let $\mathscr{A} \in \chi$ be such that for every $A \in \mathscr{A}$ we have $A \cap \varphi^{-1}(\mathscr{O}) \neq \emptyset$. It follows that $\mathscr{A} \subset \mathscr{O}$ and hence $\mathscr{O} \in \chi$. So χ converges to $f^{-1}(y)$ in the quotient topology.

COROLLARY 2.5. *f is open if and only if f^* is a homeomorphism from Y onto $f^*(Y)$, endowed with the closed convergence.*

Proof. $f^{*-1}: f^*(Y) \rightarrow Y$ mapping $f^{-1}(y)$ to y is in fact the factorization of f over the quotient set $f^*(Y)$. When $f^*(Y)$ has the quotient topology or any finer structure, continuity of f implies continuity of f^{*-1}.

THEOREM 2.6. *The quotient map $\varphi: X \rightarrow f^*(Y)$ is open if and only if the closed convergence and the quotient topology on $f^*(Y)$ coincide.*

Proof. Let φ be the quotient map and suppose it is open. From (2.4) we only have to show that the neighborhood filter χ of $f^{-1}(y)$ in the quotient topology converges to $f^{-1}(y)$ for the closed convergence. We calculate

$$\sup \chi = \bigcap_{\mathscr{A} \in \chi} \bar{E}_{\mathscr{A}} = \bigcap_{\mathscr{A} \in \chi} \overline{\varphi^{-1}(\mathscr{A})} = \bigcap_{\mathscr{A} \in \chi} \varphi^{-1}(\bar{\mathscr{A}}). \tag{2.2}$$

Now

$$\bigcap_{\mathscr{A} \in \chi} \varphi^{-1}(\bar{\mathscr{A}}) = \varphi^{-1}\left(\bigcap_{\mathscr{A} \in \chi} \bar{\mathscr{A}}\right) = \varphi^{-1}(\alpha(\chi)),$$

where $\alpha(\chi)$ is the adherence of χ in the quotient topology. Since Y is Hausdorff we have $\alpha(\chi) = f^{-1}(y)$ and

$$\sup \chi = \varphi^{-1}(f^{-1}(y)) = f^{-1}(y).$$

Next we show that $f^{-1}(y) \subset \inf \chi$. Let $x \in f^{-1}(y)$ and let V be an open neighborhood of x. The openness of φ implies that $\varphi(V)$ is an open neighborhood of $f^{-1}(y)$ in the quotient topology. We have $\varphi(V) \in \chi$ and for any $f^{-1}(z) \in f^*(Y)$ with $f^{-1}(z) \in \varphi(V)$ we have $f^{-1}(z) \cap V \neq \emptyset$.

Hence $x \in \inf \chi$. For the converse suppose that the quotient topology agrees with the closed convergence on $f^*(Y)$. Let $G \subset X$ be open and $x \in \varphi^{-1}\varphi(G)$. Let $x' \in G$ be such that $f(x) = f(x')$ and let χ be the neighborhood filter of $f^{-1}(y)$ in the quotient topology on $f^*(Y)$ where $y = f(x) = f(x')$. Since χ converges to $f^{-1}(y)$ in the closed convergence we have that $x' \in \inf \chi$. So there exists an $\mathscr{O} \in \chi$, open in the quotient topology such that for any $z \in Y$ with $f^{-1}(z) \in \mathscr{O}$ we have $f^{-1}(z) \cap G \neq \emptyset$. It follows that $f^{-1}(y) \in \mathscr{O} \subset \varphi(G)$. Therefore we have $x \in \varphi^{-1}(\mathscr{O}) \subset \varphi^{-1}\varphi(G)$. So we have shown that $\varphi^{-1}\varphi(G)$ is a neighborhood of x.

LEMMA 2.7. *For every nonempty $A \in \mathscr{P}(X)$ there is an ultrafilter χ on $\mathscr{P}(X)$ containing the collection $\mathscr{J}(A)$ of finite subsets of A and with the property that $\mathscr{F}(\chi)$ is the filter generated by A.*

Proof. Let $A \in \mathscr{P}(X)$, $A \neq \emptyset$. Let $\{\chi_i | i \in I\}$ be the family of all ultrafilters on $\mathscr{P}(X)$ containing $\mathscr{J}(A)$. Suppose that for every $i \in I$ we have $\mathscr{F}(\chi_i) \not\subset [A]$. Then for every $i \in I$ we have an $\mathscr{A}_i \in \chi_i$ such that $E_{\mathscr{A}_i} \not\supset A$. We can find a finite subset $I_0 \subset I$ such that $\bigcup_{i \in I_0} \mathscr{A}_i \supset \mathscr{J}(A)$. Otherwise the collection $\{\mathscr{A}_i{}^c | i \in I\} \cup \{\mathscr{J}(A)\}$ would generate a filter. For $i \in I_0$ we choose an $a_i \in A$ such that $a_i \notin E_{\mathscr{A}_i}$. Since the set $\{a_i | i \in I_0\}$ belongs to $\mathscr{J}(A)$, there is an index $k \in I_0$ such that $\{a_i | i \in I_0\} \in \mathscr{A}_k$. But then we have that $a_k \in E_{\mathscr{A}_k}$, which is a contradiction. It follows that there is an ultrafilter χ on $\mathscr{P}(X)$ containing $\mathscr{J}(A)$ and such that $\mathscr{F}(\chi) \subset [A]$. Since $E_{\mathscr{J}(A)} = A$ we have $\mathscr{F}(\chi) = [A]$.

THEOREM 2.8. *$f^*(Y)$ is open in $\mathscr{P}(X)$ if and only if X (and Y) are finite.*

Proof. If X is finite then X and $\mathscr{P}(X)$ are both discrete and hence $f^*(Y)$ is open.

Suppose $f^*(Y)$ is open in $\mathscr{P}(X)$. If f is a constant function then we have $f^*(Y) = \{X\}$. Let χ be an ultrafilter on $\mathscr{P}(X)$ containing $\mathscr{J}(X)$ and such that $\mathscr{F}(\chi) = [X]$ (2.7). Then we have $\{X\} \cap \mathscr{J}(X) \neq \emptyset$ since it belongs to χ. Therefore X is finite.

Now suppose f is not constant. We first prove that X is compact. Suppose $\mathscr{U}$ is an ultrafilter on X with an empty adherence. Let $f^{-1}(y)$ be a fiber of f. We consider the map

$$g_{f^{-1}(y)}: X \to \mathscr{P}(X)$$

which maps $x \in X$ to $\{x\} \cup f^{-1}(y)$. Then the image $g_{f^{-1}(y)}(\mathscr{U})$ is an ultrafilter on $\mathscr{P}(X)$ for which we have

$$\mathscr{F}(g_{f^{-1}(y)}(\mathscr{U})) = \mathscr{U} \cap [f^{-1}(y)].$$

It follows that $g_{f^{-1}(y)}(\mathscr{U})$ converges to $f^{-1}(y)$ and hence

$$f^*(Y) \in g_{f^{-1}(y)}(\mathscr{U}).$$

But then it follows that $f^{-1}(y) \in \mathscr{U}$. Since this result holds for all fibers of f, f should be constant. So we have that X is compact. Next we show that the fibers of f are open subsets of X. Let $f^{-1}(y)$ be a fiber, $x \in f^{-1}(y)$ and let $\mathscr{U}$ be an ultrafilter on X converging to x. We consider again the map $g_{f^{-1}(y)}$. The ultrafilter $g_{f^{-1}(y)}(\mathscr{U})$ converges to $f^{-1}(y)$. Since it contains $f^*(Y)$ we have $f^{-1}(y) \in \mathscr{U}$. It follows that there is at most a finite number of fibers.

Finally we prove that each fiber is finite. Let $f^{-1}(y)$ be a fiber and let χ be an ultrafilter on $\mathscr{P}(X)$ containing $\mathscr{J}(f^{-1}(y))$ and such that $\mathscr{F}(\chi) = [f^{-1}(y)]$ (2.7). Since $f^*(Y)$ is open we have

$$f^*(Y) \cap \mathscr{J}(f^{-1}(y)) \neq \emptyset.$$

Hence $f^{-1}(y)$ is finite.

THEOREM 2.9. *$f^*(Y)$ is closed in $\mathscr{P}(X)$ if and only if f is open and Y is compact.*

Proof. Suppose $f^*(Y)$ is closed in $\mathscr{P}(X)$. Then $f^*(Y)$ endowed with the closed convergence is a compact Hausdorff space. It follows that the spaces Y, $f^*(Y)$ with the quotient topology and $f^*(Y)$ with the closed convergence are all homeomorphic. From (2.3) we have that f is open. For the converse suppose that f is open and Y is compact. From (2.3) we have that f^* is continuous. Since $f^*(Y)$ is compact for the closed convergence it is closed in $\mathscr{P}(X)$.

THEOREM 2.10. *$f^*(Y)$ is never dense in $\mathscr{P}(X)$.*

Proof. Either f is constant and we have $f^*(Y) = \{X\}$ which is not dense in $\mathscr{P}(X)$, or f is not constant. Let $f^{-1}(y_1)$ and $f^{-1}(y_2)$ be different fibers and let $\mathscr{O}_1$ and $\mathscr{O}_2$ be disjoint open neighborhoods for the quotient topology on $f^*(Y)$. Let φ be the quotient map. Then $O_1 = \varphi^{-1}(\mathscr{O}_1)$ and $O_2 = \varphi^{-1}(\mathscr{O}_2)$ are disjoint open saturated subsets of X. It follows that the infimum of a filter χ on $\mathscr{P}(X)$ containing $f^*(Y)$ cannot contain points of O_1 and O_2 at the same time. It follows that $X \notin \overline{f^*(Y)}$.

Next we consider the extension of f^* to all closed subsets of Y. Let $f^{**}:\mathscr{P}(Y) \to \mathscr{P}(X)$ be the function mapping $F \in \mathscr{P}(Y)$ to $f^{-1}(F) \in \mathscr{P}(X)$. f^{**} is a one to one map. On $f^{**}(\mathscr{P}(Y))$ we consider the structure induced by the closed convergence on $\mathscr{P}(X)$.

THEOREM 2.11. *The following properties are equivalent:*
(1) *$f^{**}:\mathscr{P}(Y) \to f^{**}(\mathscr{P}(Y))$ is a homeomorphism.*
(2) *f^{**} is continuous.*
(3) *f^* is continuous.*
(4) *f is open.*

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3) since Y is a subspace of $\mathscr{P}(Y)$. (3) $\Rightarrow$ (4) by (2.3). (4) $\Rightarrow$ (1): Suppose f is open and let χ be an ultrafilter on $\mathscr{P}(Y)$ converging to some $F \in \mathscr{P}(Y)$. Then $f^{**}(\chi)$ is an ultrafilter on $\mathscr{P}(X)$. If $\chi = \dot{\emptyset}$ then $f^{**}\chi = \dot{\emptyset}$. If $\chi \neq \dot{\emptyset}$ then for $\mathscr{B} \in \chi$ we have

$$E_{f^{**}(\mathscr{B})} = \bigcup_{B \in \mathscr{B}} f^{**}(B) = \bigcup_{B \in \mathscr{B}} f^{-1}(B) = f^{-1}(E_{\mathscr{B}})$$

and hence $\mathscr{F}(f^{**}(\chi)) = f^{-1}(\mathscr{F}(\chi))$. From (2.2) it follows that

$$\begin{aligned} \sup f^{**}(\chi) &= \alpha\, \mathscr{F}(f^{**}(\chi)) = \alpha f^{-1}(\mathscr{F}(\chi)) = \bigcap_{G \in \mathscr{F}(\chi)} \overline{f^{-1}(G)} \\ &= \bigcap_{G \in \mathscr{F}(\chi)} f^{-1}(\bar{G}) = f^{-1}\left(\bigcap_{G \in \mathscr{F}(\chi)} \bar{G}\right) = f^{-1}(\alpha \mathscr{F}(\chi)) \\ &= f^{-1}(F) = f^{**}(F). \end{aligned}$$

Finally we have that $f^{**}(\chi)$ converges to $f^{**}(F)$. Next let χ be an ultrafilter on $f^{**}(\mathscr{P}(Y))$ converging to $E \in f^{**}(\mathscr{P}(Y))$. If $\chi = \dot{\emptyset}$ then $f^{**-1}(\chi) = \dot{\emptyset}$. Suppose $\chi \neq \dot{\emptyset}$. Let $F \in \mathscr{P}(Y)$ be such that $f^{**}(F) = f^{-1}(F) = E$. $\mathscr{F}(\chi)$ has a base consisting of saturated subsets of X. It follows that

$$f^{-1}f(\mathscr{F}(\chi)) = \mathscr{F}(\chi) \quad (1).$$

For $\mathscr{A} \in \chi$ we have

$$E_{f^{**-1}(\mathscr{A})} = \bigcup_{A \in \mathscr{A}} f^{**-1}(A) = \bigcup_{A \in \mathscr{A}} f(A) = f(E_{\mathscr{A}})$$

and hence

$$\mathscr{F}(f^{**-1}(\chi)) = f(\mathscr{F}(\chi)) \quad (2).$$

Since f is open we have from (2.2)

$$\alpha f^{-1}(f(\mathscr{F}(\chi)) = f^{-1}\alpha f(\mathscr{F}(\chi)) \quad (3).$$

Combining (1) (2) and (3) we finally have

$$\alpha \mathscr{F}(\chi) = f^{-1}\alpha f(\mathscr{F}(\chi)) \quad \text{and}$$
$$\alpha \mathscr{F}(f^{**-1}(\chi)) = \alpha f(\mathscr{F}(\chi)) = f\alpha \mathscr{F}(\chi) = f(E) = f^{**-1}(E).$$

It follows that $f^{**-1}(\chi)$ converges to $f^{**-1}(E)$.

3. Proper maps. In this section we suppose f is a closed map from X onto Y. We consider the extension of f to the collection of closed subsets of X. Let $\tilde{f}:\mathscr{P}(X) \to \mathscr{P}(Y)$ be the function which maps $E \in \mathscr{P}(X)$ to $f(E) \in \mathscr{P}(Y)$. I am indebted to the referee for drawing my attention to a result of [7] which allows the shortening of the proof of the next theorem.

THEOREM 3.1. *The following properties are equivalent:*
(1) *f is proper.*
(2) *$\tilde{f}$ is continuous.*
(3) *$\tilde{f}$ is proper.*

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Proof. (1) $\Rightarrow$ (2). Suppose f is proper. Then clearly $\tilde{f}$ is onto. Let χ be an ultrafilter on $\mathscr{P}(X)$ converging to some $E \in \mathscr{P}(X)$. If $\chi = \dot{\emptyset}$ then clearly $\tilde{f}(\chi) = \dot{\emptyset}$. If $\chi \neq \dot{\emptyset}$ let $\mathscr{A} \in \chi$. Then we have $E_{\tilde{f}(\mathscr{A})} = f(E_{\mathscr{A}})$. So $\mathscr{F}(\tilde{f}(\chi)) = f(\mathscr{F}(\chi))$. Since f is proper we have $\alpha f(\mathscr{F}(\chi)) = f\alpha\,\mathscr{F}(\chi)$. So $\tilde{f}(\chi)$ converges to $f(E)$ and $\tilde{f}$ is continuous.

(2) $\Rightarrow$ (1). Suppose $\tilde{f}$ is continuous and let $\mathscr{F}$ be a filter on X. There exists an ultrafilter χ on $\mathscr{P}(X)$ such that $\mathscr{F}(\chi) = \mathscr{F}$. The proof of this statement is similar to (2.7) and can be found in [**8**]. χ converges to $\alpha(\mathscr{F})$ and so $\tilde{f}(\chi)$ converges to $f(\alpha(\mathscr{F}))$. It follows that

$$f(\alpha(\mathscr{F})) = \alpha\mathscr{F}(\tilde{f}(\chi)) = \alpha f(\mathscr{F}(\chi)) = \alpha f(\mathscr{F}).$$

Hence f is proper [**1**].

(2) $\Leftrightarrow$ (3). Since $\tilde{f}$ is a map from a compact space onto a Hausdorff space the continuity and the properness are equivalent [**7**].

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